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ON CONVERGENCE THEOREMS OF AN IMPLICIT ITERATIVE PROCESS WITH ERRORS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI $I-$NONEXPANSIVE MAPPINGS

FARRUKH MUKHAMEDOV AND MANSOOR SABUROV

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DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCES, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. BOX, 141, 25710, KUANTAN,
farrukh_m@iiu.edu.my
msaburov@gmail.com
URL: http://www.iium.edu.my

ABSTRACT. In this paper we prove the weak and strong convergence of the implicit iterative process with errors to a common fixed point of a finite family $\{T_j\}_{i=1}^N$ of asymptotically quasi I_j-nonexpansive mappings as well as a family of $\{I_j\}_{j=1}^N$ of asymptotically quasi nonexpansive mappings in the framework of Banach spaces. The obtained results improve and generalize the corresponding results in the existing literature.

Key words and phrases: Implicit process with errors; Asymptotically quasi $I-$nonexpansive mapping; A common fixed point.

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1. INTRODUCTION

Let K be a nonempty subset of a real normed linear space X and $T : K \to K$ be a mapping. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in K : Tx = x\}$. Throughout this paper, we always assume that $F(T) \neq \emptyset$. Now let us recall some known definitions

Definition 1.1. A mapping $T : K \to K$ is said to be:

(i) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$;
(ii) asymptotically nonexpansive, if there exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} \lambda_n = 1$ such that $\|T^n x - T^n y\| \leq \lambda_n \|x - y\|$ for all $x, y \in K$ and $n \in \mathbb{N}$;
(iii) quasi-nonexpansive, if $\|Tx - p\| \leq \|x - p\|$ for all $x \in K,\ p \in F(T)$;
(iv) asymptotically quasi-nonexpansive, if there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} \mu_n = 1$ such that $\|T^n x - p\| \leq \mu_n \|x - p\|$ for all $x \in K,\ p \in F(T)$ and $n \in \mathbb{N}$.

Note that from the above definitions, it follows that a nonexpansive mapping must be asymptotically nonexpansive, and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive, but the converse does not hold (see [10]).

If K is a closed nonempty subset of a Banach space and $T : K \to K$ is nonexpansive, then it is known that T may not have a fixed point (unlike the case if T is a strict contraction), and even when it has, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ (the so-called *Picard sequence*) may fail to converge to such a fixed point.

In [1]-[2] Browder studied the iterative construction for fixed points of nonexpansive mappings on closed and convex subsets of a Hilbert space. Note that for the past 30 years or so, the study of the iterative processes for the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations have been flourishing areas of research for many mathematicians (see for more details [10],[6]).

In [7] Diaz and Metcalf studied quasi-nonexpansive mappings in Banach spaces. Ghosh and Debnath [11] established a necessary and sufficient condition for convergence of the Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a Banach space. The iterative approximation problems for nonexpansive mapping, asymptotically nonexpansivemapping and asymptotically quasi-nonexpansive mapping were studied extensively by Goebel and Kirk [9], Liu [18], Wittmann [30], Reich [23], Gornicki [12], Shu [26] Shoji and Takahashi [25], Tan and Xu [28] et al. in the settings of Hilbert spaces and uniformly convex Banach spaces.

There are many methods for approximating fixed points of a nonexpansive mapping. Xu and Ori [31] introduced implicit iteration process to approximate a common fixed point of a finite family of nonexpansive mappings in a Hilbert space. Namely, let H be a Hilbet space, K be a nonempty closed convex subset of H and $\{T_j\}_{j=1}^N : K \to K$ be nonexpansive mappings. Then Xu and Ori's implicit iteration process $\{x_n\}$ is defined by

$$
\left\{ \begin{array}{c} x_0 \in K, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_n; \end{array} \right. \quad n \geq 1 \tag{1.1}
$$

where $T_n = T_{n(modN)}$, $\{\alpha_n\}$ is a real sequence in $(0, 1)$. They proved the weak convergence of the sequence $\{x_n\}$ defined by (1.1) to a common fixed point $p \in F = \cap_{j=1}^N F(T_j)$.

In 2003, Sun [27] introduced the following implicit iterative sequence $\{x_n\}$

$$
\left\{ \begin{array}{c} x_0 \in K, \\ x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_{j(n)}^{k(n)} x_n; \end{array} \right. \quad n \geq 1 \tag{1.2}
$$

for a finite set of asymptotically quasi-nonexpansive self-mappings on a bounded closed convex subset K of a Banach space X with $\{\alpha_n\}$ a sequence in $(0, 1)$. Where $n = (k(n)-1)N + j(n)$,

$j(n) \in \{1, 2, \dots, N\}$, and proved the strong convergence of the sequence $\{x_n\}$ defined by (1.2) to a common fixed point $p \in F = \cap_{j=1}^N F(T_j)$.

There many papers devoted to the implicit iteration process for a finite family of asymptotically nonexpansive mappings, asymptotically quasi-expansive mappings in Banach spaces (see for example [4, 5, 14, 15, 18, 32]).

On the other hand, there are many concepts which generalize a notion of nonexpansive mapping. One of such concepts is I-nonexpansivity of a mapping T([24]). Let us recall some notions.

Definition 1.2. Let $T : K \to K$, $I : K \to K$ be two mappings of a nonempty subset K of a real normed linear space X. Then T is said to be:

(i) *$I-$nonexpansive*, if $\|Tx - Ty\| \le \|Ix - Iy\|$ for all $x, y \in K$;

(i) *asymptotically $I-$nonexpansive*, if there exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} \lambda_n = 1$ such that $\|T^n x - T^n y\| \le \lambda_n \|I^n x - I^n y\|$ for all $x, y \in K$ and $n \ge 1$;

(iii) *asymptotically quasi $I-$nonexpansive* mapping, if there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim\limits_{n\to\infty} \mu_n = 1$ such that $\|T^n x - p\| \le \mu_n \|I^n x - p\|$ for all $x \in K$, $p \in F(T) \cap F(I)$ and $n \ge 1$.

