The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 9, Issue 2, Article 11, pp. 1-9, 2012

UNITAL COMPACT HOMOMORPHISMS BETWEEN EXTENDED ANALYTIC UNIFORM ALGEBRAS

D. ALIMOHAMMADI AND M. MAYGHANI

Received 28 February, 2011; accepted 11 August, 2011; published 28 December, 2012.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, ARAK UNIVERSITY, PO BOX 38156-8-8349, ARAK, IRAN.
d-alimohammadi@araku.ac.ir
m_maighany@yahoo.com

ABSTRACT. Let X and K be compact plane sets with $K \subseteq X$. We denote by $A(X, K)$ and $A(X)$ the algebras of all continuous complex-valued functions on X which are analytic on $int(K)$ and $int(X)$, respectively. It is known that $A(X, K)$ and $A(X)$ are natural uniform algebras on X. $A(X)$ and $A(X, K)$ are called *analytic uniform algebra* and *extended analytic uniform algebra* on X, respectively. In this paper we study unital homomorphisms between extended analytic uniform algebras and investigate necessary and sufficient conditions for which these homomorphisms to be compact. We also determine the spectrum of unital compact endomorphisms of extended analytic uniform algebras.

Key words and phrases: Compact endomorphism, Compact homomorphism, Extended analytic uniform algebra.

1991 *Mathematics Subject Classification.* Primary 46J10. Secondary 46J15.

ISSN (electronic): 1449-5910

1. INTRODUCTION

We let $\mathbb{N} = \{1, 2, 3 \ldots\}$, $\mathbb{C}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, $\mathbb{D}(\lambda, r) = \{z \in \mathbb{C} : |z - \lambda| < r\}$ and $\overline{\mathbb{D}(\lambda, r)} = \{z \in \mathbb{C} : |z - \lambda| \leq r\}$ denote the set of natural numbers, the field of complex numbers, the open unit disc, the closed unit disc, the open and closed disc with center at λ and radius r, respectively. We also denote $\mathbb{D}(0, r)$ by $\mathbb{D}_r$.

Let A and B be unital commutative semi-simple Banach algebras with maximal ideal spaces $\mathcal{M}(A)$ and $\mathcal{M}(B)$. A homomorphism $T : A \to B$ is a linear map which also preserved multiplication. A homomorphism $T : A \to B$ is called *unital homomorphism* if $T1_A = 1_B$. If T is a unital homomorphism from A into B, then T is continuous and there exists a norm-continuous map $\psi : \mathcal{M}(B) \to \mathcal{M}(A)$ such that $\widehat{Tf} = \hat{f} \circ \psi$ for all $f \in A$, where $\hat{g}$ is the Gelfand transform g. In fact, φ is equal the adjoint of $T^* : B^* \to A^*$ restricted to $\mathcal{M}(B)$. Note that T^* is a weak*-weak* continuous map from B^* into A^*. Thus ψ is a continuous map from $\mathcal{M}(B)$ with the Gelfand topology into $\mathcal{M}(A)$ with the Gelfand topology.

Let A be a unital commutative semi-simple Banach algebra and let T be an endomorphism of A, a homomorphism from A into A. We denote the spectrum of T by $\sigma(T)$ and define

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

For a compact Hausdorff space X, we denote by $C(X)$ the Banach algebra of all complex-valued functions f on X with the uniform norm $\|f\|_X = sup\{|f(x)| : x \in X\}$.

Definition 1.1. Let X be a compact Hausdorff space. A *Banach function algebra* on X is a subalgebra A of $C(X)$ which contains 1_X, the constant function 1 on X, separates the points of X and is a unital Banach algebra with an algebra norm $\|\cdot\|$. If the norm of a Banach function algebra on X is $\|\cdot\|_X$, the uniform norm on X, it is called a *uniform algebra* on X.

Let A and B be Banach function algebras on X and Y, respectively. If $\varphi : Y \to X$ is a continuous mapping such that $f \circ \varphi \in B$ for all $f \in A$, and if $T : A \to B$ is defined by $Tf = f \circ \varphi$, then T is a unital homomorphism, which is called the *induced homomorphism* from A into B by φ. In particular, if $Y = X$ and $B = A$, then T is called the *induced endomorphism* of A by the self-map φ of X.

Let A be a Banach function algebra on a compact Hausdorff space X. For $x \in X$, the map $e_x : A \to \mathbb{C}$, defined by $e_x(f) = f(x)$, is an element of $\mathcal{M}(A)$ and called the *evaluation homomorphism* on A at x. Note that the map $x \mapsto e_x : X \to \mathcal{M}(A)$ is a continuous one-to-one mapping. If this map is onto, we say that A is *natural*.

We know that $C(X)$ is a natural uniform algebra on X and every Banach function algebra on X is a unital commutative semi-simple Banach algebra.

