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ON OPIAL'S INEQUALITY FOR FUNCTIONS OF N-INDEPENDENT VARIABLES

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ABSTRACT. In this paper, we introduce Opial inequalities for functions of n-independent variables. Also, we discuss some different forms of Opial inequality containing functions of n independent variables and their partial derivatives with respect to independent variables.

Key words and phrases: Opial-type inequalities; Hilbert's inequality; Integro-differential inequalities.

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1. INTRODUCTION

In (1960), Opial established the following integral inequality as follows [1]. Let $f(t)$ be a continuous function on $[a,\ b]$ such that $f\left(a\right)=f\left(b\right)=0$ and $f\left(t\right)>0$, then

$$\int_a^b \left|f\left(t\right)f'(t)\right|dt \le \frac{(b-a)}{4}\int_a^b \left|f'(t)\right|^2 dt. \tag{1.1}$$

Yang [9] obtained the following integral inequality in two independent variables

$$\int_a^b \left(\int_c^d \left|f\left(s,t\right)\right|\left|f_{12}\left(s,t\right)\right|dt\right)ds \le \frac{\left(b-a\right)\left(d-c\right)}{8}\int_a^b \left(\int_c^d \left|f_{12}\left(s,t\right)\right|^2 dt\right)ds \tag{1.2}$$

where

$$f_1\left(s,t\right)=\frac{\partial}{\partial t}f\left(s,t\right) \text{ and } f_{12}\left(s,t\right)=\frac{\partial^2}{\partial s\partial t}f\left(s,t\right). \tag{1.3}$$

During the past few years many number of papers which deal with the Opial's inequality (see [6], [7]and [8]). The main aim of this paper is to obtain some new integral inequalities in n-independent variables. Our results are considered improvement and partial generalization of those given in [2], [4], [6] and [8].

2. THE MAIN RESULTS

Throughout this paper we use the following notations. If $f(t_1,t_2,t_3,\dots,t_n)$ is a differentiable function of n variables, then

$$f_i\left(t_1,t_2,t_3,\dots,t_n\right)=\frac{\partial f(t_1,t_2,t_3,\dots,t_n)}{\partial t_i},$$

$$f_{ij}\left(t_1,t_2,t_3,\dots,t_n\right)=\frac{\partial^2 f(t_1,t_2,t_3,\dots,t_n)}{\partial t_i\partial t_j},$$

$$f_{ijk}\left(t_1,t_2,t_3,\dots,t_n\right)=\frac{\partial^3 f\left(t_1,t_2,t_3,\dots,t_n\right)}{\partial t_i\partial t_j\partial t_k},\dots,$$

$$f_{ijk\dots n}\left(t_1,t_2,t_3,\dots,t_n\right)=\frac{\partial^n f\left(t_1,t_2,t_3,\dots,t_n\right)}{\partial t_i\partial t_j\partial t_k\dots\partial t_n},$$

where $i,j,k,\dots,n=1,2,3,\dots,n$. Assume the interval $\triangle$ be such that $\triangle=[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]\times\cdots\times[a_n,b_n]$, where $a_1,a_2,a_3,\dots,a_n,b_1,b_2,b_3,\dots,b_n$ are real constants, n is a positive integer and $(t_1,t_2,t_3,\dots,t_n)\in\triangle$,

and

$$\begin{aligned}
\triangle_1 &= [a_1,s_1]\times[a_2,s_2]\times[a_3,s_3]\times\cdots\times[a_{n-1},s_{n-1}]\times[a_n,s_n],\\
\triangle_2 &= [a_1,s_1]\times[a_2,s_2]\times[a_3,s_3]\times\cdots\times[a_{n-1},s_{n-1}]\times[s_n,b_n],\\
\triangle_3 &= [a_1,s_1]\times[a_2,s_2]\times[a_3,s_3]\times\cdots\times[s_{n-1},b_{n-1}]\times[a_n,s_n],\dots,\\
\triangle_{n-1} &= [a_1,s_1]\times[s_2,b_2]\times[s_3,b_3]\times\cdots\times[s_{n-1},b_{n-1}]\times[s_n,b_n],\\
\triangle_n &= [s_1,b_1]\times[s_2,b_2]\times[s_3,b_3]\times\cdots\times[s_{n-1},b_{n-1}]\times[s_n,b_n],
\end{aligned}$$

