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SOME DISTORTION AND OTHER PROPERTIES ASSOCIATED WITH A FAMILY OF THE n-FOLD SYMMETRIC KOEBE TYPE FUNCTIONS

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ABSTRACT. In a recent work by Kamali and Srivastava [5], a certain family of the n-fold symmetric Koebe type functions was introduced and studied systematically. In an earlier investigation, Eguchi and Owa [4] had considered its special case when $n = 1$ (see also [10]). Here, in our present sequel to these earlier works, this general family of the n-fold symmetric Koebe type functions is studied further and several distortion theorems and such other properties as the radii of spirallikeness, the radii of starlikeness and the radii of convexity, which are associated with this family of the n-fold symmetric Koebe type functions, are obtained. We also provide certain criteria that embed this family of the n-fold symmetric Koebe type functions in a function class $\mathcal{G}_\lambda$ which was introduced and studied earlier by Silverman [7].

Key words and phrases: Analytic functions, Univalent functions, Starlike functions, Convex functions, Koebe type functions, Distortion theorems, Radii of spirallikeness, Radii of starikeness, Radii of convexity.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

$\mathbb{C}$ being, as usual, the set of *complex* numbers. Thus, equivalently, $\mathcal{A}$ denotes the class of functions $f(z)$ which are analytic in $\mathbb{U}$ and normalized by

$$f(0) = f'(0) - 1 = 0.$$

Suppose also that $\mathcal{S}$ denotes the subclass of functions in $\mathcal{A}$ which are *univalent* in $\mathbb{U}$ (see, for details, [3] and [10]; see also the recent works [1], [2], [8], [9], [11], and [12]).

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\kappa)$ of *starlike functions of order* κ in $\mathbb{U}$ and the class $\mathcal{K}(\kappa)$ of *convex functions of order* κ in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\kappa) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \kappa \quad (z \in \mathbb{U};\ 0 \leqq \kappa < 1) \right\} \tag{1.2}$$

and

$$\mathcal{K}(\kappa) := \left\{ f : f \in \mathcal{S} \ \text{and} \ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \kappa \ \ (z \in \mathbb{U};\ 0 \leqq \kappa < 1) \right\}. \tag{1.3}$$

It readily follows from the definitions (1.2) and (1.3) that

$$f(z) \in \mathcal{K}(\kappa) \Longleftrightarrow zf'(z) \in \mathcal{S}^*(\kappa). \tag{1.4}$$

Furthermore, for the relatively more familiar classes $\mathcal{S}^*$ and $\mathcal{K}$ of *starlike functions* in $\mathbb{U}$ and *convex functions* in $\mathbb{U}$, respectively, we have

$$\mathcal{S}^* := \mathcal{S}^*(0) \qquad \text{and} \qquad \mathcal{K} := \mathcal{K}(0).$$

In particular, the class $\mathcal{K}$ of convex functions in $\mathbb{U}$ consists of functions that map the unit disk $\mathbb{U}$ into a convex region.

A function $f(z) \in \mathcal{A}$ is said to be α-*spirallike of order*

$$\beta \qquad \left(-\frac{\pi}{2} \leqq \alpha < \frac{\pi}{2};\ 0 \leqq \beta < \cos\alpha \leqq 1\right)$$

in $\mathbb{U}$ if it satisfies the following inequality:

$$\Re\left(e^{-i\alpha} \cdot \frac{zf'(z)}{f(z)}\right) > \beta \tag{1.5}$$

$$\left(z \in \mathbb{U};\ -\frac{\pi}{2} \leqq \alpha < \frac{\pi}{2};\ 0 \leqq \beta < \cos\alpha \leqq 1\right).$$

In the special case when $\beta = 0$, geometrical characterization of the above-defined functions is that they map the unit disk $\mathbb{U}$ onto an α-*spirallike* region, that is, for each point $w_0 \in f(\mathbb{U})$, the unique α-spiral given, from w_0 to the origin, by

$$w = w_0 e^{\lambda t} \qquad \left(\lambda = e^{i\alpha};\ t \in \mathbb{R}\right)$$

lies entirely in $f(\mathbb{U})$. The class of all such functions is denoted here by $\mathcal{S}_\alpha^*(\beta)$. For $\alpha = 0$, 0-spirals are radial half-lines and the corresponding class

$$\mathcal{S}^*(\beta) := \mathcal{S}_0^*(\beta)$$

is indeed the class of starlike functions of order β in $\mathbb{U}$. Moreover, if $\alpha = \beta = 0$, we are led to the class of starlike functions in $\mathbb{U}$:

$$\mathcal{S}^* := \mathcal{S}_0^*(0) =: \mathcal{S}^*(0).$$

All of the aforementioned function classes are subclasses of the class $\mathcal{S}$ of univalent functions in $\mathbb{U}$ (see, for details, [3]; see also [10] and some recent investigations of several interesting subclasses of spirallike functions of *real* or *complex* order by Kim and Srivastava [6], Srivastava *et al.* [12], and the references cited in each of these earlier works).

It can be verified from the work of Eguchi and Owa [4] that the function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{2(\cos\alpha-\beta)\exp(i\alpha)}} \tag{1.6}$$

is the extremal function for the class $\mathcal{S}_\alpha^*(\beta)$. In the case when $\alpha = \beta = 0$, we get the well-known Koebe function. Eguchi and Owa [4] studied the generalized Koebe type function $f_*(z)$ given by

$$f_*(z) = \frac{z}{(1-z)^{b\exp(i\alpha)}} \qquad (b \in \mathbb{R};\ 0 \leqq \alpha < 2\pi)\,. \tag{1.7}$$

Since, in general, the function $f_*(z)$ defined by (1.7) is neither α-spirallike of order β in $\mathbb{U}$, nor starlike of order β in $\mathbb{U}$, nor convex of order β in $\mathbb{U}$, Eguchi and Owa [4] calculated the radii of α-spirallikeness of order β (denoted by r_{sp}), the radii of starlikeness of order β (denoted by r_{st}), and the radii of convexity of order β (denoted by r_{k}) such that

$$0 \leqq r < r_{\mathrm{sp}} \quad \Longrightarrow \quad \Re\left(e^{-i\alpha}\cdot\frac{zf_*'(z)}{f_*(z)}\right) > \beta \qquad (|z| < r), \tag{1.8}$$

$$0 \leqq r < r_{\mathrm{st}} \quad \Longrightarrow \quad \Re\left(\frac{zf_*'(z)}{f_*(z)}\right) > \beta \qquad (|z| < r), \tag{1.9}$$

and

$$0 \leqq r < r_{\mathrm{k}} \quad \Longrightarrow \quad \Re\left(1+\frac{zf_*''(z)}{f_*'(z)}\right) > \beta \qquad (|z| < r), \tag{1.10}$$

respectively. More precisely, Eguchi and Owa [4] proved the following theorems.

