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SZEGÖ LIMITS AND HAAR WAVELET BASIS

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ABSTRACT. This paper deals with Szegö type limits for multiplication operators on $L^2(R)$ with respect to Haar orthonormal basis. Similar studies have been carried out by Morrison for multiplication operators T_f using Walsh System and Legendre polynomials [14]. Unlike the Walsh and Fourier basis functions, the Haar basis functions are local in nature. It is observed that Szegö type limit exist for a class of multiplication operators T_f, $f \in L^\infty(R)$ with respect to Haar (wavelet) system with appropriate ordering. More general classes of orderings of Haar system are identified for which the Szegö type limit exist for certain classes of multiplication operators. Some illustrative examples are also provided.

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1. INTRODUCTION

The classical theorem of Szegö [3] on Toeplitz matrices states that if $\lambda_{1,N}, \lambda_{2,N}, \ldots, \lambda_{N,N}$ are the eigenvalues of the $N \times N$ section of the matrix (a_{i-j}) where $a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)dx$ is the k^{th} Fourier coefficient of the multiplier f in $L^\infty(-\pi,\pi)$ and $F : R \to R$ any continuous function on R, then

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_{k,N})}{N} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(x))dx. \tag{1.1}$$

Now recall the following notion of equidistribution of sequences of real numbers in an interval, due to Hermann Weyl.

Equidistribution of Sequences [9]. Let $(\lambda_1, \lambda_2, \ldots \lambda_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$ be two sequences of sets of real numbers in (a, b). They are said to be equally distributed if

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} F(\lambda_i)}{n} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} F(\beta_i)}{n}$$

for all continuous real functions F on (a, b).

So the classical Szegö Theorem implies that the sequences of sets of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the function values $\{f(-\pi + \frac{2v\pi}{n+1})\}$, $v = 1, 2, \ldots, n$ as $n \to \infty$ are equally distributed in (a, b) where $a =$ essinf $f(x)$ and $b =$ esssup $f(x)$.

Observe that in the classical Szegö's theorem, the Toeplitz operators and their truncations are considered with respect to Fourier basis $\{\frac{e^{inx}}{\sqrt{2\pi}}, n = 0, \pm1, \pm2, \ldots\}$ in $L^2(-\pi, \pi)$. Its applications to various fields like statistics, classical moment problem and analytic function theory have been discussed in detail in [9].

The Szegö limit provides localization of spectrum which is useful in many applications to a variety of fields such as mathematical physics, partial differential operators and signal processing. For instance such applications can be found in the work of Damanik D. and Simon B. [8], Laptev A. and Safarov Yu [12] and Houcem Gazzah, Philip A. Regalia, and Jean-Pierre Delmas [10]. In [10] Szegö limit has been used to estimate the lowest nonzero eigen value of certain covariance matrix that arises in SIMO channel identification problem based on Fourier system. There the basis used is the Fourier system. In place of that one can hope to use Haar system which will minimize the computational difficulties. Szegö's theorem is also used in medical sciences such as limited angle tomography. It is noted that the Szegö limit provides the ratio between known and unknown data which will be useful for imaging problems in tomography [16].

Morrison has proved Szegö type theorems for certain multiplication operators with respect to Walsh system as well as Legendre polynomials [14]. In [15] it has been proved that with respect to lexicographic ordering of the Haar wavelet basis in $L^2(0, 1)$, the Szegö type limit does not exist for certain multiplication operators on $L^2(0, 1)$. It has also been observed that for certain multiplication operators on $L^2(0, 1)$ and $L^2(R_+)$ the Szegö limit exist when the Haar system is ordered suitably.

This paper is committed to identify more general class of orderings of the Haar system and classes of multiplication operators for which the Szegö type limit exist.

First we highlight the importance of Haar system compared to Fourier system, Legendre polynomial and Walsh system, though it is not smooth

(i) The Haar wavelet is well localized and simple, which is useful in many applications.

(ii) The computation and implementation are easier because in time-frequency analysis using Haar wavelet the computation reduces to averaging process. [4, 11].

(iii) In many situations, the matrix of multiplication operator T_f with respect to Haar system is

more sparse than the matrices with respect to Walsh, Fourier system etc. [20].
(iv) It has been used in the analysis of fitness functions in genetic algorithms [22].