Remark 1.1. If $F(T) \cap F(I) \neq \emptyset$ then an asymptotically $I-$nonexpansive mapping is asymptotically quasi $I-$nonexpansive. But, there exists a nonlinear continuous asymptotically quasi $I-$nonexpansive mappings which is asymptotically $I-$nonexpansive.

Indeed, let us consider the following example. Let $X = \ell_2$ and $K = \{\mathbf{x} \in \ell_2 : \ \|\mathbf{x}\| \le 1\}$. Define the following mappings:

$$T(x_1, x_2, \dots, x_n, \dots) = (0, x_1^4, x_2^4, \dots, x_n^4, \dots), \tag{1.3}$$

$$I(x_1, x_2, \dots, x_n, \dots) = (0, x_1^2, x_2^2, \dots, x_n^2, \dots). \tag{1.4}$$

One see that $F(T) = F(I) = (0, 0, \dots, 0, \dots)$. Therefore, from $\sum\limits_{k=1}^{\infty} x_k^8 \le \sum\limits_{k=1}^{\infty} x_k^4$ whenever $\mathbf{x} \in K$, using (1.3),(1.4) we obtain $\|T\mathbf{x}\| \le \|I\mathbf{x}\|$ for every $\mathbf{x} \in K$. So, T is quasi I-expansive. But for $\mathbf{x}_0 = (1, 0, \dots, 0, \dots)$ and $\mathbf{y}_0 = (1/2, 0, \dots, 0, \dots)$ we have

$$\|T(\mathbf{x}_0) - T(\mathbf{y}_0)\| = \frac{15}{16}, \quad \|I(\mathbf{x}_0) - I(\mathbf{y}_0)\| = \frac{3}{4}$$

which means that T is not I-nonexpansive.

Note that best approximation properties of I-nonexpansive mappings were investigated in [24]. In [16] the weak convergence of three-step Noor iterative scheme for an I-nonexpansive mappping in a Banach space has been established. In [20] we have proved the strong convergence of an explicit iterative process for a totally asymptotically $I-$nonexpansive mappings in Banach spaces.

Very recently, in [29] the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of I-asymptotically nonexpansive mappings were studied. Let us describe the iteration scheme considered in [29]. Let K be a nonempty convex subset of a real Banach space X and $\{T_j\}_{j=1}^N : K \to K$ be a finite family of asymptotically I_j-nonexpansive mappings, $\{I_j\}_{j=1}^N : K \to K$ be a finite family of asymptotically nonexpansive mappings. Then the iteration process $\{x_n\}$ has been defined by

$$\begin{cases} x_1 \in K, & \\ x_{n+1} = & (1-\alpha_n)x_n + \alpha_n I_{j(n)}^{k(n)} y_n, \quad n \ge 1 \\ y_n = & (1-\beta_n)x_n + \beta_n T_{j(n)}^{k(n)} x_n, \end{cases} \tag{1.5}$$

here as before $n = (k(n)-1)N + j(n)$, $j(n) \in \{1, 2, \dots, N\}$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1]$. From this formula one can easily see that the employed method, indeed, is not implicit iterative processes. The used process is some kind of modified Ishikawa iteration.

Therefore, in this paper we shall extend of the implicit iterative process with errors, defined in [27, 8], to a family of I-asymptotically quasi-nonexpansive mappings defined on a uniformly convex Banach space. Namely, let K be a nonempty convex subset of a real Banach space X and $\{T_j\}_{j=1}^N : K \to K$ be a finite family of asymptotically quasi I_j-nonexpansive mappings, and $\{I_j\}_{j=1}^N : K \to K$ be a family of asymptotically quasi-nonexpansive mappings. We consider the following implicit iterative scheme $\{x_n\}$ with errors:

$$\tag{1.6} \begin{cases} x_0 \in K, & \\ x_n = & \alpha_n x_{n-1} + \beta_n T_{j(n)}^{k(n)} y_n + \gamma_n u_n \quad n \geq 1 \\ y_n = & \widehat{\alpha}_n x_n + \widehat{\beta}_n I_{j(n)}^{k(n)} x_n + \widehat{\gamma}_n v_n \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ are six sequences in $[0, 1]$ satisfying $\alpha_n+\beta_n+\gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1$ for all $n \geq 1$, as well as $\{u_n\}$, $\{v_n\}$ are bounded sequences in K.

In this paper we shall prove the weak and strong convergence of the implicit iterative process (1.6) to a common fixed points of $\{T_j\}_{j=1}^N$ and $\{I_j\}_{j=1}^N$. All results presented here generalize and extend the corresponding main results of [27], [31],[8],[13],[19].

2. PRELIMINARIES

Throughout this paper, we always assume that X is a real Banach space. We denote $F(T)$ and $D(T)$ the set of fixed points and the domain of a mapping T, respectively. Recall that a Banach space X is said to satisfy *Opial condition* [21], if for each sequence $\{x_n\}$ in X, the condition that the sequence $x_n \to x$ weakly implies that

$$\tag{2.1} \liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X$ with $y \neq x$. It is well known that (see [17]) inequality (2.1) is equivalent to

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

Definition 2.1. Let K be a closed subset of a real Banach space X and $T : K \to K$ be a mapping.

(i) A mapping T is said to be semi-closed (demi-closed) at zero, if for each bounded sequence $\{x_n\}$ in K, the conditions x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 imply $Tx = 0$.

(ii) A mapping T is said to be semi-compact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0, (n \to \infty)$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in K$ strongly.

(iii) T is called a uniformly $L-$Lipschitzian mapping, if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $x, y \in K$ and $n \geq 1$.

From the definition we immediately get the following

Proposition 2.1. *Let K be a nonempty subset of a real Banach space X, $\{T_j\}_{j=1}^N : K \to K$ and $\{I_j\}_{j=1}^N : K \to K$ be finite familis of mappings.*

(i) *If $\{T_j\}_{j=1}^N$ is a finite family of asymptotically I_j-nonexpansive (resp. asymptotically quasi I_j-nonexpansive) mappings with sequences $\{\lambda_n^{(j)}\} \subset [1, \infty)$, then there exists a sequence $\{\lambda_n\} \subset [1, \infty)$ such that $\{T_j\}_{i=1}^N$ is a finite family of asymptotically*

I_j-nonexpansive (resp. asymptotically quasi I_j-nonexpansive) mappings with a common sequence $\{\lambda_n\} \subset [1, \infty)$.