Theorem 1. (see [4, Theorem 3]) *Suppose that*

$$-\frac{\pi}{2} \leqq \alpha < \frac{\pi}{2} \qquad \textit{and} \qquad 0 \leqq \beta < \cos\alpha \leqq 1.$$

(i) *If* $b \geqq 2(\cos\alpha - \beta)$, *then*

$$r_{\mathrm{sp}} = \frac{\cos\alpha-\beta}{\beta-\cos\alpha+b} =: r_{\mathrm{sp1}}.$$

(ii) *If* $0 \leqq b \leqq 2(\cos\alpha-\beta)$, *then* $r_{\mathrm{sp}} = 1$.
(iii) *If* $b \leqq 0$, *then*

$$r_{\mathrm{sp}} = -r_{\mathrm{sp1}}.$$

Theorem 2. (see [4, Theorem 4]) *Let*

$$0 \leqq \alpha < 2\pi, \qquad 0 \leqq \beta < 1,$$

$$r_{\mathrm{st1}} = \frac{b}{2(b\cos\alpha + \beta - 1)} \quad \textit{and} \quad r_{\mathrm{st2}} = \frac{\sqrt{b^2 - 4b(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(b\cos\alpha + \beta - 1)}.$$

(i) *Let* $b \geqq 0$ *and* $\cos\alpha \geqq 0$.

(a) *If*

$$b \neq 0 \qquad \textit{and} \qquad b \neq \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\mathrm{st}} = r_{\mathrm{st1}} - r_{\mathrm{st2}}.$$

(b) *If*

$$b = \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\mathrm{st}} = \cos\alpha.$$

(c) *If* $b = 0$, *then* $r_{\mathrm{st}} = 1$.

(ii) *If* $b \geqq 0$ *and* $\cos\alpha < 0$, *then*

$$r_{\mathrm{st}} = r_{\mathrm{st1}} - r_{\mathrm{st2}}.$$

(iii) *If* $b < 0$ *and* $\cos\alpha \geqq 0$, *then*

$$r_{\mathrm{st}} = -r_{\mathrm{st1}} - r_{\mathrm{st2}}.$$

(iv) *Let* $b < 0$ *and* $\cos\alpha < 0$.

(a) *If*

$$b \neq \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\mathrm{st}} = -r_{\mathrm{st1}} - r_{\mathrm{st2}}.$$

(b) *If*

$$b = \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\mathrm{st}} = -\cos\alpha.$$

Theorem 3. (see [4, Theorem 2]) *Let* $0 \leqq \beta < 1$ *and* $\alpha = 0$, *that is*, *let*

$$f_*(z) = \frac{z}{(1-z)^b} \qquad (b \in \mathbb{R}).$$

(i) *If* $b \geqq 1$, *then*

$$r_{\mathrm{k}} = \frac{b(3-\beta) - 2(1-\beta) - \sqrt{b[b(\beta^2 - 2\beta + 5) - 4(1-\beta)]}}{2(b-1)(b-1+\beta)}.$$

(ii) *If* $b \leqq -1$, *then*

$$r_{\mathrm{k}} = \frac{-b(3-\beta) + 2(1-\beta) - \sqrt{b[b(\beta^2 - 2\beta + 5) - 4(1-\beta)]}}{2(b-1)(b-1+\beta)}.$$

The main object of this paper is to present a systematic further study of the following general family of the n-fold symmetric Koebe type functions:

$$k(z) = \frac{z}{(1-z^n)^{b\exp(i\alpha)}} \qquad (n \in \mathbb{N};\ b \in \mathbb{R};\ -\pi \leqq \alpha < \pi), \tag{1.11}$$

which was introduced by Kamali and Srivastava [5]. For this general family of the n-fold symmetric Koebe type functions, we derive several distortion theorems, that is, the estimates of

$$|k(z)|, \quad |k'(z)| \quad \text{and} \quad \left|\arg\left(k'(z)\right)\right|.$$

Further, we obtain the radii of α-spirallikeness, starlikeness and convexity of some order (using markedly diffrent techniques than those used in [4]). We also provide sufficient conditions (or criteria) that would embed this general family of the n-fold symmetric Koebe type functions in the following function class:

$$\mathcal{G}_\lambda := \left\{ f : f \in \mathcal{A} \ \text{ and } \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < \lambda \ \ (z \in \mathbb{U};\ 0 < \lambda \leqq 1) \right\}, \tag{1.12}$$

which was defined earlier by Silverman [7].

2. A SET OF DISTORTION THEOREMS

In this section, we prove several distortion theorems which provide the estimates for

$$|k(z)|, \quad |k'(z)| \quad \text{and} \quad \left|\arg\left(k'(z)\right)\right| \qquad (|z| = r < 1),$$

where the n-fold symmetric Koebe type function $k(z)$ is given by (1.11). Some of these estimates are shown to be sharp.