For instance in the analysis of fitness functions in genetic algorithm, 2^l non-zero terms are required for the expansion of a given function as linear combination of Walsh functions, where as atmost $l+1$ non-zero terms are required with the Haar expansion, where l is the size of the binary string in the solution space. It has also been observed that the Haar functions have more advantage than Walsh functions [22].
(v) John Canny [7] provided a mathematical argument for using a derivative of Gaussian kernel as an optimal edge detector. He observed that convolving an edge with a derivative of Gaussian kernel produces maxima or minima at step edges.

A similar argument may be possible in the case of Haar system too. Observe that the derivative of Walsh function is the Haar function. Hence, if Haar function is convolved with an edge, its maxima and minima will correspond to step edges. Here 'smoothing' is done by Walsh function. Though Walsh function is not smooth like Gaussian, mild 'smoothing' takes place when convolved with Walsh.

Throughout this paper the following notations have been used. For a separable Hilbert space H, let $[e_1, e_2, \ldots]$ be an orthonormal basis in H, $H_N = \text{span}[e_1, e_2, \ldots, e_N]$ and P_N be the orthogonal projection of H onto H_N. For each linear operator T on H, T_N will denote $P_N T P_N$ restricted to H_N. Let $[T] = (a_{ij})$ denote the infinite matrix (a_{ij}) of T with respect to the above basis and $[T]_N$ will denote the $N \times N$ matrix $(a_{ij})_{i,j=1\cdots N}$. For a scalar function f, T_f will denote the multiplication operator on an appropriate L^2-space.

This paper is divided into two sections. In the first section we prove that Szegö type limit exists for a general class of multiplication operators T_f with multiplier $f \in L^2(R_+)$ subject to some conditions on f. In the second section more general classes of orderings of Haar system in L^2-spaces are identified for certain classes of multiplication operators which satisfy the Szegö type limit property. Some illustrative examples are also considered.

2. SZEGÖ TYPE LIMITS

In this section we deal with an ordering of the Haar system in $L^2(R_+)$ that was considered in [20] and prove that the Szegö type limit exists for multiplication operator T_f with more general class of multipliers f. We recall this ordering for convenience.

2.1. **An ordering of the Haar wavelet basis for $L^2(R_+)$.**

Let ϕ be the characteristic function of $[0,1]$ and h_{rp} be the Haar function defined by

$$\begin{aligned} h_{r,p} &= 2^{\frac{r}{2}} \quad \frac{p}{2^r} \le x < \frac{p+1/2}{2^r} \\ &= -2^{\frac{r}{2}} \quad \frac{p+1/2}{2^r} \le x < \frac{p+1}{2^r} \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

where r & p are non negative integers. Now consider the Haar system in $L^2(R_+)$ namely $\{\varphi_r(x), h_{rp}(x),\ r,p = 0,1,2,\ldots\}$, where $\varphi_r(x) = \varphi(x-r)$.

We assign the following ordering

$$\begin{gathered}
\varphi_0, h_{00}, h_{01}\\
\varphi_1, h_{10}, h_{11}, h_{12}, h_{13}, h_{02}, h_{03}\\
\dots\\
\varphi_{r-1}, h_{r-10}, \dots, h_{r-12^r-1}, \dots, h_{02^{r-1}}, \dots, h_{02^r-1}\\
\varphi_r, h_{r0}, \dots, h_{r2^{r+1}-1}, \dots, h_{02^r}, \dots, h_{02^{r+1}-1}\\
\dots\\
\varphi_{r+s}, h_{r+s0}, \dots, h_{r+s2^{r+s+1}-1}, \dots, h_{r2^{r+s}}, \dots, h_{r+s2^{r+s+1}-1}, \dots, h_{02^{r+s}}, \dots, h_{02^{r+s+1}-1}\\
\dots
\end{gathered}$$

and the Haar system with the above ordering shall be denoted by $(\psi_k : k = 1, 2, 3, \dots)$.