(ii) *If $\{T_j\}_{j=1}^N$ is a finite family of uniformly L_j-Lipschitzian mappings, then there exists a constant $L > 0$ such that $\{T_j\}_{j=1}^N$ is a finite family of uniformly $L-$Lipschitzian mappings.*

The following lemmas play an important role in proving our main results.

Lemma 2.2 (see [26]). *Let X be a uniformly convex Banach space and b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequences in X such that*

$$\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = d, \quad \limsup_{n\to\infty} \|x_n\| \le d, \quad \limsup_{n\to\infty} \|y_n\| \le d,$$

holds some $d \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3 (see [28]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. If the following conditions is satisfied*

$$a_{n+1} \le (1+b_n)a_n + c_n, \quad n \ge 1,$$

then the limit $\lim_{n\to\infty} a_n$ exists.

3. MAIN RESULTS

In this section we shall prove our main results concerning weak and strong convergence of the sequence defined by (1.6). To formulate ones, we need some auxiliary results.

Lemma 3.1. *Let X be a real Banach space and K be a nonempty closed convex subset of X. Let $\{T_j\}_{j=1}^N : K \to K$ be a finite family of asymptotically quasi I_j-nonexpansive mappings with a common sequence $\{\lambda_n\} \subset [1, \infty)$ and $\{I_j\}_{j=1}^N : K \to K$ be a finite family of asymptotically quasi-nonexpansive mappings with a common sequence $\{\mu_n\} \subset [1, \infty)$ such that $F = \bigcap_{j=1}^N (F(T_j) \cap F(I_j)) \neq \emptyset$. Suppose $B^* = \sup_n \beta_n$, $\Lambda = \sup_n \lambda_n \ge 1$, $M = \sup_n \mu_n \ge 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ are six sequences in $[0, 1]$ which satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \quad \forall n \ge 1,$

(ii) $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1)\beta_n < \infty,$

(iii) $B^* < \dfrac{1}{\Lambda^2 M^2},$

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty.$

Then for the implicit iterative sequence $\{x_n\}$ with errors defined by (1.6) and for each $p \in F$ the limit $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Since $F = \bigcap_{j=1}^{N} (F(T_j) \cap F(I_j)) \neq \emptyset$, for any given $p \in F$, it follows from (1.6) that

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n(x_{n-1} - p) + \beta_n(T_{j(n)}^{k(n)} y_n - p) + \gamma_n(u_n - p)\| \\ &\leq (1 - \beta_n - \gamma_n)\|x_{n-1} - p\| + \beta_n\|T_{j(n)}^{k(n)} y_n - p\| + \gamma_n\|u_n - p\| \\ &\leq (1 - \beta_n - \gamma_n)\|x_{n-1} - p\| + \beta_n\lambda_{k(n)}\|I_{j(n)}^{k(n)} y_n - p\| + \gamma_n\|u_n - p\| \\ &\leq (1 - \beta_n - \gamma_n)\|x_{n-1} - p\| + \beta_n\lambda_{k(n)}\mu_{k(n)}\|y_n - p\| + \gamma_n\|u_n - p\| \\ &\leq (1 - \beta_n)\|x_{n-1} - p\| + \beta_n\lambda_{k(n)}\mu_{k(n)}\|y_n - p\| + \gamma_n\|u_n - p\|. \end{aligned}$$

Again using (1.6) we find

$$\begin{aligned} \|y_n - p\| &= \|\widehat{\alpha}_n(x_n - p) + \widehat{\beta}_n(I_{j(n)}^{k(n)} x_n - p) + \widehat{\gamma}_n(v_n - p)\| \\ &\leq (1 - \widehat{\beta}_n - \widehat{\gamma}_n)\|x_n - p\| + \widehat{\beta}_n\|I_{j(n)}^{k(n)} x_n - p\| + \widehat{\gamma}_n\|v_n - p\| \\ &\leq (1 - \widehat{\beta}_n - \widehat{\gamma}_n)\|x_n - p\| + \widehat{\beta}_n\mu_{k(n)}\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| \\ &\leq (1 - \widehat{\beta}_n)\|x_n - p\| + \widehat{\beta}_n\mu_{k(n)}\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| \\ &\leq (1 - \widehat{\beta}_n)\mu_{k(n)}\|x_n - p\| + \widehat{\beta}_n\mu_{k(n)}\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| \\ &\leq \mu_{k(n)}\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\| \\ &\leq \lambda_{k(n)}\mu_{k(n)}\|x_n - p\| + \widehat{\gamma}_n\|v_n - p\|. \end{aligned} \tag{3.1}$$

Then from (3.1) we have

$$\begin{aligned} \|x_n - p\| \leq\ & (1 - \beta_n)\|x_{n-1} - p\| + \beta_n\lambda_{k(n)}^2\mu_{k(n)}^2\|x_n - p\| \\ & + \gamma_n\|u_n - p\| + \beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n - p\|, \end{aligned}$$

so one gets

$$\begin{aligned} (1 - \beta_n\lambda_{k(n)}^2\mu_{k(n)}^2)\|x_n - p\| \leq\ & (1 - \beta_n)\|x_{n-1} - p\| \\ & + \gamma_n\|u_n - p\| + \beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n - p\|. \end{aligned} \tag{3.2}$$