Theorem 4. *Supose that*

$$\theta_1 = \frac{2}{n}\left[\arctan\left(\frac{\sqrt{1+(1-r^{2n})\tan^2\alpha}+1}{(1+r^n)\tan\alpha}\right)\right],$$

$$\theta_2 = -\frac{2}{n}\left[\arctan\left(\frac{\sqrt{1+(1-r^{2n})\tan^2\alpha}-1}{(1+r^n)\tan\alpha}\right)\right]$$

and

$$h_1(\theta) = \sqrt{1-2r^n\cos(n\theta)+r^{2n}} \ \cdot\ \exp\left[(\tan\alpha)\cdot\arctan\left(\frac{r^n\sin(n\theta)}{1-r^n\cos(n\theta)}\right)\right].$$

Also let

$$k_1 = [h_1(\theta_1)]^{-b\cos\alpha} \qquad \textit{and} \qquad k_2 = [h_1(\theta_2)]^{-b\cos\alpha}.$$

The the following estimates hold true.

(i) *If* $b\cos\alpha \geqq 0$, *then*

$$rk_1 \leqq |k(z)| \leqq rk_2 \qquad (|z| = r < 1).$$

(ii) *If* $b\cos\alpha < 0$, *then*

$$rk_2 \leqq |k(z)| \leqq rk_1 \qquad (|z| = r < 1).$$

Each of these estimates is sharp.

Proof. It can easily be verified that

$$|k(re^{i\theta})| = r[h_1(\theta)]^{-b\cos\alpha}.$$

Moreover, since $h_1(\theta)$ is continuous and $h_1(\theta) \neq 0$, we find that the condition that $h_1'(\theta) = 0$ is equivalent to the following condition:

$$\sin(n\theta) + \cos(n\theta)\tan\alpha - r^n \tan\alpha = 0$$

with solutions given by

$$\frac{2}{n}\left[\arctan\left(\frac{\sqrt{1+(1-r^{2n})\tan^2\alpha}+1}{(1+r^n)\tan\alpha}\right) + 2\ell\pi\right] \qquad (\ell \in \mathbb{Z})$$

and

$$-\frac{2}{n}\left[\arctan\left(\frac{\sqrt{1+(1-r^{2n})\tan^2\alpha}-1}{(1+r^n)\tan\alpha}\right) + 2\ell\pi\right] \qquad (\ell \in \mathbb{Z}).$$

For $\ell = 0$, we obtain the solutions θ_1 and θ_2, respectively. Therefore, $h_1(\theta)$ has the extremal values $h_1(\theta_1)$ and $h_1(\theta_2)$.

We next show that

$$h_1(\theta_1) > h_1(\theta_2) \qquad (|\theta_1| > |\theta_1|)\,.$$

Obviously, since

$$|\theta_1| > |\theta_2|,$$

we have

$$\cos|\theta_1| < \cos|\theta_2| \qquad (|\theta_1| > |\theta_2|)$$

and

$$\sqrt{1-2r^n\cos(n\theta_1)+r^{2n}} > \sqrt{1-2r^n\cos(n\theta_2)+r^{2n}}. \tag{2.1}$$

If $\alpha \geqq 0$, then

$$\tan\alpha \geqq 0 \qquad \text{and} \qquad \theta_1 > 0 > \theta_2.$$

On the other hand, if $\alpha < 0$, then

$$\tan\alpha < 0 \qquad \text{and} \qquad \theta_2 > 0 > \theta_1.$$

In both cases, we find that

$$\begin{aligned}(\tan\alpha)\cdot\arctan\left(\frac{r^n\sin(n\theta_1)}{1-r^n\cos(n\theta_1)}\right) &> 0 \\ &> (\tan\alpha)\cdot\arctan\left(\frac{r^n\sin(n\theta_2)}{1-r^n\cos(n\theta_2)}\right).\end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$h_1(\theta_1) > h_1(\theta_2) \qquad (|\theta_1| > |\theta_1|)\,.$$

Finally, in view of the monotonicity properties of the exponential function, we find for $b\cos\alpha \geqq 0$ that

$$\min|k(re^{i\theta})| = rk_1 \quad \text{and} \quad \max|k(re^{i\theta})| = rk_2 \qquad (b\cos\alpha \geqq 0).$$

In a similar manner, if $b\cos\alpha < 0$, then

$$\min|k(re^{i\theta})| = rk_2 \quad \text{and} \quad \max|k(re^{i\theta})| = rk_1.$$

Sharpness of the estimates asserted by Theorem 4 follows from the fact that $h_1(\theta)$ attains its extremal values for θ_1 and θ_2. ∎

Theorem 5. *Let*

$$a := \frac{b\sin\alpha}{1+b\cos\alpha},$$

$$\theta_3 = \frac{2}{n}\left[\arctan\left(\frac{\sqrt{1+(1-r^{2n})\cdot a^2}+1}{(1+r^n)a}\right)\right],$$

$$\theta_4 = -\frac{2}{n}\left[\arctan\left(\frac{\sqrt{1+(1-r^{2n})\cdot a^2}-1}{(1+r^n)a}\right)\right]$$

and

$$h_2(\theta) = \sqrt{1-2r^n\cos(n\theta)+r^{2n}}\ \cdot\ \exp\left[a\cdot\arctan\left(\frac{r^n\sin(n\theta)}{1-r^n\cos(n\theta)}\right)\right].$$

Suppose also that

$$k_3 = [h_2(\theta_3)]^{-b\cos\alpha-1}, \qquad k_4 = [h_2(\theta_4)]^{-b\cos\alpha-1},$$

$$k_5 = 1-r^n\sqrt{1-2bn\cos\alpha+b^2n^2}$$

and

$$k_6 = 1+r^n\sqrt{1-2bn\cos\alpha+b^2n^2}.$$

Then the following estimates hold true.