Remark 2.2. One can easily determine the position of h_{rp} and φ_r, for a given r and p from the above triangular form of arrangement as given below.

$$\begin{aligned}
\psi_k &= \phi_r \quad \text{if } k = r(2^r + 1) + 1\\
&= h_{rp},\ p < 2^{r+1} \quad \text{if } k = r(2^r + 1) + p + 2\\
&= h_{rp},\ p \geq 2^{r+s},\ s \in N, \quad \text{if } k = (r + 2s)(2^{r+s} + 1) + p - s + 2
\end{aligned}$$

Now we analyze the behavior of certain multiplication operators with respect to the above ordering of the Haar system. First of all we recall the following theorem [20, theorem 2] which is needed to prove the main result of this section.

Theorem 2.3. *Let T_{f_n} be the multiplication operator on $L^2(R_+)$ with*

$$f_n = \sum_{k=1}^{n} \alpha_k \psi_k,\ n = (m + 1)(2^{m+1} + 1),$$

and α_k, s are real for each k. With respect to the above ordered Haar system the following asymptotic formula holds for any continuous function F on R,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_{k,N}^{(n)})}{N} = \lim_{M\to\infty} \frac{1}{M} \int_0^M F[f_n(x)]dx,$$

where $\lambda_{1,N}^{(n)}, \lambda_{2,N}^{(n)}, \dots, \lambda_{N,N}^{(n)}$ are the eigenvalues of $(T_{f_n})_N$.

Now we prove the following theorem.

Theorem 2.4. *Let T_{f_n} be the multiplication operator on $L^2(R_+)$ where $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$, α_k's are real for each k. With respect to the ordered Haar system 2.1, and for any continuous function F on R we have*

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_{k,N}^{(n)})}{N} = \lim_{M\to\infty} \frac{1}{M} \int_0^M F[f_n(x)]dx,$$

where $\lambda_{1,N}^{(n)}, \lambda_{2,N}^{(n)}, \dots, \lambda_{N,N}^{(n)}$ are the eigenvalues of $(T_{f_n})_N$.

Proof. From the above theorem we have the result for the operators T_{f_n} with multiplier $f_n = \sum_{k=1}^{n} \alpha_k \psi_k, n = (m + 1)(2^{m+1} + 1)$, m any positive integer where $\psi_1, \psi_2, \dots, \psi_n$ ordered as 2.1. From this we deduce the result for operators T_{f_n} where $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$ in the following way.

Now if $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$, $n \neq (m+1)(2^{m+1}+1)$ for any m, without loss of generality we may assume that n is of the form $(m+1)(2^{m+1}+1)$ for some positive integer m by taking $\alpha_k = 0$ for $k = n+1, n+2, \ldots, (m+1)(2^{m+1}+1)$. By doing this only an increase in the multiplicity of the eigenvalue zero takes place. This completes the proof. ∎

Next we consider the asymptotic distribution of eigenvalues of multiplication operators on $L^2(R_+)$ with multiplier $f = \sum_{k=1}^{\infty} \alpha_k \psi_k$, $\sum |\alpha_k|^2 < \infty$ in $L^2(R_+)$.

Let $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$. Let $(\lambda_{1,N}, \lambda_{2,N}, \lambda_{3,N}, \ldots, \lambda_{N,N})$ and $(\lambda_{1,N}^{(n)}, \lambda_{2,N}^{(n)}, \lambda_{3,N}^{(n)}, \ldots, \lambda_{N,N}^{(n)})$ be the eigenvalues of $[T_f]_N$ and, $[T_{f_n}]_N$ respectively.

We bypass the straightforward proof of the following proposition.

Proposition 2.5. *With f_n, and f as above, $\|T_{f_n} - T_f\| \to 0$ and $\|(T_{f_n})_N - (T_f)_N)\| \to 0$ for all N, where $\|\cdot\|$ is the usual operator norm, provided $f_n \to f$ uniformly on R.*

Since $\|(T_{f_n})_N - (T_f)_N\| \to 0$, as $N \to \infty$, the following proposition is an immediate consequence of Weyl's Theorem [18].