By condition (iii) we obtain $\beta_n\lambda_{k(n)}^2\mu_{k(n)}^2 \leq B^*\Lambda^2 M^2 < 1$, and therefore

$$1 - \beta_n\lambda_{k(n)}^2\mu_{k(n)}^2 \geq 1 - B^*\Lambda^2 M^2 > 0.$$

Hence from (3.2) we derive

$$\begin{aligned}
\|x_n-p\| &\leq \frac{1-\beta_n}{1-\beta_n\lambda_{k(n)}^2\mu_{k(n)}^2}\|x_{n-1}-p\| \\
&\qquad+\frac{\gamma_n\|u_n-p\|+\beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n-p\|}{1-\beta_n\lambda_{k(n)}^2\mu_{k(n)}^2} \\
&= \left(1+\frac{(\lambda_{k(n)}^2\mu_{k(n)}^2-1)\beta_n}{1-\beta_n\lambda_{k(n)}^2\mu_{k(n)}^2}\right)\|x_{n-1}-p\| \\
&\qquad+\frac{\gamma_n\|u_n-p\|+\beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n-p\|}{1-\beta_n\lambda_{k(n)}^2\mu_{k(n)}^2} \\
&\leq \left(1+\frac{(\lambda_{k(n)}^2\mu_{k(n)}^2-1)\beta_n}{1-B^*\Lambda^2M^2}\right)\|x_{n-1}-p\| \\
&\qquad+\frac{\gamma_n\|u_n-p\|+\beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n-p\|}{1-B^*\Lambda^2M^2}.
\end{aligned}$$

Let

$$b_n=\frac{(\lambda_{k(n)}^2\mu_{k(n)}^2-1)\beta_n}{1-B^*\Lambda^2M^2},\qquad c_n=\frac{\gamma_n\|u_n-p\|+\beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n-p\|}{1-B^*\Lambda^2M^2}.$$

Then the last inequality has the following form

$$\|x_n-p\| \leq (1+b_n)\|x_{n-1}-p\|+c_n. \tag{3.3}$$

From the condition (ii) we find

$$\begin{aligned}
\sum_{n=1}^{\infty}b_n &= \frac{1}{1-B^*\Lambda^2M^2}\sum_{n=1}^{\infty}(\lambda_{k(n)}^2\mu_{k(n)}^2-1)\beta_n \\
&= \frac{1}{1-B^*\Lambda^2M^2}\sum_{n=1}^{\infty}(\lambda_{k(n)}\mu_{k(n)}-1)(\lambda_{k(n)}\mu_{k(n)}+1)\beta_n \\
&\leq \frac{\Lambda M+1}{1-B^*\Lambda^2M^2}\sum_{n=1}^{\infty}(\lambda_{k(n)}\mu_{k(n)}-1)\beta_n<\infty,
\end{aligned}$$

and boundedness of the sequences $\{\|u_n-p\|\}$, $\{\|v_n-p\|\}$ with (iv) implies

$$\begin{aligned}
\sum_{n=1}^{\infty}c_n &= \sum_{n=1}^{\infty}\frac{\gamma_n\|u_n-p\|+\beta_n\lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n\|v_n-p\|}{1-B^*\Lambda^2M^2} \\
&\leq \frac{1}{1-B^*\Lambda^2M^2}\sum_{n=1}^{\infty}\gamma_n\|u_n-p\|+\frac{B^*\Lambda M}{1-B^*\Lambda^2M^2}\sum_{n=1}^{\infty}\widehat{\gamma}_n\|v_n-p\|<\infty.
\end{aligned}$$

Now taking $a_n=\|x_{n-1}-p\|$ in (3.3) we obtain

$$a_{n+1}\leq(1+b_n)a_n+c_n,$$

and according to Lemma 2.3 the limit $\lim\limits_{n\to\infty}a_n$ exists. This means the limit

$$\lim_{n\to\infty}\|x_n-p\|=d \tag{3.4}$$

exists, where $d\geq 0$ is a constant. This completes the proof. ∎

Now we are ready to prove a general criteria of strong convergence of (1.6).

Theorem 3.2. *Let X be a real Banach space and K be a nonempty closed convex subset of X. Let $\{T_j\}_{j=1}^N : K \to K$ be a finite family of uniformly L_1-Lipschitzian asymptotically quasi I_j-nonexpansive mappings with a common sequence $\{\lambda_n\} \subset [1,\infty)$ and $\{I_j\}_{j=1}^N : K \to K$ be a finite family of uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mappings with a common sequence $\{\mu_n\} \subset [1,\infty)$ such that $F = \bigcap_{j=1}^N (F(T_j) \cap F(I_j)) \neq \emptyset$. Suppose $B^* = \sup_n \beta_n$, $\Lambda = \sup_n \lambda_n \geq 1$, $M = \sup_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ are six sequences in $[0,1]$ which satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \quad \forall n \geq 1,$

(ii) $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1)\beta_n < \infty,$

(iii) $B^* < \dfrac{1}{\Lambda^2 M^2},$

(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty.$

Then the implicit iterative sequence $\{x_n\}$ with errors defined by (1.6) *converges strongly to a common fixed point in F if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0. \tag{3.5}$$

Proof. The necessity of condition (3.5) is obvious. Let us proof the sufficiency part of the theorem.

Since $T_j, I_j : K \to K$ are uniformly L_1, L_2-Lipschitzian mappings, respectively, T_j and I_j are continuous mappings, for each $j = \overline{1,N}$. Therefore, the sets $F(T_j)$ and $F(I_j)$ are closed, for each $j = \overline{1,N}$. Hence $F = \bigcap_{j=1}^N (F(T_j) \cap F(I_j))$ is a nonempty closed set.

For any given $p \in F$, we have (see (3.3))

$$\|x_n - p\| \leq (1 + b_n)\, \|x_{n-1} - p\| + c_n, \tag{3.6}$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Hence, we find

$$d(x_n, F) \leq (1 + b_n)\, d(x_{n-1}, F) + c_n \tag{3.7}$$

So, the inequality (3.7) with Lemma 2.3 implies the existence of the limit $\lim_{n\to\infty} d(x_n, F)$. By condition (3.5), one gets

$$\lim_{n\to\infty} d(x_n, F) = \liminf_{n\to\infty} d(x_n, F) = 0.$$

Let us prove that the sequence $\{x_n\}$ converges to a common fixed point of $\{T_j\}_{j=1}^N$ and $\{I_j\}_{j=1}^N$.