(i) *If* $b\cos\alpha+1 \geqq 0$, *then*

$$k_3k_5 \leqq |k'(z)| \leqq k_4k_6 \qquad (|z|=r<1).$$

(ii) *If* $b\cos\alpha+1<0$, *then*

$$k_4k_5 \leqq |k'(z)| \leqq k_3k_6 \qquad (|z|=r<1).$$

Proof. Since

$$k'(z) = \frac{1+(bne^{i\alpha}-1)\,z^n}{(1-z^n)^{b\exp(i\alpha)+1}}, \tag{2.3}$$

which, for $z=re^{i\theta}$, yields

$$\left|k'\left(re^{i\theta}\right)\right| = [h_2(\theta)]^{b\cos\alpha+1}\cdot h_3(\theta),$$

where

$$h_3(\theta) = \left|1+\left(bne^{i\alpha}-1\right)\cdot r^n\cdot e^{in\theta}\right|.$$

Just as in the proof of Theorem 4, it can be shown that the inequality:

$$1+b\cos\alpha \geqq 0$$

implies that

$$\min\left|[h_2(\theta)]^{b\cos\alpha+1}\right| = k_3 \quad \text{and} \quad \max\left|[h_2(\theta)]^{b\cos\alpha+1}\right| = k_4.$$

Similarly, the inequality:

$$1+b\cos\alpha<0$$

implies that

$$\min\left|[h_2(\theta)]^{b\cos\alpha+1}\right| = k_4 \quad \text{and} \quad \max\left|[h_2(\theta)]^{b\cos\alpha+1}\right| = k_3.$$

Since

$$\min|[h_3(\theta)| = k_5 \quad \text{and} \quad \max|h_3(\theta)| = k_6,$$

we easily arrive at the estimates asserted by Theorem 5. ∎

Remark 1. The estimates in Theorem 5 are not sharp, since the minima and the maxima used in the proof of Theorem 5 are (in general) obtained for different values of θ. Owing to the obviously complicated calculations, we leave the finding of sharp estimates as an *open* problem.

When $r \to 1-$, Theorem 4 and Theorem 5 yield the following corollary.

Corollary 1. *Assume that*

$$a := \frac{b\sin\alpha}{1+b\cos\alpha},$$

$$a_1 = 2\cdot|\cos\alpha|\cdot e^{\alpha\tan\alpha} \qquad \textit{and} \qquad a_2 = \frac{2}{\sqrt{1+a^2}}\cdot e^{a\cdot\arctan(a)}.$$

(i) *If* $b\cos\alpha \geqq 0$, *then*

$$|k(z)| \geqq a_1^{-b\cos\alpha} \qquad (|z|=1).$$

(ii) *If* $b\cos\alpha < 0$, *then*

$$|k(z)| \leqq a_1^{-b\cos\alpha} \qquad (|z|=1).$$

(iii) *If* $b\cos\alpha + 1 \geqq 0$, *then*

$$|k'(z)| \geqq a_2^{-b\cos\alpha-1} \qquad (|z|=1).$$

(iv) *If* $b\cos\alpha + 1 < 0$, *then*

$$|k'(z)| \leqq a_2^{-b\cos\alpha-1} \qquad (|z|=1).$$

The estimates asserted by (i) *and* (ii) *are sharp.*

Proof. The estimates asserted by Corollary 1 follow from the facts that

$$\lim_{r\to1^-}\frac{n\theta_1}{2} = \frac{\pi}{2} - \alpha, \qquad \lim_{r\to1^-} h_1(\theta_1) = a_1,$$

$$\lim_{r\to1^-}\frac{n\theta_3}{2} = \arctan\left(\frac{1}{a}\right), \qquad \lim_{r\to1^-} h_2(\theta_3) = a_2,$$

$$\lim_{r\to1^-}\frac{n\theta_2}{2} = \lim_{r\to1^-}\frac{n\theta_4}{2} = 0 \qquad \text{and} \qquad \lim_{r\to1^-} h_1(\theta_2) = \lim_{r\to1^-} h_2(\theta_4) = 0.$$

∎

Theorem 6 below provides sharp estimate of $\left|\arg\left(k'(z)\right)\right|$ in the case when $\alpha = 0$. The estimation of $\left|\arg\left(k'(z)\right)\right|$ in more general cases would involve complicated calculations.

Theorem 6. *Let* $\alpha = 0$, *that is*, *let*

$$k(z) = \frac{z}{(1-z^n)^b} \qquad (b \in \mathbb{R}\setminus\{0\}.$$

Suppose that

$$0 \leqq r < 1 \qquad \textit{and} \qquad r^n\cdot|1-bn| < 1,$$

$$A = -2bnr^{2n}(1-bn),$$

$$B = bnr^n\left[r^{2n}(1-bn)(3+n-bn)+n+1\right]$$

and

$$C = -bnr^{2n}\left[r^{2n}(1-bn)^2+1+2n-bn^2\right].$$

Assume also that s_0 *is the unique solution of the following equation*:

$$As^2 + Bs + C = 0 \qquad (-1 < s_0 < 1). \tag{2.4}$$

Then the following sharp estimate holds true:

$$\left|\arg\left(k'(z)\right)\right| \leqq |h_4(\theta_0)| \qquad (|z| = r), \tag{2.5}$$

where

$$\theta_0 = \frac{1}{n}\arccos(s_0)$$

and

$$h_4(\theta) = \arctan\left(\frac{r^n(bn-1)\sin(n\theta)}{1 + r^n(bn-1)\cos(n\theta)}\right) - (b+1)\arctan\left(\frac{r^n\sin(n\theta)}{r^n\cos(n\theta) - 1}\right).$$

Proof. For $z = re^{i\theta}$, we find from (2.3) that

$$\arg\left(k'(re^{i\theta})\right) = \left(1 + (bn-1)r^n e^{in\theta}\right) - (b+1)\arg\left(1 - r^n e^{in\theta}\right) =: h_4(\theta).$$

Since

$$k'(\overline{z}) = \overline{k'(z)} \qquad (z \in \mathbb{U}),$$

we conclude that

$$h_4(\theta) = \arg\left(k'(re^{i\theta})\right)$$

attains its minimal and maximal values for some $\theta_* \geqq 0$ and $-\theta_*$. Further, since the function $h_4(\theta)$ is continuous and

$$h_4(0) = h_4\left(\frac{\pi}{n}\right) = 0,$$

we also conclude that

$$\theta_* \in \left(0, \frac{\pi}{n}\right),$$

that is, that

$$\cos(n\theta_*) \neq \pm 1.$$

It also implies that $h_4'(\theta_*) = 0$, which is equivalent to

$$g(s_*) = As_*^2 + Bs_* + C = 0 \qquad \text{and} \qquad s_* = \cos(n\theta_*).$$

This quadratic equation has two real roots and only one of them lies in the interval $(-1, 1)$ because

$$g(-1)\cdot g(1) = b^2n^2r^{2n}(r^{2n}-1)[r^{2n}(bn-1)^2 - (n+1)^2][r^{2n}(bn-1)^2 - 1] < 0.$$

Therefore, we have

$$\cos(n\theta_*) = s_0,$$

that is,

$$\theta_* = \frac{1}{n}\arccos(s_0) =: \theta_0.$$

∎

For $bn = 1$, we deduce Corollary 2 below.