where $a_1\le s_1\le b_1,\ a_2\le s_2\le b_2,\ a_3\le s_3\le b_3,\ \dots,\ a_{n-1}\le s_{n-1}\le b_{n-1},\ a_n\le s_n\le b_n$. Let $f\left(t_1,t_2,t_3,\dots,t_n\right)$ satisfies the following hypotheses.

(H_1) Let $f\left(t_1,t_2,t_3,\dots,t_n\right),\ f_1\left(t_1,t_2,t_3,\dots,t_n\right),\ f_{12}\left(t_1,t_2,t_3,\dots,t_n\right),$

$f_{123}(t_1,t_2,t_3,\ldots,t_n)$, $\ldots$, $f_{123\ldots(n-1)}(t_1,t_2,t_3,\ldots,t_n)$ be continuous functions on $\triangle$, and

$$\begin{aligned} f(a_1,t_2,t_3,\ldots,t_n) &= f(b_1,t_2,t_3,\ldots,t_n) \\ &= f(t_1,a_2,t_3,\ldots,t_n) = f(t_1,b_2,t_3,\ldots,t_n) \\ &= f(t_1,t_2,a_3,\ldots,t_n) = f(t_1,t_2,b_3,\ldots,t_n)\ldots \\ &= f(t_1,t_2,t_3,\ldots,a_n) = f(t_1,t_2,t_3,\ldots,b_n) = 0. \end{aligned}$$

(H_2) Let

$$\begin{aligned} f_1(a_1,t_2,t_3,\ldots,t_n) &= f_1(b_1,t_2,t_3,\ldots,t_n) \\ &= f_1(t_1,a_2,t_3,\ldots,t_n) = f_1(t_1,b_2,t_3,\ldots,t_n) \\ &= f_1(t_1,t_2,a_3,\ldots,t_n) = f_1(t_1,t_2,b_3,\ldots,t_n) = \ldots \\ &= f_1(t_1,t_2,t_3,\ldots,a_n) = f_1(t_1,t_2,t_3,\ldots,b_n) = 0. \end{aligned}$$

Using the above assumptions we introduce the following results.

Theorem 2.1. *Assume that (H_1) and (H_2) hold. Then the following inequality holds*

$$\begin{aligned} &\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\ldots \\ &\quad (\int_{a_n}^{b_n}|f(t_1,t_2,t_3,\ldots,t_n)|\,|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n)\ldots)dt_3)dt_2)dt_1 \\ (2.1)\quad &\le \frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)\ldots(b_n-a_n)}{2^{n+1}} \\ &\quad\times\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\ldots(\int_{a_n}^{b_n}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|^2dt_n)\ldots)dt_3)dt_2)dt_1. \end{aligned}$$

Proof. We consider the following 2^n cases.

Case 1. Let $(t_1,t_2,t_3,\ldots,t_n)\in\triangle_1$, and we define the following function

$$\begin{aligned} (2.2)\quad & h(s_1,s_2,s_3,\ldots,s_n) \\ &= \int_{a_1}^{s_1}(\int_{a_2}^{s_2}(\int_{a_3}^{s_3}(\ldots(\int_{a_n}^{s_n}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n)\ldots)dt_3)dt_2)dt_1. \end{aligned}$$

Thus, from 2.2 we obtain

$$(2.3)\quad h_1(s_1,s_2,s_3,\ldots,s_n) = \int_{a_2}^{s_2}(\int_{a_3}^{s_3}(\ldots(\int_{a_n}^{s_n}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n)\ldots)dt_3)dt_2,$$

and

$$\begin{aligned} (2.4)\quad & h_{12}(s_1,s_2,s_3,\ldots,s_n) \\ &= \int_{a_3}^{s_3}(\int_{a_4}^{s_4}(\ldots(\int_{a_5}^{s_5}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n)\ldots)dt_4)dt_3. \end{aligned}$$