In fact, due to $1 + t \leq \exp(t)$ for all $t > 0$, and from (3.6), one finds

$$\|x_n - p\| \leq \exp(b_n)\|x_{n-1} - p\| + c_n. \tag{3.8}$$

Hence, for any positive integer $m, n,$ from (3.8) we find

$$\begin{aligned}
\|x_{n+m}-p\| &\leq \exp(b_{n+m})\|x_{n+m-1}-p\|+c_{n+m} \\
&\leq \exp(b_{n+m}+b_{n+m-1})\|x_{n+m-2}-p\| \\
&\quad +c_{n+m}+c_{n+m-1}\exp(b_{n+m}) \\
&\leq \exp(b_{n+m}+b_{n+m-1}+b_{n+m-2})\|x_{n+m-3}-p\| \\
&\quad +c_{n+m}+c_{n+m-1}\exp(b_{n+m})+c_{n+m-2}\exp(b_{n+m}+b_{n+m-1}) \\
&\quad \vdots \\
&\leq \exp\left(\sum_{i=n+1}^{n+m} b_i\right)\|x_n-p\|+c_{n+m}+\sum_{j=n+1}^{n+m-1} c_j\exp\left(\sum_{i=j+1}^{n+m} b_i\right) \\
&\leq \exp\left(\sum_{i=n+1}^{n+m} b_i\right)\|x_n-p\| \\
&\quad +c_{n+m}\exp\left(\sum_{i=n+1}^{n+m} b_i\right)+\sum_{j=n+1}^{n+m-1} c_j\exp\left(\sum_{i=n+1}^{n+m} b_i\right) \\
&\leq \exp\left(\sum_{i=n+1}^{n+m} b_i\right)\left(\|x_n-p\|+\sum_{j=n+1}^{n+m} c_j\right) \\
&\leq \exp\left(\sum_{i=1}^{\infty} b_i\right)\left(\|x_n-p\|+\sum_{j=n+1}^{\infty} c_j\right) \\
&\leq W\left(\|x_n-p\|+\sum_{j=n+1}^{\infty} c_j\right), \qquad (3.9)
\end{aligned}$$

for all $p \in F$, where $W=\exp\left(\sum_{i=1}^{\infty} b_i\right)<\infty$.

Since $\lim_{n\to\infty} d(x_n, F)=0$ and $\sum_{j=1}^{\infty} c_j<\infty$, for any given $\varepsilon>0$, there exists a positive integer number n_0 such that

$$d(x_{n_0}, F)<\frac{\varepsilon}{2W}, \qquad \sum_{j=n_0+1}^{\infty} c_j<\frac{\varepsilon}{2W}.$$

Therefore there exists $p_1 \in F$ such that

$$\|x_{n_0}-p_1\|<\frac{\varepsilon}{2W}, \qquad \sum_{j=n_0+1}^{\infty} c_j<\frac{\varepsilon}{2W}.$$

Consequently, for all $n \geq n_0$ from (3.9) we have

$$\begin{aligned}
\|x_n-p_1\| &\leq W\left(\|x_{n_0}-p_1\|+\sum_{j=n_0+1}^{\infty} c_j\right) \\
&< W\cdot\frac{\varepsilon}{2W}+W\cdot\frac{\varepsilon}{2W} \\
&= \varepsilon,
\end{aligned}$$

this means that the sequence $\{x_n\}$ converges strongly to a common fixed point p_1 of $\{T_j\}_{j=1}^N$ and $\{I_j\}_{j=1}^N$. This completes the proof. ∎

To prove main results we need one more an auxiliary result.

Proposition 3.3. *Let X be a real uniformly convex Banach space and K be a nonempty closed convex subset of X. Let $\{T_j\}_{j=1}^N : K \to K$ be a finite family of uniformly L_1-Lipschitzian asymptotically quasi I_j-nonexpansive mappings with a common sequence $\{\lambda_n\} \subset [1,\infty)$ and $\{I_j\}_{j=1}^N : K \to K$ be a finite family of uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mappings with a common sequence $\{\mu_n\} \subset [1,\infty)$ such that*

$$F = \bigcap_{j=1}^{N} (F(T_j) \cap F(I_j)) \neq \emptyset.$$

Suppose $B_* = \inf_n \beta_n$, $B^* = \sup_n \beta_n$, $\Lambda = \sup_n \lambda_n \geq 1$, $M = \sup_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ are six sequences in $[0,1]$ which satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \quad \forall n \geq 1,$

(ii) $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1)\beta_n < \infty,$

(iii) $0 < B_* \leq B^* < \dfrac{1}{\Lambda^2 M^2} < 1,$

(iv) $0 < \widehat{B}_* = \inf_n \widehat{\beta}_n \leq \sup_n \widehat{\beta}_n = \widehat{B}^* < 1,$

(v) $\sum_{n=1}^{\infty} \gamma_n < \infty, \quad \sum_{n=1}^{\infty} \widehat{\gamma}_n < \infty.$

Then the implicit iterative sequence $\{x_n\}$ with errors defined by (1.6) *satisfies the following*

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - I_j x_n\| = 0, \qquad \forall j = \overline{1,N}.$$

Proof. First, we shall prove that

$$\lim_{n\to\infty} \|x_n - T_{j(n)}^{k(n)} x_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - I_{j(n)}^{k(n)} x_n\| = 0.$$

According to Lemma 3.1 for any $p \in F$ we have

$$\lim_{n\to\infty} \|x_n - p\| = d. \tag{3.10}$$

So, the sequence $\{x_n\}$ is bounded in K.