Proposition 1.1. *Let X and Y be compact Hausdorff spaces. Let A and B be natural Banach function algebras on X and Y, respectively. Then every unital homomorphism $T : A \to B$ is induced by a unique continuous map $\varphi : Y \to X$. In particular, if X is a compact plane set and the coordinate function Z belongs to A, then $\varphi = TZ$ and so $\varphi \in B$.*

Proof. Let $T : A \longrightarrow B$ be a unital homomorphism. Since A and B are unital commutative semi-simple Banach algebras, there exists a continuous map $\psi : \mathcal{M}(B) \longrightarrow \mathcal{M}(A)$ such that $\widehat{Tf} = \hat{f} \circ \psi$ for all $f \in A$. The naturality of the Banach function algebra A on X implies that the map $J_A : X \longrightarrow \mathcal{M}(A)$, defined by $J_A(x) = e_x$, is a homeomorphism and so $J_A^{-1} : \mathcal{M}(A) \longrightarrow X$ is continuous. Since B is a Banach function algebra on Y, the map $J_B : Y \longrightarrow \mathcal{M}(B)$, defined by $J_B(y) = e_y$, is continuous. We now define the map $\varphi : Y \longrightarrow X$ by $\varphi = J_A^{-1} \circ \psi \circ J_B$.

Clearly, φ is continuous. Let $f \in A$. Since

$$(Tf)(y) = \widehat{Tf}(e_y) = (\hat{f} \circ \psi)(J_B(y)) = (\hat{f} \circ J_A)(\varphi(y)) \\ = \hat{f}(e_{\varphi(y)}) = e_{\varphi(y)}(f) = f(\varphi(y)) \\ = (f \circ \varphi)(y). \tag{1.1}$$

for all $y \in Y$, we have $Tf = f \circ \varphi$. Therefore, T is induced by φ.

Now, let X be a compact plane set and let $Z \in A$. Then $\varphi = Z \circ \varphi = TZ$, and so $\varphi \in B$. ∎

Corollary 1.2. *Let X be a compact Hausdorff space and let A be a natural Banach function algebra on X. Then every unital endomorphism T of A is induced by a unique continuous self-map φ of X. In particular, if X is a compact plane set and A contains the coordinate function Z, then $\varphi = TZ$ and so $\varphi \in A$.*

Note that if A is a natural uniform algebra on a compact Hausdorff space X and T is a unital endomorphism of A, then $||T|| \leq 1$.

Let X be a compact plane set. We denote by $A(X)$ the algebra of all continuous complex-valued functions on X which are analytic on $int(X)$, the interior of X, and call *analytic uniform algebra* on X. It is known that $A(X)$ is a natural uniform algebra on X (see [3]).

F. behrouzi and H. Mahyar studied compact endomorphisms of certain natural subalgebras of $A(X)$, for certain compact plane set X [1].

Let X and K be compact plane sets such that $K \subseteq X$. We denote by $A(X, K)$ the algebra of all continuous complex-valued functions f on X that are analytic on $int(K)$ and called *extended analytic uniform algebra* on X with respect to K. Clearly, $A(X, K) = A(X)$ if $K = X$ and $A(X, K) = C(X)$ if $int(K)$ is empty. We know that $A(X, K)$ is a natural uniform algebra on X (see [3]).

In this paper, we study unital homomorphisms between extended analytic uniform algebras and investigate sufficient and necessary conditions for which these homomorphisms to be compact. We also determine the spectrum of unital compact endomorphisms of these algebras.

2. UNITAL COMPACT HOMOMORPHISMS

We first give a sufficient condition for which a continuous map $\varphi : X_2 \to X_1$ induces a unital homomorphism T from a subalgebra B_1 of $A(X_1, K_1)$ into a subalgebra B_2 of $A(X_2, K_2)$.

Proposition 2.1. *Let X_j and K_j be compact plane sets with $int(K_j) \neq \emptyset$ and $K_j \subseteq X_j$, and let B_j be a subalgebra of $A(X_j, K_j)$ which is a natural Banach function algebra on X_j under an algebra norm $\|\cdot\|_j$, where $j \in \{1, 2\}$. If $\varphi \in B_2$ with $\varphi(X_2) \subseteq int(K_1)$, then φ induces a unital homomorphism $T : B_1 \to B_2$. Moreover, if $Z \in B_1$, then $\varphi = TZ$.*

Proof. The naturality of Banach function algebra B_2 on X_2 implies that $\sigma_{B_2}(h) = h(X_2)$, where $\sigma_A(h)$ is the spectrum of $h \in A$ in the Banach algebra A. Let $f \in B_1$. Since $\varphi \in B_2$, $\varphi(X_2) \subseteq int(K_1)$ and f is analytic on $int(K_1)$, we conclude that f is analytic on an open neighborhood of $\sigma_{B_2}(\varphi)$. By using the Functional Calculus Theorem [3, Chapter I, Theorem 5.1], there exists $g \in B_2$ such that $\hat{g} = f \circ \hat{\varphi}$ on $\mathcal{M}(B_2)$. It follows that

$$g(z) = e_z(g) = \hat{g}(e_z) = f(\hat{\varphi}(e_z)) \\ = f(e_z(\varphi)) = f(\varphi(z)) = (f \circ \varphi)(z), \tag{2.1}$$

for all $z \in X_2$ and so $g = f \circ \varphi$. Therefore, $f \circ \varphi \in B_2$. This implies that the map $T : B_1 \to B_2$ defined by $Tf = f \circ \varphi$, is a unital homomorphism from B_1 into B_2, which is induced by φ. Now let $Z \in B_1$. Then $\varphi = TZ$ by Proposition 1.1. ∎