Therefore, if we continue in these differentiations, we get

$$(2.5)\quad h_{123\ldots(n-1)}(s_1,s_2,s_3,\ldots,s_n) = \int_{a_n}^{s_n}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n.$$

Now, under the condition (H_1), we have

$$(2.6)\quad |f(s_1,s_2,s_3,\ldots,s_n)| \le \int_{a_1}^{s_1}|f_1(t_1,s_2,s_3,\ldots,s_n)|\,dt_1,$$

$$(2.7)\qquad |f_1(s_1,s_2,s_3,\dots,s_n)| \le \int_{a_2}^{s_2} |f_{12}(s_1,t_2,s_3,\dots,s_n)|\,dt_2,$$

$$(2.8)\qquad |f_{12}(s_1,s_2,s_3,\dots,s_n)| \le \int_{a_3}^{s_3} |f_{123}(s_1,s_2,t_3,\dots,s_n)|\,dt_3.$$

Also, we get

$$(2.9)\qquad \left|f_{12\dots(n-1)}(s_1,s_2,s_3,\dots,s_n)\right| \le \int_{a_n}^{s_n} |f_{12\dots n}(s_1,s_2,s_3,\dots,t_n)|\,dt_n.$$

It follows from, 2.7, 2.8 and 2.9 that

$$(2.10)\qquad \begin{aligned} & |f(s_1,s_2,s_3,\dots,s_n)| \\ \le\ & \int_{a_1}^{s_1} |f_1(t_1,s_2,s_3,\dots,s_n)|\,dt_1 \\ \le\ & \int_{a_1}^{s_1}(\int_{a_2}^{s_2} |f_{12}(t_1,t_2,s_3,\dots,s_n)|\,dt_2)dt_1 \\ \le\ & \int_{a_1}^{s_1}(\int_{a_2}^{s_2}(\int_{a_3}^{s_3} |f_{123}(t_1,t_2,t_3,\dots,s_n)|\,dt_3)dt_2)dt_1,\dots, \\ \le\ & \int_{a_1}^{s_1}(\int_{a_2}^{s_2}(\int_{a_3}^{s_3}(\dots(\int_{a_n}^{s_n} |f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1 \\ =\ & h(s_1,s_2,s_3,\dots,s_n), \end{aligned}$$

$$(2.11)\qquad |f_1(s_1,s_2,s_3,\dots,s_n)| \le h_1(s_1,s_2,s_3,\dots,s_n),$$

$$(2.12)\qquad |f_{12}(s_1,s_2,s_3,\dots,s_n)| \le h_{12}(s_1,s_2,s_3,\dots,s_n),$$

and

$$(2.13)\qquad \left|f_{123\dots(n-1)}(s_1,s_2,s_3,\dots,s_n)\right| \le h_{123\dots(n-1)}(s_1,s_2,s_3,\dots,s_n).$$

Now, let I_1 be such that

$$I_1 = \int_{a_1}^{s_1}(\int_{a_2}^{s_2}(\int_{a_3}^{s_3}(\dots(\int_{a_n}^{s_n} |f(t_1,t_2,t_3,\dots,t_n)|\,|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1.$$

From 2.5 and 2.10, we get

$$\begin{aligned} I_1 \le\ & \int_{a_1}^{s_1}\left(\int_{a_2}^{s_2}\left(\int_{a_3}^{s_3}(\dots \right.\right. \\ & \left.\left.\left(\int_{a_n}^{s_n} h((t_1,t_2,t_3,\dots,t_n))\,|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n\right)\dots\right)dt_3\right)dt_2\Bigg)dt_1 \\ \le\ & \int_{a_1}^{s_1} h(t_1,s_2,s_3,\dots,s_n) \\ & \times(\int_{a_2}^{s_2}(\int_{a_3}^{s_3}(\dots(\int_{a_n}^{s_n}(|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1 \\ =\ & \int_{a_1}^{s_1} h(t_1,s_2,s_3,\dots,s_n) \\ & \times(\int_{a_2}^{s_2}(\int_{a_3}^{s_3}(\dots(\int_{a_{n-1}}^{s_{n-1}}(h_{123\dots(n-1)}(t_1,t_2,t_3,\dots,t_{n-1},s_n)\,dt_{n-1})\dots)dt_3)dt_2)dt_1. \end{aligned}$$