It follows from (1.6) that

$$\begin{aligned} \|x_n - p\| &= \|(1-\beta_n)(x_{n-1} - p + \gamma_n(u_n - x_{n-1})) \\ &\qquad + \beta_n(T_{j(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}))\|. \end{aligned} \tag{3.11}$$

Due to condition (v) and boundedness of the sequences $\{u_n\}$ and $\{x_n\}$ we have

$$\begin{aligned} \limsup_{n\to\infty} \|x_{n-1} &- p + \gamma_n(u_n - x_{n-1})\| \leq \\ &\leq \limsup_{n\to\infty} \|x_{n-1} - p\| + \limsup_{n\to\infty} \gamma_n \|u_n - x_{n-1}\| = d. \end{aligned} \tag{3.12}$$

By means of asymptotically quasi I_j-nonexpansivity of T_j and asymptotically quasi-nonexpansivity of I_j from (3.1) and boundedness of $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ with condition (v) we obtain

$$\begin{aligned}\limsup_{n\to\infty} & \| \quad T_{j(n)}^{k(n)} y_n - p + \gamma_n(u_n - x_{n-1}) \quad \| \\ & \leq \limsup_{n\to\infty} \lambda_{k(n)}\mu_{k(n)}\|y_n - p\| + \limsup_{n\to\infty} \gamma_n \|u_n - x_{n-1}\| \\ & \leq \limsup_{n\to\infty} \|y_n - p\| \\ & \leq \limsup_{n\to\infty} \lambda_{k(n)}\mu_{k(n)}\|x_n - p\| + \limsup_{n\to\infty} \widehat{\gamma}_n \|v_n - p\| = d. \end{aligned} \tag{3.13}$$

Now using (3.12), (3.13) and applying to Lemma 2.2 to (3.11) one finds

$$\lim_{n\to\infty} \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| = 0. \tag{3.14}$$

From (1.6), (3.14) and condition (v) we infer that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = \lim_{n\to\infty} \|\beta_n(T_{j(n)}^{k(n)} y_n - x_{n-1}) + \gamma_n(u_n - x_{n-1})\| = 0. \tag{3.15}$$

From (3.15) one can get

$$\lim_{n\to\infty} \|x_n - x_{n+j}\| = 0, \qquad j = \overline{1, N}. \tag{3.16}$$

On the other hand, we have

$$\begin{aligned} \|x_{n-1} - p\| & \leq \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| + \|T_{j(n)}^{k(n)} y_n - p\| \\ & \leq \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| + \lambda_{k(n)}\mu_{k(n)}\|y_n - p\|, \end{aligned}$$

which means

$$\|x_{n-1} - p\| - \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| \leq \lambda_{k(n)}\mu_{k(n)}\|y_n - p\|.$$

The last inequality with (3.1) implies that

$$\begin{aligned} \|x_{n-1} - p\| - \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| & \leq \lambda_{k(n)}\mu_{k(n)}\|y_n - p\| \leq \\ & \leq \lambda_{k(n)}^2 \mu_{k(n)}^2 \|x_n - p\| + \lambda_{k(n)}\mu_{k(n)}\widehat{\gamma}_n \|v_n - p\|. \end{aligned}$$

Then condition (v) and (3.14), (3.10) with the Squeeze theorem yield

$$\lim_{n\to\infty} \|y_n - p\| = d \tag{3.17}$$

Again from (1.6) we can see that

$$\begin{aligned} \|y_n - p\| = \|(1 - \widehat{\beta}_n)(x_n - p & + \widehat{\gamma}_n(v_n - x_n)) \\ & + \widehat{\beta}_n(I_{j(n)}^{k(n)} x_n - p + \widehat{\gamma}_n(v_n - x_n))\|. \end{aligned} \tag{3.18}$$

From (3.10) with condition (v) one finds

$$\begin{aligned} \limsup_{n\to\infty} \|x_n & - p + \widehat{\gamma}_n(v_n - x_n)\| \leq \\ & \leq \limsup_{n\to\infty} \|x_n - p\| + \limsup_{n\to\infty} \widehat{\gamma}_n \|v_n - x_n\| = d. \end{aligned}$$

and

$$\begin{aligned} \limsup_{n\to\infty} \|I_{j(n)}^{k(n)} x_n & - p + \widehat{\gamma}_n(v_n - x_n)\| \\ & \leq \limsup_{n\to\infty} \mu_{k(n)}\|x_n - p\| + \limsup_{n\to\infty} \widehat{\gamma}_n \|v_n - x_n\| \\ & \leq \limsup_{n\to\infty} \|x_n - p\| = d. \end{aligned}$$

Now applying Lemma 2.2 to (3.18) we obtain

$$\lim_{n\to\infty} \|x_n - I_{j(n)}^{k(n)} x_n\| = 0. \tag{3.19}$$

Consider

$$\begin{aligned} \|x_n - T_{j(n)}^{k(n)} x_n\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| + \|T_{j(n)}^{k(n)} y_n - T_{j(n)}^{k(n)} x_n\| \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| + L_1\|y_n - x_n\| \\ &= \|x_n - x_{n-1}\| + \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| \\ &\quad + L_1\|\widehat{\beta}_n(I_{j(n)}^{k(n)} x_n - x_n) + \widehat{\gamma}_n(v_n - x_n)\| \\ &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_{j(n)}^{k(n)} y_n\| \\ &\quad + L_1\widehat{\beta}_n\|I_{j(n)}^{k(n)} x_n - x_n\| + L_1\widehat{\gamma}_n\|v_n - x_n\|. \end{aligned}$$

Then from (3.14), (3.15), (3.19) and condition (v) we get

$$\lim_{n\to\infty} \|x_n - T_{j(n)}^{k(n)} x_n\| = 0. \tag{3.20}$$

Now we prove that

$$\lim_{n\to\infty} \|x_n - T_j x_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - I_j x_n\| = 0, \qquad \forall j = \overline{1, N}.$$

For each $n > N$ we have $n \equiv n - N(\mathrm{mod} N)$ and $n = (k(n) - 1)N + j(n)$, hence $n - N = ((k(n) - 1) - 1)N + j(n) = (k(n - N) - 1)N + j(n - N)$, i.e.