Corollary 2.2. *Let X and K be compact plane sets with $int(K) \neq \emptyset$ and $K \subseteq X$. Let B be a subalgebra of $A(X, K)$ which is a natural Banach function algebra on X under an algebra norm $\|\cdot\|_B$. If $\varphi \in B$ with $\varphi(X) \subseteq int(K)$, then φ induces a unital endomorphism T of B. Moreover, if $Z \in B$, then $\varphi = TZ$.*

Proposition 2.3. *For $j \in \{1, 2\}$ suppose that $z_j \in \mathbb{C}$, $0 < r_j < R_j$, $G_j = \mathbb{D}(z_j, R_j)$, $\Omega = \mathbb{D}(z_j, r_j)$, $X_j = \overline{G_j}$ and $K_j = \overline{\Omega_j}$. Then for each $\rho \in (r_1, R_1]$ there exists a continuous map $\varphi_\rho : X_2 \to X_1$ with $\varphi_\rho(X_2) = \overline{\mathbb{D}(z_1, \rho)}$ such that $\varphi_\rho \in A(X_2, K_2)$ and φ_ρ does not induce any homomorphism from $A(X_1, K_1)$ to $A(X_2, K_2)$.*

Proof. Let $\rho \in (r_1, R_1]$. We define the map $\varphi_\rho : X_2 \to X_1$ by

$$\varphi_\rho(z) = \begin{cases} z_1 + \frac{\rho(z-z_2)}{r_2} & |z - z_2| \leq r_2, \\ z_1 + \frac{\rho(z-z_1)}{|z-z_2|} & r_2 < |z - z_2| \leq R_2. \end{cases}$$

Clearly, φ_ρ is a continuous mapping, $\varphi_\rho(X_2) = \overline{\mathbb{D}_\rho(z_1, \rho)}$ and $\varphi_\rho \in A(X_2, K_2)$. We now define the function $f_\rho : X_1 \to \mathbb{C}$ by

$$f_\rho(z) = \begin{cases} \frac{\rho(z-z_1)}{r_1} & |z - z_1| \leq r_1, \\ \frac{\rho(z-z_1)}{|z-z_1|} & r_1 < |z - z_1| \leq R_1. \end{cases}$$

Then $f_\rho \in A(X_1, K_1)$. Since $0 < \frac{r_1 r_2}{\rho} < r_2$ and

$$(f_\rho \circ \varphi_\rho)(z) = \begin{cases} \frac{\rho^2}{r_1 r_2}(z - z_2) & |z - z_2| \leq \frac{r_1 r_2}{\rho}, \\ \frac{\rho(z-z_2)}{|z-z_2|} & \frac{r_1 r_2}{\rho} < |z - z_2| \leq R_2, \end{cases}$$

we conclude that $f_\rho \circ \varphi_\rho \notin A(X_2, K_2)$. Therefore, φ_ρ does not induce any homomorphism from $A(X_1, K_1)$ to $A(X_2, K_2)$. ∎

Corollary 2.4. *Suppose that $\lambda \in \mathbb{C}$, $0 < r < R$, $G = \mathbb{D}(\lambda, R)$, $\Omega = \mathbb{D}(\lambda, r)$, $X = \overline{G}$ and $K = \overline{\Omega}$. Then for each $\rho \in (r, R]$, there exists a continuous self-map φ_ρ of X with $\varphi_\rho(X) = \overline{\mathbb{D}(\lambda, \rho)}$ such that $\varphi_\rho \in A(X, K)$ and φ_ρ does not induce any endomorphism of $A(X, K)$.*

We now give a sufficient condition for which a unital homomorphism from a uniform subalgebra B_1 of $A(X_1, K_1)$ into a uniform subalgebra B_2 of $A(X_2, K_2)$ to be compact.

Theorem 2.5. *Suppose that X_j and K_j are compact plane sets with $int(K) \neq \emptyset$, and $K_j \subseteq X_j$ and B_j is a natural uniform subalgebra of $A(X_j, K_j)$ where $j \in \{1, 2\}$. Let $\varphi : X_2 \to X_1$ be a continuous mapping. If φ is constant or $\varphi \in B_2$ with $\varphi(X_2) \subseteq int(K_1)$, then φ induces a unital compact homomorphism $T : B_1 \to B_2$.*

Proof. If $\varphi : X_2 \to X_1$ is constant, then the map $T : B_1 \to B_2$ defined by $Tf = f \circ \varphi$ is a unital homomorphism from B_1 into B_2 with $dim\, T(B_1) \leq 1$, and so T is compact.