Then, we have

$$\begin{aligned} I_1 &\leq \int_{a_1}^{s_1} h\left(t_1, s_2, s_3, \ldots, s_n\right) \left(\int_{a_2}^{s_2} h_{12}\left(t_1, t_2, s_3, \ldots, s_n\right) dt_2 \right) dt_1 \\ &\leq \int_{a_1}^{s_1} h\left(t_1, s_2, s_3, s_4, s_5\right) h_1\left(t_1, s_2, s_3, s_4, s_5\right) dt_1 \\ &= \frac{1}{2} h^2\left(s_1, s_2, s_3, \ldots, s_n\right) \\ &= \frac{1}{2} \left(\int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\ldots (\int_{a_n}^{s_n} \left| f_{123\ldots n}\left(t_1, t_2, t_3, \ldots, t_n\right) \right| dt_n) \ldots) dt_3) dt_2) dt_1 \right)^2 . \end{aligned}$$

Now, by applying Schwarz's inequality, we get

$$\begin{aligned} I_1 \leq{} & \frac{\left(s_1 - a_1\right)\left(s_2 - a_2\right)\left(s_3 - a_3\right) \ldots \left(s_n - a_n\right)}{2} \\ & \times \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\ldots \\ & \quad (\int_{a_n}^{s_n} \left| f_{123\ldots n}\left(t_1, t_2, t_3, \ldots, t_n\right) \right|^2 dt_n) \ldots) dt_3) dt_2) dt_1 \end{aligned} \tag{2.14}$$

Case 2. Let $(t_1, t_2, t_3, \ldots, t_n) \in \triangle_2$, and we define

$$\begin{aligned} & h\left(s_1, s_2, s_3, \ldots, s_n\right) \\ &= \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\ldots (\int_{s_n}^{b_n} \left| f_{123\ldots n}\left(t_1, t_2, t_3, \ldots, t_n\right) \right| dt_n) \ldots) dt_3) dt_2) dt_1 . \end{aligned} \tag{2.15}$$

By similar manner as in the first case we obtain

$$\begin{aligned} I_2 \leq{} & \frac{\left(s_1 - a_1\right)\left(s_2 - a_2\right)\left(s_3 - a_3\right) \ldots \left(b_n - s_n\right)}{2} \\ & \times \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\ldots \\ & \quad (\int_{s_n}^{b_n} \left| f_{123\ldots n}\left(t_1, t_2, t_3, \ldots, t_n\right) \right|^2 dt_n) \ldots) dt_3) dt_2) dt_1 . \end{aligned} \tag{2.16}$$

Case 3. Let $(t_1, t_2, t_3, \ldots, t_n) \in \triangle_3$, and we define

$$\begin{aligned} & h\left(s_1, s_2, s_3, \ldots, s_n\right) \\ &= \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\ldots (\int_{s_{n-1}}^{b_{n-1}} (\int_{a_n}^{s_n} \left| f_{123\ldots n}\left(t_1, t_2, t_3, \ldots, t_n\right) \right| dt_n) \ldots) dt_3) dt_2) dt_1 . \end{aligned} \tag{2.17}$$

Then, we get

$$\begin{aligned} I_3 \leq{} & \frac{\left(s_1 - a_1\right)\left(s_2 - a_2\right)\left(s_3 - a_3\right) \ldots \left(b_{n-1} - s_{n-1}\right)\left(s_n - a_n\right)}{2} \\ & \times \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\ldots \\ & \quad (\int_{s_{n-1}}^{b_{n-1}} (\int_{a_n}^{s_n} \left| f_{123\ldots n}\left(t_1, t_2, t_3, \ldots, t_n\right) \right|^2 dt_n) \ldots) dt_3) dt_2) dt_1 \end{aligned} \tag{2.18}$$