Corollary 2. *Let* $\alpha = 0$, *that is*, *let*

$$k(z) = \frac{z}{(1-z^n)^b} \qquad (b \in \mathbb{R}\setminus\{0\}.$$

Suppose that $0 \leqq r < 1$. *If* $bn = 1$, *then*

$$\left|\arg\left(k'(z)\right)\right| \leqq (b+1)\cdot\arctan\left(\frac{r^n}{\sqrt{1-r^{2n}}}\right) \qquad (|z| = r). \tag{2.6}$$

This estimate is sharp.

Proof. For $bn = 1$, we use the notations which are already introduced in Theorem (6). We thus find that

$$A = 0, \;\; B = r^n(n+1), \;\; C = -r^{2n}(n+1), \;\; s_0 = r^n \quad \text{and} \quad \theta_0 = \frac{1}{n}\arccos(r^n).$$

The estimate given by (2.6) now follows easily from the estimate (2.5) asserted by Theorem 6. ∎

When $r \to 1-$ in Theorem 6, we obtain the following corollary.

Corollary 3. *Let* $\alpha = 0$, *that is, let*

$$k(z) = \frac{z}{(1-z^n)^b} \qquad (b \in \mathbb{R} \setminus \{0\}.$$

If $0 < bn < 2$, *then the following sharp estimate holds true*:

$$\left|\arg\left(k'(z)\right)\right| \leqq (b+1)\frac{\pi}{2} \qquad (|z| = 1).$$

Furthermore,

$$k(z) \notin \mathcal{R}_\beta = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \left|\arg\left(f'(z)\right)\right| < \frac{\beta\pi}{2} \;\; (z \in \mathbb{U};\; 0 < \beta \leqq 1) \right\}$$

and

$$k(z) \notin \mathcal{R}(\gamma) = \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\big(f'(z)\big) > \gamma \;\; (z \in \mathbb{U};\; 0 \leqq \gamma < 1) \right\}.$$

Proof. Corollary 3 follows directly from Theorem 6 in light of the fact that, for $r = 1$, $r^n \cdot |1 - bn| < 1$ is equivalent to $0 < bn < 2$, and by verifying that the equation (2.4) has the following solutions:

$$s_1 = 1 \qquad \text{and} \qquad s_2 = \frac{1}{2}\left(n - bn + 1 - \frac{n+1}{bn-1}\right) \qquad (|s_2| \geqq 1).$$

Therefore, we get

$$\theta_0 = 0 \qquad \text{and} \qquad |h_4(0)| = (b+1)\frac{\pi}{2}.$$

Moreover, $bn > 0$ implies that $b > 0$. We thus arrive readily at each of the following assertions:

$$k(z) \notin \mathcal{R}_\beta \qquad \text{and} \qquad k(z) \notin \mathcal{R}(\gamma)$$

of Corollary 3. ∎

3. Radii of Spirallikeness, Starlikeness and Convexity

We begin this section with the relation between the n-fold symmetric Koebe type function $k(z)$ and the class $\mathcal{S}^*_\alpha(\beta)$ of α-spirallike functions of order β in $\mathbb{U}$.

Theorem 7. *If* $\alpha \neq \gamma$ *and* $b \neq 0$, *then* $k(z) \notin \mathcal{S}^*_\gamma(\beta)$.

Proof. By means of straightforward calculations, we see that

$$\frac{zk'(z)}{k(z)} = 1 + \frac{bne^{i\alpha}z^n}{1-z^n} \equiv h_5(z) \tag{3.1}$$

and that, for $z = e^{i\theta}$,

$$\Re\left(e^{-i\gamma}\frac{e^{i\theta}k'(e^{i\theta})}{k(e^{i\theta})}\right) = \cos\gamma - \frac{bn}{2} \cdot \left[\sin(\alpha-\gamma)\cot\left(\frac{n\theta}{2}\right) + \cos(\alpha-\gamma)\right].$$

Therefore, if $\alpha \neq \gamma$ and $b \neq 0$, then

$$h_5(\mathbb{U}) = \mathbb{R}.$$

∎

From Theorem 7 naturally arises the interesting question of finding the radii of α-spirallikeness of some order, as described in (1.9). Such a set of radii exists, since every analytic function $f(z)$ maps, one-to-one, a small disk onto a small disk, that is, there always exists small enough disk

$$\mathbb{U}_r = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < r\}$$

such that $f(\mathbb{U}_r)$ is a convex, starlike or α-spirallike region.