Let $\varphi : X_2 \to X_1$ be a nonconstant mapping with $\varphi \in B_2$ and $\varphi(X_2) \subseteq int(K_1)$. Then the map $T : B_1 \to B_2$ defined by $Tf = f \circ \varphi$ is a unital homomorphism from B_1 into B_2 by Proposition 2.1. To prove the compactness of T, let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in B_1 with $\|f_n\|_{X_1} \leq 1$. By Montel's theorem, $\{f_n\}_{n=1}^\infty$ has a subsequence $\{f_{n_j}\}_{j=1}^\infty$ which is uniformly convergent on every compact subset of $int(K_1)$, in particular on $\varphi(X_2)$. Therefore, $\{f_{n_j} \circ \varphi\}_{j=1}^\infty$ is uniformly convergent on X_2, i.e. $\{Tf_{n_j}\}_{j=1}^\infty$ is convergent uniformly on X_2. It follows that $\{Tf_{n_j}\}_{j=1}^\infty$ is a Cauchy sequence in $(B_2, \|\cdot\|_{X_2})$. Since $(B_2, \|\cdot\|_{X_2})$ is a complete normed space, we conclude that $\{Tf_{n_j}\}_{j=1}^\infty$ is convergent in B_2. Hence T is compact. ∎

Corollary 2.6. *Suppose that X and K are compact plane sets with $int(K) \neq \emptyset$, and $K \subseteq X$. Let B be a natural uniform subalgebra of $A(X, K)$ and φ be a self-map of X. If φ is constant or $\varphi \in B$ and $\varphi(X) \subseteq int(K)$, then φ induces a unital compact endomorphism of B.*

We now give a necessary condition for which a unital homomorphism T from a uniform subalgebra B_1 of $A(X_1, K_1)$ into a uniform subalgebra B_2 of $A(X_2, K_2)$ to be compact.

Definition 2.1. (a) A plane set X at $c \in \partial X$, the boundary of X, has an internal circular tangent if there exists a disc D in the complex plane such that $c \in \partial D$ and $\overline{D} \setminus \{c\} \subseteq int(X)$.

(b) A plane set X is called strongly accessible from the interior, if it has an internal circular tangent at each point of its boundary.

Such sets include the closed unit disc $\overline{\mathbb{D}}$ and $\overline{\mathbb{D}(z_0, r)} \setminus \bigcup_{k=1}^{n} \mathbb{D}(z_k, r_k)$, where closed discs $\overline{\mathbb{D}(z_k, r_k)}$ are mutually disjoint in $\mathbb{D}(z_0, r)$.

(c) A compact plane set X has peak boundary with respect to $B \subseteq C(X)$ if for each $c \in \partial X$ there exists a nonconstant function $h \in B$ such that $\|h\|_X = h(c) = 1$.

Example 2.7. The closed unit disc $\overline{\mathbb{D}}$ has peak boundary with respect to $A(\overline{\mathbb{D}})$. Because if $c \in \partial\overline{\mathbb{D}}$, then the function $h : \overline{\mathbb{D}} \to \mathbb{C}$ defined by $h(z) = \frac{1}{2}(1 + \bar{c}z)$, belongs to $A(\overline{\mathbb{D}})$, and satisfies in $\|h\|_{\overline{\mathbb{D}}} = h(c) = 1$.

Example 2.8. Let X be a compact plane set such that $\mathbb{C} \setminus X$ has smooth boundary. If $R_0(X) \subseteq B \subseteq C(X)$, then X has peak boundary with respect to B, where $R_0(X)$ is the algebra of all rational functions with poles off X.

Proof. Let $z_0 \in \mathbb{C} \setminus X$. Since $\mathbb{C} \setminus X$ is strongly accessible from the interior each $c \in \partial(\mathbb{C} \setminus X)$ there exists a $\delta > 0$ such that $|c - z_0| = \delta$ and $\overline{\mathbb{D}}(z_0, \delta) \setminus \{c\} \subseteq int(\mathbb{C} \setminus X)$. Now, define the function $h : X \to \mathbb{C}$ by

$$h(z) = \frac{\delta^2}{\overline{(c - z_0)}(z - z_0)}.$$

Then $h \in R_0(X)$ and so $h \in B$. Moreover, $\|h\|_X = h(c) = 1$. ∎

Theorem 2.9. *Let X_1 be a compact plane set such that $G_1 = int(X_1)$ is connected, $\overline{G_1} = X_1$ and X_1 has peak boundary with respect to $A(X_1)$. Suppose that $\Omega_1 \subseteq G_1$ is a bounded domain in the complex plane and let $K_1 = \overline{\Omega_1}$. Let Ω_2 be a bounded domain in the complex plane and let $K_2 = \overline{\Omega_2}$. Suppose that X_2 is a compact plane set such that $K_2 \subseteq X_2$. If $T : A(X_1, K_1) \to A(X_2, K_2)$ is a unital compact homomorphism, then T is induced by a continuous mapping $\varphi : X_2 \to X_1$ such that φ is constant on K_2 or $\varphi(K_2) \subseteq G_1 = int(X_1)$.*