Proposition 2.6. *Let $\lambda_{1,N}^{(n)} \geq \lambda_{2,N}^{(n)} \geq \cdots \geq \lambda_{N,N}^{(n)}$ and $\lambda_{1,N} \geq \lambda_{2,N} \geq \cdots \geq \lambda_{N,N}$ be the eigenvalues, arranged in decreasing order of the matrices $[T_{f_n}]_N$ and $[T_f]_N$ respectively, then, $\lambda_{k,N}^{(n)} \to \lambda_{k,N}$ as $n \to \infty$ uniformly for all values of $k = 1, 2, \ldots, N$.*

Remark 2.7. Since $(T_{f_n})_N$ and $(T_f)_N$ are self adjoint, using the upper semicontinuity and lower semicontinuity [13] of the eigenvalues, we have the following result.

Let

$$A = \Lambda_N(f_n) = \{\lambda_{1,N}^{(n)}, \lambda_{2,N}^{(n)}, \ldots, \lambda_{N,N}^{(n)}\}$$
$$B = \Lambda_N(f) = \{\lambda_{1,N}, \lambda_{2,N}, \ldots, \lambda_{N,N})$$

Then the Hausdorff distance between A and B goes to zero as $n \to \infty$. That is

$$h(\Lambda_N(f_n), \Lambda_N(f)) \to 0.$$

The main result of this section is given in the following theorem and it gives the asymptotic distribution of eigenvalues of the multiplication operator T_f.

Theorem 2.8. *If $f_n \to f$ uniformly on compact subsets E of R and $|F_n| = |f_n - f|$ is uniformly bounded, then T_f satisfies the Szegö type limit property, where $|\cdot|$ denote the sup norm of scalar valued functions.*

Proof. The proof follows immediately if we establish the following results.
(i) $T_{f_n} \to T_f$ pointwise.
(ii) $(T_{f_n})_N \to (T_f)_N$ as $n \to \infty$ uniformly for all N
(iii) The limit (double limit) of the double sequence (a_{Nn}) where

$$a_{N,n} = \frac{F(\lambda_{1,N}^{(n)}) + F(\lambda_{2,N}^{(n)}) + \cdots F(\lambda_{N,N}^{(n)})}{N}, \; N, n = 1, 2, \ldots$$

exist as $N, n \to \infty$.
Proof of (i)

Since $|F_n| = |f_n - f|$ is uniformly bounded, we have $|F_n(x)| \leq \beta$ and since $f_n(x)$ converges uniformly to f on a compact set E,

$$|F_n(x)| \to 0 \quad \forall x \in E.$$

Now we show that $T_{f_n} \to T_f$ point wise.
Consider $T_{f_n} - T_f = T_{f_n - f} = T_{F_n}$. Then,

$$\|T_{F_N}\| = \sup_{\|\xi\|=1} (\|T_{F_n}(\xi)\|), \ \xi \in L^2(R_+)$$

Hence for any $0 < N_0 < \infty$ we have,

$$\|T_{F_n}(\xi)\|^2 = \int_0^{N_0} |F_n(x)|^2 |\xi(x)|^2 dx + \int_{N_0}^{\infty} |F_n(x)|^2 |\xi(x)|^2 dx. \tag{2.1}$$

Let $\epsilon > 0$ be given. Since $\xi(x) \in L^2(R_+)$, N_0 can be chosen such that,

$$\int_{N_0}^{\infty} |\xi(x)|^2 dx \leq \frac{\epsilon}{2\beta^2}.$$

$$\int_{N_0}^{\infty} |F_n(x)\xi(x)|^2 dx \leq \int_{N_0}^{\infty} \beta^2 |\xi(x)|^2 dx,$$

$$\leq \beta^2 \int_{N_0}^{\infty} |\xi(x)|^2 dx < \frac{\epsilon}{2} \quad \forall n. \tag{2.2}$$

Let $E = [0, N_0]$ be the compact set.
Since $F_n \to 0$ uniformly on E, we have for every $\epsilon > 0$ there exists N_1 such that

$$|F_n(x)| < \frac{\epsilon}{2} \quad \forall n \geq N_1, \ \text{ and } \forall \ x \in E$$

$$\text{Therefore } \int_0^{N_0} |F_n(x)|^2 |\xi(x)|^2 dx < \frac{\epsilon}{2} \quad \forall n \geq N_1. \tag{2.3}$$

By substituting equations (2.2) and (2.3) in (2.1), we have $T_{F_N} \to 0$.
Proof of (ii)

$$\begin{aligned} \|(T_{f_n})_N - (T_f)_N\| &= \|P_N T_{f_n} P_N - P_N T_f P_N\| \\ &= \|P_N (T_{f_n} - T_f) P_N\| \\ &\leq \|P_N\| \ \|(T_{f_n} - T_f) P_N\| \\ &\leq \|(T_{f_n} - T_f) P_N\| \end{aligned}$$

Since $T_{f_n} \to T_f$ point wise on R and P_N is compact, we have $\|(T_{f_n} - T_f)P_N\| \to 0$ uniformly on R[2].