$$k(n - N) = k(n) - 1, \qquad j(n - N) = j(n).$$

So letting $T_n := T_{j(n)(\mathrm{mod} N)}$ we obtain

$$\begin{aligned} \|x_n - T_n x_n\| &\leq \|x_n - T_{j(n)}^{k(n)} x_n\| + \|T_{j(n)}^{k(n)} x_n - T_{j(n)} x_n\| \\ &\leq \|x_n - T_{j(n)}^{k(n)} x_n\| + L_1\|T_{j(n)}^{k(n)-1} x_n - x_n\| \\ &\leq \|x_n - T_{j(n)}^{k(n)} x_n\| + L_1\|T_{j(n)}^{k(n)-1} x_n - T_{j(n-N)}^{k(n)-1} x_{n-N}\| \\ &\quad + L_1\|T_{j(n-N)}^{k(n)-1} x_{n-N} - x_{n-N}\| + L_1\|x_{n-N} - x_n\| \\ &\leq \|x_n - T_{j(n)}^{k(n)} x_n\| + L_1^2\|x_n - x_{n-N}\| \\ &\quad + L_1\|T_{j(n-N)}^{k(n-N)} x_{n-N} - x_{n-N}\| + L_1\|x_{n-N} - x_n\| \\ &\leq \|x_n - T_{j(n)}^{k(n)} x_n\| + L_1(L_1 + 1)\|x_n - x_{n-N}\| \\ &\quad + L_1\|T_{j(n-N)}^{k(n-N)} x_{n-N} - x_{n-N}\| \end{aligned}$$

which with (3.16), (3.20) implies

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.21}$$

Analogously, one has

$$\begin{aligned} \|x_n - I_n x_n\| &\leq \|x_n - I_{j(n)}^{k(n)} x_n\| + L_2(L_2 + 1)\|x_n - x_{n-N}\| \\ &\quad + L_2\|I_{k(n-N)}^{k(n-N)} x_{n-N} - x_{n-N}\|. \end{aligned}$$

which with (3.16), (3.19) implies

$$\lim_{n\to\infty} \|x_n - I_n x_n\| = 0. \tag{3.22}$$

For any $j = \overline{1, N}$, from (3.16) and (3.21) we have

$$\begin{aligned}\|x_n - T_{n+j}x_n\| &\leq \|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| + \|T_{n+j}x_{n+j} - T_{n+j}x_n\| \\ &\leq (1 + L_1)\|x_n - x_{n+j}\| + \|x_{n+j} - T_{n+j}x_{n+j}\| \to 0 \ \ (n \to \infty),\end{aligned}$$

which implies that the sequence

$$\bigcup_{j=1}^{N} \big\{ \|x_n - T_{n+j}x_n\| \big\}_{n=1}^{\infty} \to 0 \ \ (n \to \infty). \tag{3.23}$$

Analogously we have

$$\|x_n - I_{n+j}x_n\| \leq (1 + L_2)\|x_n - x_{n+j}\| + \|x_{n+j} - I_{n+j}x_{n+j}\| \to 0 \ \ (n \to \infty),$$

and

$$\bigcup_{j=1}^{N} \big\{ \|x_n - I_{n+j}x_n\| \big\}_{n=1}^{\infty} \to 0 \ \ (n \to \infty). \tag{3.24}$$

According to

$$\begin{aligned}\big\{ \|x_n - T_jx_n\| \big\}_{n=1}^{\infty} &= \big\{ \|x_n - T_{n+(j-n)}x_n\| \big\}_{n=1}^{\infty} \\ &= \big\{ \|x_n - T_{n+j_n}x_n\| \big\}_{n=1}^{\infty} \subset \bigcup_{l=1}^{N} \big\{ \|x_n - T_{n+l}x_n\| \big\}_{n=1}^{\infty},\end{aligned}$$

and

$$\begin{aligned}\big\{ \|x_n - I_jx_n\| \big\}_{n=1}^{\infty} &= \big\{ \|x_n - I_{n+(j-n)}x_n\| \big\}_{n=1}^{\infty} \\ &= \big\{ \|x_n - I_{n+j_n}x_n\| \big\}_{n=1}^{\infty} \subset \bigcup_{l=1}^{N} \big\{ \|x_n - I_{n+l}x_n\| \big\}_{n=1}^{\infty},\end{aligned}$$

where $j - n \equiv j_n (\mathrm{mod} N)$, $j_n \in \{1, 2, \cdots, N\}$, from (3.23), (3.24) we find

$$\lim_{n\to\infty} \|x_n - T_jx_n\| = 0, \qquad \lim_{n\to\infty} \|x_n - I_jx_n\| = 0, \qquad \forall j = \overline{1, N}. \tag{3.25}$$

∎

Now we are ready to formulate one of main result concerning weak convergence of the sequence $\{x_n\}$.

Theorem 3.4. *Let X be a real uniformly convex Banach space satisfying Opial condition and K be a nonempty closed convex subset of X. Let $E : X \to X$ be an identity mapping, $\{T_j\}_{j=1}^{N} : K \to K$ be a finite family of uniformly L_1-Lipschitzian asymptotically quasi I_j-nonexpansive mappings with a common sequence $\{\lambda_n\} \subset [1, \infty)$ and $\{I_j\}_{j=1}^{N} : K \to K$ be a finite family of uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mappings with a common sequence $\{\mu_n\} \subset [1, \infty)$ such that*

$$F = \bigcap_{j=1}^{N} (F(T_j) \cap F(I_j)) \neq \emptyset.$$

Suppose $B_* = \inf_n \beta_n$, $B^* = \sup_n \beta_n$, $\Lambda = \sup_n \lambda_n \geq 1$, $M = \sup_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ are six sequences in $[0, 1]$ which satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \quad \forall n \geq 1,$

(ii) $\sum_{n=1}^{\infty} (\lambda_n\mu_n - 1)\beta_n < \infty,$

(iii) $0 < B_* \leq B^* < \dfrac{1}{\Lambda^2 M^2} < 1$,

(iv) $0 < \widehat{B}_* = \inf\limits_n \widehat{\beta}_n \leq \sup\limits_n \widehat{\beta}_n = \widehat{B}^* < 1$,

(v) $\sum\limits_{n=1}^{\infty} \gamma_n < \infty, \quad \sum\limits_{n=1}^{\infty} \widehat{\gamma}_n < \infty$.