Proof. Since $A(X_1, K_1)$ and $A(X_2, K_2)$ are natural uniform algebras on X_1 and X_2, respectively, we conclude that T is induced by a unique continuous mapping $\varphi : X_2 \to X_1$. Moreover, $\varphi \in A(X_2, K_2)$ since the coordinate function Z is in $A(X_1, K_1)$. Since φ is a nonconstant analytic function on Ω_2, we deduce that $\varphi(\Omega_2)$ is an open set in the complex plane and $\varphi(\Omega_2) \subseteq X_1$. Thus,

$$\varphi(\Omega_2) \subseteq int(X_1) = G_1. \tag{2.2}$$

If $\varphi(X_2) \nsubseteq G_1$, then there exists a $c \in \partial X_2$ such that $\varphi(c) \in \partial X_1$. Since X_1 has peak boundary with respect to $A(X_1)$, there exists a nonconstant function $h \in A(X_1)$ such that

$$\|h\|_{X_1} = h(\varphi(c)) = 1. \tag{2.3}$$

We now define the sequence $\{f_n\}_{n=1}^{\infty}$ by $f_n = h^n$. Then $f_n \in A(X_1) \subseteq A(X_1, K_1)$ and $\|f_n\|_{X_1} = 1$ for all $n \in \mathbb{N}$. The compactness of T implies that $\{f_n\}_{n=1}^{\infty}$ has a subsequence

$\{f_{n_j}\}_{j=1}^{\infty}$ such that the sequence $\{Tf_{n_j}\}_{j=1}^{\infty}$ is converges to a function g in $A(X_2, K_2)$. Since $(Tf_{n_j})(c) = f_{n_j}(\varphi(c)) = (h(\varphi(c)))^{n_j} = 1$ for all $j \in \mathbb{N}$, we have $g(c) = 1$. On the other hand, $|h(z)| < 1$ for all $z \in \Omega_1$ since $\|h\|_{\overline{\Omega_1}} \leq \|h\|_{X_1} = 1$, h is a nonconstant analytic function on G_1 and h is a continuous complex-valued function on $\overline{\Omega_1}$. Let $w \in \Omega_2$. Then $\varphi(w) \in \Omega_1$ and so $|h(\varphi(w))| < 1$. Therefore,

$$g(w) = \lim_{j\to\infty}(Tf_{n_j})(w) = \lim_{j\to\infty}(h \circ \varphi)^{n_j}(w) = 0.$$

The continuity of g on $\overline{\Omega_2}$ implies that $g(w) = 0$ for all $w \in \overline{\Omega_2} = K_2$. This follows that $g(c) = 0$, contradicting to $g(c) = 1$. Consequently, $\varphi(K_2) \subseteq G_1 = int(X_1)$. ∎

Corollary 2.10. *Let X be a compact plane set such that $G = int(X)$ is connected and $\overline{G} = X$. Suppose that $\Omega \subseteq G$ is a bounded domain in the complex plane and let $K = \overline{\Omega}$. Suppose that X has peak boundary with respect to $A(X)$. If T is a unital compact endomorphism of $A(X, K)$ then T is induced by a continuous self-map φ of X such that φ is constant on K or $\varphi(K) \subseteq G = int(X)$.*

Lemma 2.11. *Let Ω be a domain in the complex plane and let φ be a one-to-one analytic function on Ω. If f is a continuous complex-valued function on $\varphi(\Omega)$ such that $fo\varphi$ is an analytic function on Ω, then f is an analytic function on $\varphi(\Omega)$.*

Proof. By ([2];Chapter IV, Theorem 7.5 and Corollary 7.6), we deduce that $\varphi(\Omega)$ is a domain in the complex plane, $\varphi'(z) \neq 0$ for all $z \in \Omega$ and $\varphi^{-1} : \varphi(\Omega) \to \Omega$ is an analytic function on $\varphi(\Omega)$. Since $f = f \circ \varphi \circ \varphi^{-1}$, we conclude that f is analytic on $\varphi(\Omega)$. ∎

Lemma 2.12. *Let G and Ω be bounded domains in the complex plane with $\Omega \subseteq G$ and let $X = \overline{G}$, $K = \overline{\Omega}$. Then for each $c \in G \setminus K$, there exists a function f_c in $A(X, K)$ such that f_c is not analytic at c.*

Proof. Let $c \in G \setminus K$. Then there exists an $r > 0$ such that

$$\{z \in \mathbb{C} : |z - c| \leq r\} \subseteq G \setminus K.$$

We now define the function $f_c : X \to \mathbb{C}$ by

$$f_c(z) = \begin{cases} z - c & z \in X, |z - c| \geq r, \\ \frac{(1+r)(z-c)}{1+|z-c|} & z \in X, |z - c| < r. \end{cases}$$

It is easily seen that $f_c \in A(X, K)$ and f_c is not analytic at c. ∎

Definition 2.2. Let X and K be compact plane sets with $K \subseteq X$. We say that K has K-peak boundary with respect to $B \subseteq A(X, K)$ if for each $c \in \partial K$ there is a function h in B such that h is nonconstant on K and $\|h\|_X = h(c) = 1$.

Example 2.13. Let $K = \overline{\mathbb{D}_r}$, where $0 < r \leq 1$. Then K has K-peak boundary with respect to $A(\overline{\mathbb{D}}, K)$.