Proof of (iii)

Let

$$Y_N^n = \frac{F(\lambda_{1,N}^{(n)}) + F(\lambda_{2,N}^{(n)}) + \cdots + F(\lambda_{N,N}^{(n)})}{N}, \quad N = 1, 2, \ldots$$

denote the row sequences and

$$Z_n^N = \frac{F(\lambda_{1,N}^{(n)}) + F(\lambda_{2,N}^{(n)}) + \cdots + F(\lambda_{N,N}^{(n)})}{N}, \quad n = 1, 2, \ldots$$

denote the column sequences of $a_{n,N}$. By theorem 2.4 $\lim_{N\to\infty} Z_n^N = Z_n$ and by continuity of F

$$\lim_{n\to\infty} Y_N^n = Y_N = \frac{\sum_1^N F(\lambda_{k,N})}{N}.$$

Now proposition 2.6 implies that

$$\lim_{n\to\infty} \lambda_{k,N}^{(n)} = \lambda_{k,N}.$$

Let $\epsilon > 0$ be given. Consider,

$$
\begin{aligned}
|a_{N,n} - Y_N| &= |\frac{\sum_{k=1}^{N} F(\lambda_{k,N}^{(n)})}{N} - \frac{\sum_{k=1}^{N} F(\lambda_{k,N})}{N}| \\
&= \frac{1}{N}|\sum_{k=1}^{N} F(\lambda_{k,N}^{(n)}) - \sum_{k=1}^{N} F(\lambda_{k,N})| \\
&\leq \frac{1}{N}\sum_{k=1}^{N} |F(\lambda_{k,N}^{(n)}) - F(\lambda_{k,N})| \qquad (2.4)
\end{aligned}
$$

From the proposition 2.6, there exists $\tilde{N}$ such that

$$|F(\lambda_{k,N}^{(n)}) - F(\lambda_{k,N})| < \epsilon \quad n \geq \tilde{N} \text{ and } \forall k.$$

Therefore equation (2.4) reduces to

$$|a_{N,n} - Y_N| < \epsilon \quad \forall n \geq \tilde{N}, N = 1, 2, \ldots$$

Hence using the Integrated limit Theorem [18] the double limit exists. We observe that $\frac{1}{M}\int F(f(x))dx = \int F d\mu_M$, μ_M is the measure on $[a,b]$ defined by $\mu_M(A) = (\frac{1}{M} m \circ f^{-1}|_{[0,M]})(A)$ for any Borel set $A \subset [a,b]$, where m denote the Lebesgue measure. Therefore if μ is the measure on $[a,b]$ defined by $\lim\limits_{M\to\infty} \mu_M$ then,

$$\lim_{M\to\infty} \frac{1}{M}\int_0^M F(f(x))dx = \int_{[a,b]} F d\mu.$$

Therefore if $\rho_n = \frac{\sum_{i=1}^{n} \delta_{\lambda_i}}{n}$ where δ_i denote the Dirac delta measure concentrated at i, then this says that $\rho_N \to \mu$ weakly, as $n \to \infty$. ∎

3. MORE GENERAL ORDERINGS OF HAAR SYSTEM

In this section we identify different classes of orderings of Haar system in $L^2(R_+)$ and in $L^2(R)$ so that for certain multiplication operators the Szegö type limit exist. Also we have given examples for orderings other than the orderings mentioned earlier. Throughout this paper $H_{ar} = \{\varphi_r(x), h_{rp}(x), r, p \in Z_+ \cup \{0\}\}$ will denote the Haar system in $L^2(R_+)$.

We consider the multiplication operator T_f on $L^2(R_+)$ with respect to H_{ar} equipped with some special class of orderings. Now we define the following classes of orderings for H_{ar} which will depend very much on the multipliers f chosen.