If the mappings $E - T_j$ and $E - I_j$ are semi-closed at zero for every $j = \overline{1,N}$, then the implicit iterative sequence $\{x_n\}$ with errors defined by (1.6) *converges weakly to a common fixed point of finite families of asymptotically quasi I_j-nonexpansive mappings $\{T_j\}_{j=1}^N$ and asymptotically quasi-nonexpansive mappings $\{I_j\}_{j=1}^N$.*

Proof. Let $p \in F$, the according to Lemma 3.1 the sequence $\{\|x_n - p\|\}$ converges. This provides that $\{x_n\}$ is bounded. Since X is uniformly convex, then every bounded subset of X is weakly compact. From boundedness of $\{x_n\}$ in K, we find a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$. Hence from (3.25), it follows that

$$\lim_{n_k \to \infty} \|x_{n_k} - T_j x_{n_k}\| = 0, \ \lim_{n_k \to \infty} \|x_{n_k} - I_j x_{n_k}\| = 0, \qquad \forall j = \overline{1,N}.$$

Since the mappings $E - T_j$ and $E - I_j$ are semi-closed at zero, therefore we have $T_j q = q$ and $I_j q = q$, for all $j = \overline{1,N}$, which means $q \in F$.

Finally, we prove that $\{x_n\}$ converges weakly to q. In fact, suppose the contrary, then there exists some subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in F$.

Taking $p = q$ and $p = q_1$ and using the same argument given in the proof of (3.4), we can prove that the limits $\lim\limits_{n\to\infty} \|x_n - q\|$ and $\lim\limits_{n\to\infty} \|x_n - q_1\|$ exist, and we have

$$\lim_{n\to\infty} \|x_n - q\| = d, \quad \lim_{n\to\infty} \|x_n - q_1\| = d_1,$$

where d, d_1 are two nonnegative numbers. By virtue of the Opial condition of X, one finds

$$\begin{aligned} d &= \limsup_{n_k\to\infty} \|x_{n_k} - q\| < \limsup_{n_k\to\infty} \|x_{n_k} - q_1\| = \\ &= \limsup_{n_j\to\infty} \|x_{n_j} - q_1\| < \limsup_{n_j\to\infty} \|x_{n_j} - q\| = d \end{aligned}$$

This is a contradiction. Hence $q_1 = q$. This implies that $\{x_n\}$ converges weakly to q. This completes the proof of Theorem 3.4. ∎

Next, we prove strong convergence theorem

Theorem 3.5. *Let X be a real uniformly convex Banach space and K be a nonempty closed convex subset of X. Let $\{T_j\}_{j=1}^N : K \to K$ be a finite family of uniformly L_1-Lipschitzian asymptotically quasi I_j-nonexpansive mappings with a common sequence $\{\lambda_n\} \subset [1,\infty)$ and $\{I_j\}_{j=1}^N : K \to K$ be a finite family of uniformly L_2-Lipschitzian asymptotically quasi-nonexpansive mappings with a common sequence $\{\mu_n\} \subset [1,\infty)$ such that*

$$F = \bigcap_{j=1}^N (F(T_j) \cap F(I_j)) \neq \emptyset.$$

Suppose $B_* = \inf\limits_n \beta_n$, $B^* = \sup\limits_n \beta_n$, $\Lambda = \sup\limits_n \lambda_n \geq 1$, $M = \sup\limits_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\widehat{\alpha}_n\}$, $\{\widehat{\beta}_n\}$, $\{\widehat{\gamma}_n\}$ are six sequences in $[0,1]$ which satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = \widehat{\alpha}_n + \widehat{\beta}_n + \widehat{\gamma}_n = 1, \quad \forall n \geq 1$,

(ii) $\sum_{n=1}^{\infty}(\lambda_n\mu_n - 1)\beta_n < \infty$,

(iii) $0 < B_* \leq B^* < \dfrac{1}{\Lambda^2 M^2} < 1$,

(iv) $0 < \widehat{B}_* = \inf_n \widehat{\beta}_n \leq \sup_n \widehat{\beta}_n = \widehat{B}^* < 1$,

(v) $\sum_{n=1}^{\infty}\gamma_n < \infty, \quad \sum_{n=1}^{\infty}\widehat{\gamma}_n < \infty.$

If at least one mapping of the mappings $\{T_j, I_j\}_{j=1}^N$ is semi-compact, then the implicit iterative sequence $\{x_n\}$ with errors defined by (1.6) *converges strongly to a common fixed point of finite families of asymptotically quasi I_j-nonexpansive mappings $\{T_j\}_{j=1}^N$ and asymptotically quasi-nonexpansive mappings $\{I_j\}_{j=1}^N$.*

Proof. Without any loss of generality, we may assume that T_1 is semi-compact. This with (3.25) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^*$ strongly and $x^* \in K$. Since T_j, I_j are continuous, then from (3.25), for all $j = \overline{1, N}$ we find

$$\|x^* - T_jx^*\| = \lim_{n_k\to\infty}\|x_{n_k} - T_jx_{n_k}\| = 0, \quad \|x^* - I_jx^*\| = \lim_{n_k\to\infty}\|x_{n_k} - I_jx_{n_k}\| = 0.$$

This shows that $x^* \in F$. According to Lemma 3.1 the limit $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Then

$$\lim_{n\to\infty}\|x_n - x^*\| = \lim_{n_k\to\infty}\|x_{n_k} - x^*\| = 0,$$

which means $\{x_n\}$ converges to $x^* \in F$. This completes the proof. ∎

Remark 3.1. If we take $\gamma_n = 0$, for all $n \in \mathbb{N}$ and $I_j = E$, $j = 1, 2, \ldots, N$ then the above theorem becomes Theorem 3.3 due to Sun [27]. If $I_j = E$, $j = 1, 2, \ldots, N$ and $\{T_j\}_{j=1}^N$ are asymptotically nonexpansive, then we got main results of [4, 13]. If $j = 1$ and $\gamma_n = 0$ for all $n \in \mathbb{N}$, then we recover the result of [19].

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