Similarly, we continue up to the following cases:

Case ($2^{\mathbf{n}}-1$). Let $(t_1,t_2,t_3,\dots,t_n)\in\triangle_{2^n-1}$, and we define

$$h(s_1,s_2,s_3,\dots,s_n) = \int_{a_1}^{s_1}(\int_{s_2}^{b_2}(\int_{s_3}^{b_3}(\dots(\int_{s_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1. \tag{2.19}$$

Then, we get

$$I_{2^n-1} \le \frac{(s_1-a_1)(b_2-s_2)(b_3-s_3)\dots(b_n-s_n)}{2} \times\int_{a_1}^{s_1}(\int_{s_2}^{b_2}(\int_{s_3}^{b_3}(\dots (\int_{s_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|^2\,dt_n)\dots)dt_3)dt_2)dt_1. \tag{2.20}$$

Case ($2^{\mathbf{n}}$). Let $(t_1,t_2,t_3,\dots,t_n)\in\triangle_{2^n}$, and we define

$$h(s_1,s_2,s_3,\dots,s_n) = \int_{s_1}^{b_1}(\int_{s_2}^{b_2}(\int_{s_3}^{b_3}(\dots(\int_{s_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1. \tag{2.21}$$

Then, we obtain

$$I_{2^n-1} \le \frac{(b_1-s_1)(b_2-s_2)(b_3-s_3)\dots(b_n-s_n)}{2} \times\int_{s_1}^{b_1}(\int_{s_2}^{b_2}(\int_{s_3}^{b_3}(\dots (\int_{s_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1. \tag{2.22}$$

Finally, let $s_1=\frac{a_1+b_1}{2}$, $s_2=\frac{a_2+b_2}{2}$, $s_3=\frac{a_3+b_3}{2}$, $\dots,s_n=\frac{a_n+b_n}{2}$, it follows from 2.14, 2.16, 2.18, 2.20 and 2.22 that

$$\begin{aligned}&\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\dots(\int_{a_n}^{b_n}|f(t_1,t_2,t_3,\dots,t_n)| \\ &|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1 \\ \le\ & \frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)\dots(b_n-a_n)}{2^{n+1}} \\ &\times\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\dots(\int_{a_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|^2dt_n)\dots)dt_3)dt_2)dt_1.\end{aligned}$$

This completes the proof. ∎

Remark 2.1. (1) Put $n=1$ in 2.1, we get Opial's inequality as in [5]

$$\int_a^b|f(t)|\,|f_1(t)|\,dt\le\frac{(b-a)}{4}\int_a^b|f_1(t)|^2\,dt.$$

(2) If $n=2$ in 2.1, then

$$\int_{a_1}^{b_1}(\int_{a_2}^{b_2}|f(t_1,t_2)|\,|f_{12}(t_1,t_2)|\,dt_2)dt_1\le\frac{(b_1-a_1)(b_2-a_2)}{8}\int_{a_1}^{b_1}(\int_{a_2}^{b_2}|f_{12}(t_1,t_2)|^2dt_2)dt_1,$$

which is Yang's inequality [9].

Theorem 2.2. *Assume that* (H_1) *and* (H_2) *hold. Then*

$$
\begin{aligned}
& \int_{a_1}^{b_1} (\int_{a_2}^{b_2} (\int_{a_3}^{b_3} (\dots (\int_{a_n}^{b_n} |f(t_1, t_2, t_3, \dots, t_n)|^m \\
& \times |f_{123\dots n}(t_1, t_2, t_3, \dots, t_n)| \, dt_n) \dots) dt_3) dt_2) dt_1 \\
(2.23) \leq \; & \frac{1}{m+1} \left[\frac{(b_1 - a_1)(b_2 - a_2)(b_3 - a_3) \dots (b_n - a_n)}{2^n} \right]^m \\
& \times \int_{a_1}^{b_1} (\int_{a_2}^{b_2} (\int_{a_3}^{b_3} (\dots (\int_{a_n}^{b_n} |f_{123\dots n}(t_1, t_2, t_3, \dots, t_n)|^{m+1} dt_n) \dots) dt_3) dt_2) dt_1.
\end{aligned}
$$