Theorem 8. *Assume that*

$$-\frac{\pi}{2} \leqq \alpha < \frac{\pi}{2} \qquad \textit{and} \qquad 0 \leqq \beta < \cos\alpha \leqq 1.$$

(i) *If* $bn \geqq 2(\cos\alpha - \beta)$, *then*

$$r_{\rm sp}^n = \frac{\cos\alpha - \beta}{\beta - \cos\alpha + bn} \equiv r_{\rm sp2}.$$

(ii) *If* $0 \leqq bn \leqq 2(\cos\alpha - \beta)$, *then*

$$r_{\rm sp} = 1.$$

(iii) *If* $b \leqq 0$, *then*

$$r_{\rm sp}^n = -r_{\rm sp2}.$$

Proof. Since

$$k(z) = z^{1-n} \cdot f_*(z^n),$$

it is fairly easy to verify that

$$\frac{zk'(z)}{k(z)} = 1 - n + n \cdot \frac{z^n f_*'(z^n)}{f_*(z^n)}$$

and

$$\Re\left(e^{-i\alpha} \cdot \frac{zk'(z)}{k(z)}\right) = (1-n)\cos\alpha + n \cdot \Re\left(e^{-i\alpha} \cdot \frac{z^n f_*'(z^n)}{f_*(z^n)}\right) > \beta.$$

Therefore, we have

$$\Re\left(e^{-i\alpha} \cdot \frac{z^n f_*'(z^n)}{f_*(z^n)}\right) > \frac{\beta - (1-n)\cos\alpha}{n}.$$

Thus, upon setting

$$\beta \mapsto \frac{\beta - (1-n)\cos\alpha}{n}$$

in Theorem 1, we get the result asserted by Theorem 8. ∎

Since

$$r_{\rm sp} = 1 \iff k(z) \in \mathcal{S}_\alpha^*(\beta),$$

we can easily deduce Corollary 4 below.

Corollary 4. *Suppose that*

$$-\frac{\pi}{2} \leqq \alpha < \frac{\pi}{2} \qquad \textit{and} \qquad 0 \leqq \beta < \cos\alpha \leqq 1.$$

Then the n-fold symmetric Koebe type function $k(z)$ is α-spirallike of order β in $\mathbb{U}$, that is, $k(z) \in \mathcal{S}_\alpha^*(\beta)$, if and only if

$$0 \leqq bn \leqq 2(\cos\alpha - \beta).$$

For the radii of starlikeness of order β, by using Theorem 2 and similar technique as in the proof of Theorem 8, we now prove Theorem 9 below.

Theorem 9. *Assume that*

$$0 \leqq \alpha < 2\pi, \qquad 0 \leqq \beta < 1, \qquad r_{\text{st}3} = \frac{bn}{2(bn\cos\alpha + \beta - 1)}$$

and

$$r_{\text{st}4} = \frac{\sqrt{b^2n^2 - 4bn(1-\beta)\cos\alpha + 4(1-\beta)^2}}{2(bn\cos\alpha + \beta - 1)}.$$

(i) *Let* $b \geqq 0$ *and* $\cos\alpha \geqq 0$.

(a) *If*

$$b \neq 0 \qquad \textit{and} \qquad bn \neq \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\text{st}}^n = r_{\text{st}3} - r_{\text{st}4}.$$

(b) *If*

$$bn = \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\text{st}}^n = \cos\alpha.$$

(c) *If* $b = 0$, *then* $r_{\text{st}} = 1$.

(ii) *If*

$$b \geqq 0 \qquad \textit{and} \qquad \cos\alpha < 0,$$

then

$$r_{\text{st}}^n = r_{\text{st}3} - r_{\text{st}4}.$$

(iii) *If*

$$b < 0 \qquad \textit{and} \qquad \cos\alpha \geqq 0,$$

then

$$r_{\text{st}}^n = -r_{\text{st}3} - r_{\text{st}4}.$$

(iv) *Let* $b < 0$ *and* $\cos\alpha < 0$.

(a) *If*

$$bn \neq \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\text{st}}^n = -r_{\text{st}3} - r_{\text{st}4}.$$

(b) *If*

$$bn = \frac{1-\beta}{\cos\alpha},$$

then

$$r_{\text{st}}^n = -\cos\alpha.$$

Proof. We first observe that

$$\Re\left(\frac{zk'(z)}{k(z)}\right) = 1 - n + n\cdot\Re\left(\frac{z^n f_*'(z^n)}{f_*(z^n)}\right) > \beta$$

or, equivalently, that

$$\Re\left(\frac{z^n f_*'(z^n)}{f_*(z^n)}\right) > \frac{\beta - 1 + n}{n}.$$

Now, in Theorem 2, we set

$$\beta \mapsto \frac{\beta - 1 + n}{n}.$$

We then obtain the result asserted by Theorem 9. ∎

Next, by recalling the fact that

$$r_{\mathrm{st}} = 1 \iff k(z) \in \mathcal{S}^*(\beta),$$

we are led to the following corollary.

Corollary 5. *Let* $0 \leqq \beta < 1$. *Then the* n*-fold symmetric Koebe type function* $k(z)$ *is starlike of order* β, *that is*, $k(z) \in \mathcal{S}^*(\beta)$, *if and only if one of the following conditions is satisfied*:

(i) $b = 0$;
(ii) $\alpha = 0$ *and* $bn = 1 - \beta$;
(iii) $\alpha = \pi$ *and* $bn = \beta - 1$.

Proof. Corollary 5 follows directly from Theorem 9 because $bn = 1 - \beta$ implies that $b > 0$, and that $bn = \beta - 1$ implies that $b < 0$. Cases (i)(a), (ii), (iii) and (iv)(a) do not lead to $r_{\mathrm{st}} = 1$ because, under the conditions specified there, we cannot have

$$r_{\mathrm{st}3} - r_{\mathrm{st}4} = 1 \qquad \text{and} \qquad - r_{\mathrm{st}3} - r_{\mathrm{st}4} = 1.$$

∎

Finally, for the radii of convexity of order β, we choose to cover only the case when $\alpha = 0$ and $|b| \geqq 1$. As before, the more general cases would involve complicated calculations, and so we leave them as open problems.