Proof. We first assume that $r = 1$. For each $c \in \partial K$ we define the function $h_c : \overline{\mathbb{D}} \to \mathbb{C}$ by $h_c(z) = \frac{1}{2}(1 + \overline{c}z)$. Then $h_c \in A(\overline{\mathbb{D}}, K)$, $\|h_c\|_{\overline{\mathbb{D}}} = h_c(c) = 1$.
We now assume that $0 < r < 1$. For each $c \in \partial K$, set $\lambda = \frac{(1+r)c}{r}$. Then $\lambda \in \mathbb{C} \setminus \overline{\mathbb{D}}$. We define the function $h_c : \overline{\mathbb{D}} \to \mathbb{C}$ by

$$h_c(z) = \begin{cases} -\frac{r}{\overline{c}(z-\lambda)} & z \in \overline{\mathbb{D}}, |z - \lambda| \geq 1, \\ -\frac{r|z-\lambda|}{\overline{c}(z-\lambda)} & z \in \overline{\mathbb{D}}, |z - \lambda| < 1. \end{cases}$$

Then $h_c \in A(\overline{\mathbb{D}}, K)$ and $\|h_c\|_{\overline{\mathbb{D}}} = h_c(c) = 1$. ∎

Theorem 2.14. *Let X_1 be a compact plane set such that $G_1 = int(X_1)$ is connected and $\overline{G_1} = X_1$. Suppose that K_1 is a compact subset of X_1 such that $\Omega_1 = int(K_1)$ is connected, $K_1 = \overline{\Omega_1}$ and K_1 has K_1-peak boundary with respect to $A(X_1, K_1)$. Let Ω_2 be a bounded domain in the complex plane and let $K_2 = \overline{\Omega_2}$. Suppose that $T : A(X_1, K_1) \to A(X_2, K_2)$ is a unital homomorphism and $\varphi = TZ$. If T is compact and φ is one-to-one on Ω_2, then $\varphi(K_2) \subseteq \Omega_1 = int(K_1)$.*

Proof. We first show that $\varphi(\Omega_2) \subseteq K_1$. Since $\varphi \in A(X_2, K_2)$ and φ is one-to-one on Ω_2, we conclude that $\varphi(\Omega_2)$ is an open set in the complex plane. Thus $\varphi(\Omega_2) \subseteq int(X_1) = G_1$. Suppose $\varphi(\Omega_2) \nsubseteq K_1$. Then there exists $\lambda \in \Omega_2$ such that $\varphi(\lambda) \in G_1 \setminus K_1$. Since K_1 has K_1-boundary with respect to $A(X_1, K_1)$, there exists a function $f_{\varphi(\lambda)}$ in $A(X_1, K_1)$ such that $f_{\varphi(\lambda)}$ is not analytic at $\varphi(\lambda)$. But $f_{\varphi(\lambda)} \circ \varphi = Tf_{\varphi(\lambda)} \in A(X_2, K_2)$, so that $f_{\varphi(\lambda)} \circ \varphi$ is analytic on Ω_2. Since φ is a one-to-one analytic mapping on Ω_2 and φ_λ is continuous on $\varphi(\Omega_2)$, we conclude that $f_{\varphi(\lambda)}$ is analytic on $\varphi(\Omega_2)$ by Lemma 2.11. This contradicts to that $f_{\varphi(\lambda)}$ is not analytic at $\varphi(\lambda)$. Hence $\varphi(\Omega_2) \subseteq K_1$, so that $\varphi(\Omega_2) \subseteq int(K_1) = \Omega_1$ since $\varphi(\Omega_2)$ is an open set in the complex plane. This implies that

$$\varphi(K_2) = \varphi(\overline{\Omega_2}) \subseteq \overline{\varphi(\Omega_2)} \subseteq \overline{\Omega_1} = K_1.$$

We now show that $\varphi(K_2) \subseteq \Omega_1$. If $\varphi(K_2) \nsubseteq \Omega_1$, then there exists $c \in \partial K_2$ such that $\varphi(c) \in \partial K_1$ since $\varphi(\Omega_2) \subseteq \Omega_1$ and $\varphi(K_2) \subseteq K_1$. This implies that there exists a function $h \in A(X_1, K_1)$ such that h is nonconstant on K_1 and

$$\|h\|_{X_1} = h(\varphi(c)) = 1.$$

We now define the sequence $\{f_n\}_{n=1}^\infty$ in $A(X_1, K_1)$ by $f_n = h^n$. Then $\|f_n\|_{X_1} = 1$ for all $n \in \mathbb{N}$. The compactness of T implies that there exists a subsequence $\{f_{n_j}\}_{j=1}^\infty$ of the sequence $\{f_n\}_{n=1}^\infty$ and a function g in $A(X_2, K_2)$ such that $Tf_{n_j} \to g$ in $A(X_2, K_2)$. Since

$$Tf_{n_j}(c) = f_{n_j}(\varphi(c)) = h(\varphi(c))^{n_j} = 1,$$

for all $j \in \mathbb{N}$, we conclude that $g(c) = 1$. But h is a nonconstant analytic function on Ω_1, so that $|h(w)| < 1$ for all $w \in \Omega_1$. This implies that $|h(\varphi(z))| < 1$ for all $z \in \Omega_2$. Therefore,

$$(Tf_{n_j})(z) = (h(\varphi(z)))^{n_j} \to 0, \text{as } \ j \to \infty,$$

for all $z \in \Omega_2$. This follows that $g(z) = 0$ for all $z \in \Omega_2$. The continuity of g on $\overline{\Omega_2}$ implies that $g(z) = 0$ for all $z \in K_2 = \overline{\Omega_2}$. Therefore, $g(c) = 0$ contradicting to $g(c) = 1$. Hence, $\varphi(K_2) \subseteq \Omega_1$. ∎