The following specification of ordering may be useful for certain chosen multipliers.

3.1. **Ordering when $f = h_{00}$.** Let T_f be the multiplication operator on $L^2(R_+)$ where $f = h_{00}$. It can be easily seen that H_{ar} itself is a complete orthonormal system of eigenvectors in $L^2(R_+)$ of T_f. Let M_j be the eigenspace associated with λ_j, $j = 1, 2, 3$. Let $H_j = H_{ar} \cap M_j$. For a sequence (j_n) of positive integers, let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be partitions of H_j and $H_{ar} \cap M_j^c$ respectively such that $|A_n| = nj_n$ and $|B_n| = j_n$. Now order the elements in H_{ar} according to the arrangement specified by the sequence $B_1, A_1, B_2, A_2, \ldots$.

It is to be mentioned that there is no restriction on how the elements in A_n or B_n are ordered.

3.2. **Ordering when** $f = \sum_{k=1}^{n} \alpha_k \psi_k$**,** $n < \infty$**.** Let T_f be the multiplication operator with $f = \sum_{k=1}^{n} \alpha_k \psi_k$ where $\psi_k(x) = h_{rp}(x)$ or $\varphi_r(x)$ and assume that the support of $f = [0, 2^t]$ for a non-negative integer t. Let $M_0 = \{h_{rp}(x), \varphi_r(x)/\text{whose support } \not\subset [0, 2^t]\}$. For a sequence of positive integers (j_n), let $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ be partitions of M_0 and $H_{ar} \cap M_0^c$ respectively such that $|A_n| = nj_n$ and $|B_n| = j_n$. Then $H = (B_1, A_1, B_2, A_2, \ldots)$ is an ordered basis for $L^2(R_+)$.

Theorem 3.3. *Let T_f be the multiplication operator on $L^2(R_+)$ where $f = h_{00}$. Then with respect to H_{ar} with ordering 3.1, the Szegö type limit exists.*

Proof. To prove the theorem it is enough to show that $\frac{N_j}{N} \to 1$ as $N \to \infty$, where N_j is the multiplicity of eigenvalue λ_j of $(T_f)_N$. Let N be a positive integer. Then for some n depending on N,

$$N = \sum_{k=1}^{n-1} ((k+1)j_k) + K, \quad K \leq j_n + K_1 \text{ where } K_1 < nj_n.$$

Then $N_j = \sum_{k=1}^{n-1} kj_k + K_1$ where K_1 is defined as above.

$$\lim_{n\to\infty} \frac{N_i}{N} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} kj_k + K_1}{\sum_{k=1}^{n-1} (k+1)j_k + K} = 1$$

∎

Example 3.4. The ordering 3.1 gives three classes of ordering for multiplication operator on $L^2(R_+)$ where $f = h_{00}$. The only eigen values are $\lambda_j = 1, -1, 0$.

For example when $\lambda_j = 0$ we may define for each positive integer t.
Let $A_n = (h_{n-t2^{n-t}}, \ldots, h_{n-t2^{n-t+1}-1}, h_{n-t-12^{n-t}}, \ldots h_{0,2^{n-t}}, \ldots, h_{0,2^{n-t+1}-1}, \varphi_{n+1})$, $n \geq t$.
$B_n = (h_{n0}, \ldots, h_{n2^{n-t+1}-1}, h_{n-12^{n-t}}, \ldots, h_{n-t+1,2^{n-t+1}-1})$, $n \geq t$ and for each t, let
$A = (\varphi_0, \varphi_1, \ldots, \varphi_{t-1}, h_{t-1,0}, h_{t-2,0}, \ldots, h_{00}, \varphi_t)$. Now an ordering of the Haar system is induced by the following arrangement namely $(A, B_t, A_t, B_{t+1}, A_{t+1}, \ldots)$ is a basis for $L^2(R_+)$, for which Szegö type limit exist. Each A_n and B_n are finite sets of cardinality $|A_n| = (n - t + 1)2^{n-t} + 1$, $|B_n| = (t + 1)2^{n-t}$ for $n = t$, $t + 1, \ldots$ and $|A| = 2t + 1$. It is immaterial how the members of A_n, B_n, and A are ordered. It is clear that $A_n \subset M_0$ and $B_n \subset H_{ar} \cap M_0^c$ for $n = t, t + 1, \ldots$