Theorem 10. *Let* $0 \leqq \beta < 1$ *and* $\alpha = 0$, *that is*, *let*

$$k(z) = \frac{z}{(1 - z^n)^b} \qquad (b \in \mathbb{R}).$$

(i) *If* $b \geqq 1$, *then*

$$r_{\mathrm{k}}^n = \frac{bn(2 + n - \beta) - 2(1 - \beta) - n\sqrt{b[b(\beta^2 - 2n\beta + n^2 + 4n) - 4(1 - \beta)]}}{2(bn - 1)(bn - 1 + \beta)} =: r_{\mathrm{k}1}.$$

(ii) *If* $b \leqq -1$, *then*

$$r_{\mathrm{k}}^n = \frac{-bn(2 + n - \beta) + 2(1 - \beta) - n\sqrt{b[b(\beta^2 - 2n\beta + n^2 + 4n) - 4(1 - \beta)]}}{2(bn - 1)(bn - 1 + \beta)}.$$

Proof. We start from the following relation:

$$1 + \frac{zk''(z)}{k'(z)} = 1 - bn + n(b + 1) \cdot h_5(z) - n \cdot h_6(z) =: h_7(z),$$

where

$$h_5(z) = \frac{1}{1 - z^n} \qquad \text{and} \qquad h_6(z) = \frac{1}{1 + (bn - 1)z^n},$$

so that

$$\begin{aligned} \min\{h_5(re^{i\theta}) : \theta \in [0, 2\pi)\} &= (1 + r^n)^{-1} \qquad (n\theta = \pi), \\ \max\{h_5(re^{i\theta}) : \theta \in [0, 2\pi)\} &= (1 - r^n)^{-1} \qquad (n\theta = 0), \end{aligned}$$

and

$$\min\{h_6(re^{i\theta}) : \theta \in [0, 2\pi)\} = \begin{cases} \dfrac{1}{(1 + r^n(bn - 1))} & (bn \geqq 1;\ n\theta = 0) \\[2ex] \dfrac{1}{(1 - r^n(bn - 1))} & (bn < 1;\ n\theta = \pi) \end{cases}$$

and

$$\min\{h_6(re^{i\theta}) : \theta \in [0, 2\pi)\} = \begin{cases} \dfrac{1}{(1 + r^n(bn-1))} & (bn \geqq 1;\ n\theta = 0) \\[2ex] \dfrac{1}{(1 - r^n(bn-1))} & (bn < 1;\ n\theta = \pi). \end{cases}$$

We now let $b \geqq 1$. Then $bn \geqq 1$ and we find that

$$\min\{h_7(re^{i\theta}) : \theta \in [0, 2\pi)\} = 1 - bn + \frac{n(b+1)}{1+r^n} - \frac{n}{1-(bn-1)r^n} =: h_8(r) \qquad (n\theta = \pi).$$

Further, $h_8(0) = 1 > \beta$ and $h_8(r)$ on the interval $[0, 1]$ has exactly one point of discontinuity given by

$$r_*^n = \frac{1}{bn-1}.$$

Moreover, $h_8(r)$ on the interval $[0, r_*)$ is a decreasing function with the range $(-\infty, 1]$. Therefore, the following implication:

$$0 \leqq r < r_k \quad \Longrightarrow \quad \Re\left(1 + \frac{zk''(z)}{k'(z)}\right) > \beta \qquad (|z| < r)$$

holds true only when r_{k} is the unique solution of the equation $h_8(r) = \beta$, that is, when $r_{\mathrm{k}}^n = r_{\mathrm{k}1}$.

In a similar manner, we can prove the assertion given in Part (ii) of Theorem 10. ∎

Remark 2. By setting $n = 1$ in Theorem 8, Theorem 9 and Theorem 10, we can deduce the above-mentioned Theorem 1, Theorem 2 and Theorem 3, respectively, in this paper.

4. A SET OF NECESSARY AND SUFFICIENT CONDITIONS FOR THE FUNCTION CLASS $\mathcal{G}_\lambda$ OF SILVERMAN

In our next theorem, we shows that, in general, the n-fold symmetric Koebe type function $k(z)$ is not in the class $\mathcal{G}_\lambda$ which was introduced (for any $\lambda \in \mathbb{R}$) by Silverman [7].

Theorem 11. *If* $b(bn - 2\cos\alpha) > 0$, *then* $k(z) \notin \mathcal{G}_\lambda$.

Proof. We begin by noting that

$$\frac{1 + \left(\dfrac{zk''(z)}{k'(z)}\right)}{\left(\dfrac{zk'(z)}{k(z)}\right)} - 1 = \frac{bn^2 e^{i\alpha} z^n}{(1+\delta z^n)^2} \qquad \left(\delta := bne^{i\alpha} - 1\right).$$

If $b(bn - 2\cos\alpha) > 0$, then

$$|\delta| > 1, \quad z_0 = \sqrt[n]{-\frac{1}{\delta}} \in \mathbb{U} \quad \text{and} \quad \left|\frac{1 + \left(\dfrac{z_0 k''(z_0)}{k'(z_0)}\right)}{\left(\dfrac{z_0 k'(z_0)}{k(z_0)}\right)} - 1\right| > \lambda \qquad (\lambda \in \mathbb{R}).$$

∎

Remark 3. In light of Theorem 11, we are now motivated to find $r_{\text{sil}} \in (0,1]$ such that

$$0 \leqq r < r_{\text{sil}} \quad \Longrightarrow \quad \left| \frac{1 + \left(\dfrac{zk''(z)}{k'(z)}\right)}{\left(\dfrac{zk'(z)}{k(z)}\right)} - 1 \right| < \lambda \qquad (|z| < r), \tag{4.1}$$

which, together with other available information, will lead us to the necessary and sufficient conditions for $k(z) \in \mathcal{G}_\lambda$. In fact, since every analytic function maps, one-to-one, a small disk onto a small disk, such r_{sil} as defined by (4.1) above exists.