Corollary 2.15. *Let X be a compact plane set such that $G = int(X)$ is connected and $\overline{G} = X$. Let K be a compact subset of X such that $\Omega = int(K)$ is connected and $K = \overline{\Omega}$. Suppose that K has K-peak boundary with respect to $A(X, K)$. Let T be a unital endomorphism of $A(X, K)$ and let $\varphi = TZ$. If T is compact and φ is one-to-one on Ω, then $\varphi(K) \subseteq \Omega = int(K)$.*

3. Spectrum of Unital Compact Endomorphisms

In this section we determine the spectrum of a unital compact endomorphism of a natural uniform subalgebra of $A(X, K)$.

The following result is a modification of ([4], Theorem 1.7) for unital compact endomorphisms of natural Banach function algebras.

Theorem 3.1. *Let X be a compact Hausdorff space and B be a natural Banach function algebra on X. If T is a unital compact endomorphism of B induced by a self-map $\varphi : X \to X$, then $\bigcap_{n=0}^\infty \varphi_n(X)$ is finite and if X is connected, $\bigcap_{n=0}^\infty \varphi_n(X)$ is singleton where φ_n is nth iterate of*

φ, i.e, $\varphi_0(x) = x$ and $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$. If $\bigcap_{n=0}^{\infty} \varphi_n(X) = \{x_0\}$, then x_0 is a fixed point for φ. In fact, if $F = \bigcap_{n=0}^{\infty} \varphi_n(X)$, then $\varphi(F) = F$.

Theorem 3.2. *Let G and Ω be bounded domains in the complex plane with $\Omega \subseteq G$ and let $X = \overline{G}$ and $K = \overline{\Omega}$. Suppose that B is subalgebra of $A(X, K)$ containing the coordinate function Z which is a natural Banach function algebra on X with an algebra norm $\|\cdot\|_B$. Let T be a unital compact endomorphism of B induced by a self-map φ of X. If $\varphi(X) \subseteq int(K)$ with a fixed point z_0, then*

$$\sigma(T) = \{0, 1\} \cup \{(\varphi'(z_0))^n : n \in \mathbb{N}\}.$$

Proof. Clearly 0 and also $1 \in \sigma(T)$ since $T(1_X) = 1_X$. If φ is constant then the proof is complete. Let $\lambda \in \sigma(T) \setminus \{0, 1\}$. The compactness of T implies that there exists $f \in B \setminus \{0\}$ such that $Tf = f \circ \varphi = \lambda f$. Since $\varphi(z_0) = z_0 \in int(K)$, $f(z_0) = 0$. We claim that $f^{(j)}(z_0) \neq 0$ for some $j \in \mathbb{N}$. If $f^{(n)}(z_0) = 0$ for all $n \in \mathbb{N}$, then $f = 0$ on an open disc with center z_0 and so $f = 0$ on Ω by maximum modulus principle. It follows that $f = 0$ on X since $\varphi(X) \subseteq \Omega$ and $\lambda f(z) = f(\varphi(z))$ for all $z \in X$. This contradicts to $f \neq 0$. Hence, our claim justified. Let $m = min\{n \in \mathbb{N} : f^{(n)}(z_0) \neq 0\}$. Then $f^{(k)}(z_0) = 0$ for all $k \in \{0, ..., n-1\}$ and $f^{(m)}(z_0) \neq 0$. By m times differentiation of $f \circ \varphi = \lambda f$, we have $(\varphi'(z_0))^m f^{(m)}(\varphi(z_0)) = \lambda f^{(m)}(z_0)$, therefore $\lambda = (f'(z_0))^m$.Then $\sigma(T) \setminus \{0, 1\} \subseteq \{(\varphi'(z_0))^n : n \in \mathbb{N}\}$.
Conversely, first we show that if $\lambda \in \sigma(T)$ with $|\lambda| = 1$, then $\lambda = 1$. Let $\lambda \in \sigma(T)$ and $|\lambda| = 1$. The compactness of T implies that there exists $g \in B \setminus \{0\}$ such that $g \circ \varphi = \lambda g$. It follows that $|g \circ \varphi| = |g|$. Since $\varphi(K) \subseteq int(K) = \Omega$ and g is analytic on the domain Ω we conclude that g is constant on Ω by maximum modulus principle. Since $\varphi(X) \subseteq \Omega$, $g \circ \varphi = \lambda g$ and $\lambda \in \mathbb{C} \setminus \{0\}$, we deduce that g is constant on X. Applying again $g \circ \varphi = \lambda g$ implies that $\lambda = 1$.