where $m \geq 0$ *is a constant.*

Proof. We consider the same cases as in Theorem 2.1
Case 1. Let $(t_1, t_2, t_3, \dots, t_n) \in \triangle_1$, and we let

$$
\begin{aligned}
(2.24) J_1 = \; & \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\dots (\\
& \int_{a_n}^{s_n} |f(t_1, t_2, t_3, \dots, t_n)|^m |f_{123\dots n}(t_1, t_2, t_3, \dots, t_n)| \, dt_n) \dots) dt_3) dt_2) dt_1.
\end{aligned}
$$

From 2.2, 2.3, 2.5 and 2.10 we have

$$
\begin{aligned}
J_1 \leq \; & \int_{a_1}^{s_1} h(t_1, s_2, s_3, \dots, s_n)^m \\
& \times (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\dots (\int_{a_{n-1}}^{s_{n-1}} (h_{123\dots(n-1)}(t_1, t_2, t_3, \dots, t_n)) dt_{(n-1)}) \dots) dt_3) dt_2) dt_1. \\
\leq \; & \int_{a_1}^{s_1} h(t_1, s_2, s_3, \dots, s_n)^m (\int_{a_2}^{s_2} (h_{12}(t_1, t_2, s_3, \dots, s_n) \, dt_2) dt_1 \\
\leq \; & \int_{a_1}^{s_1} h(t_1, s_2, s_3, \dots, s_n)^m \, h_1(t_1, s_2, s_3, \dots, s_n) \, dt_1 \\
= \; & \frac{1}{m+1} \left(\int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\dots (\int_{a_n}^{s_n} |f_{123\dots n}(t_1, t_2, t_3, \dots, t_n)| \, dt_n) \dots) dt_3) dt_2) dt_1 \right)^{m+1}.
\end{aligned}
$$

Now, applying Schwarz's inequality, we have

$$
\begin{aligned}
J_1 \leq \; & \frac{1}{m+1} [(s_1 - a_1)(s_2 - a_2)(s_3 - a_3) \dots (s_n - a_n)]^m \\
& \times \int_{a_1}^{s_1} (\int_{a_2}^{s_2} (\int_{a_3}^{s_3} (\dots (\int_{a_n}^{s_n} |f_{123\dots n}(t_1, t_2, t_3, \dots, t_n)|^{m+1} dt_n) \dots) dt_3) dt_2) dt_1.
\end{aligned}
$$

Proof of the remaining cases follows by same manner as those given in the proof of Theorem 2.1 so the result follows. ∎

Remark 2.2. (1) We note that for the case $m = 1$ the inequality (2.23) reduced to

$$\begin{aligned}
&\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\ldots(\int_{a_n}^{b_n}|f(t_1,t_2,t_3,\ldots,t_n)| \\
&\times|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n)\ldots)dt_3)dt_2)dt_1 \\
\leq\;& \frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)\ldots(b_n-a_n)}{2^{n+1}} \\
&\times\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\ldots(\int_{a_n}^{b_n}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|^2dt_n)\ldots)dt_3)dt_2)dt_1.
\end{aligned}$$

This is the same result of Theorem 2.1.

2. Put $n=1$ in (2.23), we find that

$$\int_a^b |f(t)|^m\,|f_1(t)|\,dt \leq \frac{1}{m+1}\left[\frac{(b-a)}{2}\right]^m\int_a^b |f_1(t)|^{m+1}dt.$$