We can arrange this ordered basis in the following way.

$$\begin{array}{cc}
\varphi_0, \varphi_1, \ldots, \varphi_{t-1}, \varphi_t, h_{t-10}, \ldots, h_{10}, h_{00} & (A) \\
h_{t0}, h_{t1}, h_{t-11}, \ldots, h_{11} & (B_t) \\
h_{01}, \varphi_{t+1} & (A_t) \\
h_{t+10}, h_{t+11}, h_{t+12}, h_{t+13}, h_{t2}, h_{t3}, \ldots h_{22} & (B_{t+1}) \\
h_{12}, h_{13}, h_{02}, h_{03}, \varphi_{t+2} & (A_{t+1}) \\
\cdots & \cdots \\
h_{n0}, \ldots, h_{n2^{n-t+1}-1}, \ldots, h_{n-t+1\, 2^{n-t+1}-1} & (B_n) \\
h_{n-t2^{n-t}}, \ldots, h_{n-t2^{n-t+1}-1}, \ldots h_{02^{n-t}-1}, \varphi_{n+1} & (A_n) \\
\cdots & \cdots
\end{array}$$

Remark 3.5. In the above example for $t = 0, 1, \ldots$, we get a collection of orderings for which Szegö type limit exist. In particular when $t = 0$ the ordering reduces to the ordering given in 2.1, where $j_n = 2^n$, $|B_n| = 2^n$, $|A_n| = (n + 1)2^n + 1$.

Theorem 3.6. *Let T_f be the multiplication operator on $L^2(R_+)$ where $f = \sum_{k=1}^{n} \alpha_k \psi_k$. Then with respect to H_{ar} with ordering 3.2, the Szegö type limit exist.*

Proof.

$$(T_f)_N(\psi_k(x)) = 0 \Leftrightarrow (i)\ \psi_k(x) = \varphi_r(x) \forall k \geq 2^t \text{ where } (T_f)_N = P_N T_f P_N$$
$$(ii)\ \psi_k(x) = h_{rp}(x) \Leftrightarrow [\frac{p}{2^r}, \frac{p+1}{2^r}] \not\subset [0, 2^t]$$

Therefore M_0 is the eigenspace corresponding to the eigenvalue zero and the rest of the proof is similar to the proof of the above theorem 3.3. ∎

Example 3.7. The above theorem indicates that there are variety of orderings for which Szegö type limit exist. For example let H_{ar}, f and M_0 be defined as in the ordering 3.2 and $j_n = 2^{n+t}$, where t is a fixed non-negative integer. Define

$$A_n = \{h_{n2^{n+t}}, \ldots, h_{n2^{n+t+1}-1}, h_{n-12^{n+t}}, \ldots, h_{02^{n+t+1}-1}\}$$
$$B_n = \{\varphi_n, h_{n0}, \ldots, h_{n2^{n+t}-1}\},\ \varphi_n \in A_{n-1} \text{ if } n \geq 2^t$$

such that

$$|A_n| = (n+1)2^{n+t} \quad \text{if } n < 2^t - 1$$
$$= (n+1)2^{n+t} + 1 \quad \text{if } n \geq 2^t - 1$$

and

$$|B_n| = 2^{n+t} + 1 \quad \text{if } n < 2^t$$
$$= 2^{n+t} \quad \text{if } n \geq 2^t$$

respectively. Then the ordering $\{B_0, A_0, B_1, A_1, \ldots\}$ is a basis for $L^2(R_+)$ for which Szegö type limit exist. The above ordered Haar system can be arranged in the following way,

$$\varphi_0, h_{00}, h_{01}, h_{02}, \ldots, h_{02^{t+1}-1}(B_0, A_0)$$
$$\varphi_1, h_{10}, \ldots, h_{12^{t+2}-1}, h_{02^{t+1}}, \ldots, h_{02}, h_{02^{t+2}-1}(B_1, A_1)$$
$$\ldots$$
$$\varphi_{r-1}, h_{r-10}, \ldots, h_{r-12^{r+t}-1}, h_{r-12^{r+t-1}-1}, \ldots, h_{r-22^{r+t}-1}, \ldots, h_{02^{r+t-1}}, \ldots, h_{02^{r+t}-1}(B_{r-1}, A_{r-1})$$
$$\varphi_r, h_{r0}, \ldots, h_{r2^{r+t+1}-1}, h_{r-12^{r+t}}, \ldots, h_{r-12^{r+t+1}-1}, \ldots, h_{02^{r+t}}, \ldots, h_{02^{t+t+1}-1}(B_r, A_r)$$
$$\ldots$$