We now claim that $\varphi'(z_0) \in \sigma(T)$. If $\varphi'(z_0) \notin \sigma(T)$, then there exists a nonzero linear operator $S : B \to B$ such that

$$(T - \varphi'(z_0)I)S = I. \tag{3.1}$$

Since $Z - z_0 1_X \in B$, $h = S(Z - z_0 1_X) \in B$ and so

$$h \circ \varphi - \varphi'(z_0)h = Z - z_0 1_X,$$

by 3.1. By differentiation at z_0, we have

$$0 = h'(\varphi(z_0))\varphi'(z_0) - \varphi'(z_0)h'(z_0) = 1,$$

this is a contradiction. Hence, our claim is justified.

We now show that $(\varphi'(z_0))^n \in \sigma(T)$ for all $n \in \mathbb{N}$. If $\varphi'(z_0) = 0$ or $|\varphi'(z_0)| = 1$, the proof is complete. Suppose $\varphi'(z_0) \neq 0$ and $|\varphi'(z_0)| \neq 1$. If $(\varphi'(z_0))^j \notin \sigma(T)$ for some $j \in \mathbb{N}$ with $j > 1$, then there exists a nonzero linear operator $S_j : B \to B$ such that

$$(T - (\varphi'(z_0))^j I)S_j = I. \tag{3.2}$$

Since $(Z - z_0 1_X)^j \in B$, $h_j = S_j(Z - z_0 1_X)^j \in B$ and so

$$h_j \circ \varphi - (\varphi'(z_0))^j h_j = (Z - z_0 1_X)^j,$$

by 3.2. By $j - 1$ times differentiation at z_0, we have

$$h_j(z_0) = h_j'(z_0) = ... = h_j^{(j-1)}(z_0) = 0,$$

and j times differentiation at z_0, we have

$$0 = (\varphi'(z_0))^j h_j^{(j)}(\varphi(z_0)) - (\varphi'(z_0))^j h_j^{(j)}(z_0) = j!,$$

this is a contradiction. ∎

Corollary 3.3. *Let B and T satisfy the conditions of Theorem 3.2. Let F be a finite set such that $\varphi(F) = F$. Then there exist $z_0 \in F$ and $m \in \mathbb{N}$ such that*

$$\{\lambda^m : \lambda \in \sigma(T)\} = \{0, 1\} \cup \{(\varphi'_m(z_0))^n : n \in \mathbb{N}\}.$$

Proof. Since F is a finite set and $\varphi(F) = F$, there exist $z_0 \in F$ and $m \in \mathbb{N}$ such that $\varphi_m(z_0) = z_0$. Since $\varphi(X) \subseteq int(K)$, so $z_0 \in int(K)$. If φ is constant the proof is complete. When φ is not constant, we define $\widetilde{T} : B \to B$ by $\widetilde{T}f = f \circ \varphi_m$. Then $\widetilde{T}$ is a compact endomorphism of B by Theorem 2.5 and $\varphi_m(z_0) = z_0$. By Theorem 3.2,

$$\sigma(\widetilde{T}) = \{0, 1\} \cup \{(\varphi'_m(z_0))^n : n \in \mathbb{N}\}.$$

Since $Tf = f \circ \varphi$ and $\widetilde{T}f = f \circ \varphi_m$, we have $\widetilde{T} = T^m$. By Spectral Mapping Theorem, $\sigma(T^m) = \{\lambda^m : \lambda \in \sigma(T)\}$. Therefore,

$$\{\lambda^m : \lambda \in \sigma(T)\} = \{0, 1\} \cup \{(\varphi'_m(z_0))^n : n \in \mathbb{N}\}.$$

This completes the proof. ∎

Corollary 3.4. *Let G and Ω be bounded domains in the complex plane with $\Omega \subseteq G$ and let $X = \overline{G}$ and $K = \overline{\Omega}$. Suppose that φ is a self-map of X such that $\varphi(X) \subseteq K$ and $\varphi(z_0) = z_0$ for some $z_0 \in int(K)$. If $T : A(X, K) \to A(X, K)$ is the endomorphism of $A(X, K)$ induced by φ, then T is compact and*

$$\sigma(T) = \{0, 1\} \cup \{(\varphi'(z_0))^n : n \in \mathbb{N}\}.$$

References

[1] F. BEHROUZI and H. MAHYAR, Compact endomorphisms of certain analytic Lipschitz algebras, *Bull. Belg. Math. Soc. Simon Steven*, **1**2(2005), No. 2, pp. 301–312.

[2] J. B. CONWAY, Functions of One Complex Variable, *Second Edition, Springer-Velag, New York*, (1978).

[3] T. W. GAMELIN, Uniform Algebras, *Second ed., Chelsea Press, New York*, (1984).

[4] H. KAMOWITZ, Compact endomrphisms of Banach algebras , *Pacific J. Math.* **8**9 (2)(1980), pp. 313–325.

[5] W. RUDIN, Functional Analysis, *Second ed. McGraw-Hill, New York*, (1991).