Theorem 12. *Suppose that*

$$0 < \lambda \leqq 1 \qquad \textit{and} \qquad \delta = bne^{i\alpha} - 1.$$

(i) *If* $b = 0$, *then* $r_{\text{sil}} = 1$.
(ii) *If* $|b|n = 1$ *and* $bn\cos\alpha = 1$, *then*

$$r_{\text{sil}}^n = \frac{\lambda}{n}.$$

(iii) *Let* $b \neq 0$. *If* $n \cdot |b| \neq 1$ *or* $bn\cos\alpha \neq 1$, *then*

$$r_{\text{sil}}^n = \frac{2\lambda|\delta| + n^2 \cdot |b| - \sqrt{4\lambda n^2 \cdot |\delta| \cdot |b| + b^2 n^4}}{2\lambda|\delta|^2} =: r_{\text{sil1}}.$$

Proof. We begin by recalling the fact that

$$\left| \frac{1 + \left(\frac{zk''(z)}{k'(z)}\right)}{\left(\dfrac{zk'(z)}{k(z)}\right)} - 1 \right| = \frac{n^2 \cdot |b| \cdot |z|^n}{|1 + \delta z^n|^2} =: h_9(z). \tag{4.2}$$

The following cases correspond to those in Theorem 12.

(i) If $b = 0$, then

$$h_9(z) = 0 < 1 \qquad (z \in \mathbb{U}),$$

which yields $r_{\text{sil}} = 1$.

(ii) If the conditions stated in Part (ii) of Theorem 12 are satisfied, then

$$\delta = 0, \quad bne^{i\alpha} = 1 \quad \text{and} \quad h_9(re^{i\theta}) = nr^n,$$

which leads us to the following assertion:

$$r_{\text{sil}}^n = \frac{\lambda}{n} \qquad (0 < \lambda \leqq 1).$$

(iii) Let $b \neq 0$. If $n \cdot |b| \neq 1$ or $bn\cos\alpha \neq 1$, then

$$\delta \neq 0 \quad \text{and} \quad \max\{h_9(re^i\theta) : \theta \in [0, 2\pi)\} = \frac{n^2 \cdot |b| \cdot r^n}{(1 - r^n|\delta|)^2} \leqq \lambda \tag{4.3}$$

$$(0 < \lambda \leqq 1),$$

which is equivalent to the following inequalities:

$$|\delta| r^n < 1 \quad \text{and} \quad \lambda s^2 \cdot |\delta|^2 - \left(2\lambda|\delta| + n^2 \cdot |b|\right) s + \lambda \geqq 0 \tag{4.4}$$

$$(s = r^n;\ 0 < \lambda \leqq 1).$$

The corresponding quadratic equation has two different real roots given by

$$s_{1,2} = \frac{2\lambda|\delta| + n^2 \cdot |b| \pm \sqrt{4\lambda n^2 \cdot |\delta| \cdot |b| + b^2 n^4}}{2\lambda|\delta|^2},$$

irrespective of the values of λ, δ and n. Suppose that s_1 is the *smaller* root. Then, since $\lambda|\delta|^2 > 0$, we conclude that the inequality (4.4) holds true if and only if

$$s \leqq s_1 \qquad \text{or} \qquad s \geqq s_2.$$

Moreover, it is easily verified in this case that

$$0 < s_1 < 1 \leqq s_2,$$

which implies that the inequality (4.4) holds true if and only if

$$0 \leqq r^n \leqq s_1.$$

Finally, if $0 \leqq r^n \leqq s_1$, then

$$r^n \cdot |\delta| < 1,$$

which allows us to conclude that the inequality (4.3) holds true if and only if

$$0 \leqq r^n \leqq s_1,$$

that is, that

$$r_{\text{sil}}^n = s_1 = r_{\text{sil1}}.$$

∎

Since $k(z) \in \mathcal{G}_\lambda$ if and only if $r_{\text{sil}} = 1$, we have the following corollary.

Corollary 6. *Each of the following assertions holds true*:

(i) *If* $b = 0$, *then*

$$k(z) \in \mathcal{G}_\lambda \qquad (0 < \lambda \leqq 1).$$

(ii) *If* $n = |b| = b\cos\alpha = \lambda = 1$, *then* $k(z) \in \mathcal{G}_1$.

(iii) *If*

$$b \neq 0, \quad bn < \cos\alpha \quad \textit{and} \quad \lambda = \frac{n^2 \cdot |b|}{\left(1 - \sqrt{b^2n^2 - 2bn\cos\alpha + 1}\right)^2} \leqq 1,$$

then $k(z) \in \mathcal{G}_\lambda$.

Proof. If $b = 0$, then $k(z) = z$ and Part (i) of Corollary 6 follows readily. If, on the other hand,

$$n = |b| = b\cos\alpha = \lambda = 1,$$

then

$$n \cdot |b| = 1 \qquad \text{and} \qquad bn\cos\alpha = 1$$

and, by appying Theorem 12(ii), we find that

$$r_{\text{sil}} = \sqrt[n]{\frac{\lambda}{n}} = 1,$$

which completes the proof of Part (ii) of Corollary 6. In the case of Part (iii) of Corollary 6, by means of $bn < \cos\alpha$, and the notations used in the proof of Theorem 12, we find that $|\delta| < 1$ and, furthermore, that

$$\max\{h_9(re^i\theta) : \theta \in [0, 2\pi)\} = \frac{n^2 \cdot |b|}{(1 - |\delta|)^2} = \lambda \qquad (0 < \lambda \leqq 1). \tag{4.5}$$

∎

5. CONCLUDING REMARKS AND OBSERVATIONS

Our present investigation is motivated essentially by a recent work by Kamali and Srivastava [5], which dealt with the following family of the n-fold symmetric Koebe type functions:

$$k(z) = \frac{z}{(1-z^n)^{b\exp(i\alpha)}} \qquad (b \in \mathbb{R};\ -\pi \leqq \alpha < \pi;\ n \in \mathbb{N}),$$

whose special case when $n = 1$ was investigated in an earlier paper by Eguchi and Owa [4]. We have discussed and presented here a systematic further study of this general family of the n-fold symmetric Koebe type functions. In particular, we have proved several distortion theorems and such other properties for this general family of the n-fold symmetric Koebe type functions as the radii of spirallikeness, the radii of starlikeness and the radii of convexity. For this family of the n-fold symmetric Koebe type functions, we have also demonstrated certain criteria that embed this family of the n-fold symmetric Koebe type functions in a function class $\mathcal{G}_\lambda$ which was introduced and studied earlier by Silverman [7].

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