This ordered Haar basis can be represented as a sequence $[\psi_k : k = 1, 2, \ldots]$ where,

$$\psi_k = \varphi_r \text{ if } k = r(2^{r+t} + 1) + 1$$
$$= h_{rp}, p < 2^{r+t+1} \text{ if } k = r(2^{r+t} + 1) + p + 2$$
$$= h_{rp}, p \geq 2^{r+s+t+1} \text{ if } k = (r + 2s)(2^{r+t+s} + 1) + p - s + 2, s = 1, 2, \ldots$$

Remark 3.8. In the above ordering if $t = 0$, then the ordering reduces to the ordering given in 2.1 for which $j_n = 2^n$.

Now we consider the case of multiplication operators in $L^2(R)$. The following remark gives a class of orderings in $L^2(R)$ for which Szegö type limit exist.

Remark 3.9. Let $[\psi_k : k = 1, 2, \ldots]$, $[\eta_k : k = 1, 2, \ldots]$ be any ordered Haar system in $L^2(R_+)$ and in $L^2(R_-)$ respectively for which Szegö type limit exist for certain multiplication operators. Then with respect to the ordering $[\psi_1, \psi_2, \ldots, \psi_{n_1}, \eta_1, \eta_2, \ldots, \eta_{n_2}]$ in $L^2(R)$, Szegö type limit exist for the multiplication operators on $L^2(R)$ with the same multiplier.

We conclude the paper with the following example.

Example 3.10. We give below an ordering of the Haar system for the space $L^2(R)$ in which case Szegö type limit exist for certain multiplication operators.

The ordered Haar basis for $L^2(R)$ is given by the filling arrangement $\varphi_0, h_{00}, h_{01}, h_{0-1}$, φ_1, $\ldots, h_{r0}, \ldots, h_{0-(2^r-1)}, \ldots$. This can be written in the triangular form as

$$\varphi_0, h_{00}, h_{01}, h_{0-1}$$
$$\varphi_1, h_{10}, h_{11}, h_{12} h_{13}, h_{03}, \varphi_{-1}, h_{1-1}, \ldots, h_{0-2}, h_{0-1}$$
$$\varphi_2, h_{20}, \ldots, h_{27}, h_{14}, \ldots, h_{17}, h_{04}, \ldots, h_{07}, \varphi_{-1}, h_{1-1}, \ldots, h_{0-1}$$
$$\ldots$$
$$\varphi_r, h_{r0}, \ldots, h_{r2^{r+1}-1}, h_{r-12^r}, h_{r-12^r}, \ldots, h_{0,2^{r+1}-1}, \varphi_{-r}, h_{r-1}, \ldots, h_{r-(2^{r+1}-1)}, \ldots, h_{0-(2^{r+1}-1)}$$
$$\ldots$$

Let us denote this basis by $\{\omega_k : k \in N\}$. Then from the above triangular form the ψ_k's are as follows.

$$w_k = \begin{cases} c\phi_r & \text{if } k = r(2^{r+1}+1)+1 \\ h_{rp}, p < 2^{r+1}-1 & \text{if } k = r(2^{r+1}+1)+p+1 \\ h_{rp}, p \geq 2^{r+s+1}-1, s \in N, & \text{if } k = (2r+3s)(2^{r+s}+1)+r+s+p+1 \\ \phi_{-r} & \text{if } k = (3r+2)2^r+r+1 \\ h_{r,-p}, p < 2^{r+1}-1 & \text{if } k = (3r+2)2^r+p+1 \\ h_{r,-p}, p \geq 2^{r+s+1}-1, & \text{if } k = (3r+4s+2)2^{r+s}+r+s+p+1 \end{cases}$$

where $\phi_r(x)$ and h_{rp} are defined as before.

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