Theorem 2.3. *If (H_1), (H_2) and (H_3) hold. Then*

$$\begin{aligned}
&\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\ldots(\int_{a_n}^{b_n}|f(t_1,t_2,t_3,\ldots,t_n)|\,|f_1(t_1,t_2,t_3,\ldots,t_n)| \\
&\ldots|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|\,dt_n)\ldots)dt_3)dt_2)dt_1 \\
\leq\;& \frac{1}{4}\left[\frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)\ldots(b_n-a_n)}{2^n}\right]^3 \\
&\times\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\ldots(\int_{a_5}^{b_5}|f_{123\ldots n}(t_1,t_2,t_3,\ldots,t_n)|^4dt_n)\ldots)dt_3)dt_2)dt_1 \\
&+\left[\frac{(b_2-a_2)(b_3-a_3)\ldots(b_n-a_n)}{2^{n-1}}\right] \\
&\times\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\int_{a_4}^{b_4}(\int_{a_5}^{b_5}|f_{123\ldots n}(S_1,t_2,t_3,\ldots,t_n)|^2dt_n)\ldots)dt_3)dt_2.
\end{aligned} \tag{2.25}$$

Proof. The proof of Theorem 2.3 is similar to the Proof of Theorem 2.1 so it is omitted. ∎

Theorem 2.4. *Let (H_1), (H_2) and (H_3) hold. Then*

$$
\begin{aligned}
&\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\dots(\int_{a_n}^{b_n}|f(t_1,t_2,t_3,\dots,t_n)|^{m_1}|f_1(t_1,t_2,t_3,\dots,t_n)|^{m_2}\\
&\times|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|\,dt_n)\dots)dt_3)dt_2)dt_1\\
\le\;&\frac{1}{2m_2+2}\left[\frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)\dots(b_n-a_n)}{2^n}\right]^{m_1+1}\\
&\times\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\dots\\
&\quad((\int_{a_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|^{2m_1+2}dt_n)\dots)dt_3)dt_2)dt_1 \qquad (2.26)\\
&+\left[\frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)\dots(b_n-a_n)}{2^{n-1}}\right]^{2m_2-1}\\
&\times\int_{a_2}^{b_2}(\int_{a_3}^{b_3}(\dots((\int_{a_n}^{b_n}|f_{123\dots n}(t_1,t_2,t_3,\dots,t_n)|^{2m_2}dt_n)\dots)dt_3)dt_2,
\end{aligned}
$$

where $m_1, m_2 \ge 0$ *are constants.*

Remark 2.3. (1) Put $n = 2$ in 2.25, we get

$$
\begin{aligned}
&\int_{a_1}^{b_1}(\int_{a_2}^{b_2}|f(t_1,t_2)|\,|f_1(t_1,t_2)|\,|f_{12}(t_1,t_2)|\,dt_2)dt_1)\\
\le\;&\frac{1}{4}\left[\frac{(b_1-a_1)(b_2-a_2)}{4}\right]^3\int_{a_1}^{b_1}(\int_{a_2}^{b_2}|f_{12}(t_1,t_2)|^4dt_2)dt_1\\
&+\left[\frac{(b_1-a_1)(b_2-a_2)}{2}\right]\int_{a_2}^{b_2}|f_{12}(S_1,t_2)|^2dt_2.
\end{aligned}
$$

Which is Yang's result [9].

(2) Let $n = 3$ in 2.25, we obtain

$$
\begin{aligned}
&\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}|f(t_1,t_2,t_3)|\,|f_1(t_1,t_2,t_3)|\,|f_{123}(t_1,t_2,t_3)|\,dt_3)dt_2)dt_1\\
\le\;&\frac{1}{4}\left[\frac{(b_1-a_1)(b_2-a_2)(b_3-a_3)}{8}\right]^3\int_{a_1}^{b_1}(\int_{a_2}^{b_2}(\int_{a_3}^{b_3}|f_{123}(t_1,t_2,t_3)|^4dt_3)dt_2)dt_1\\
&+\left[\frac{(b_2-a_2)(b_3-a_3)}{4}\right]\int_{a_2}^{b_2}(\int_{a_3}^{b_3}|f_{123}(S_1,t_2,t_3)|^2dt_3)dt_2.
\end{aligned}
$$

This is the result of Salem [8].

(3) Put $m_1 = 1$ and $m_2 = 1$ we obtain the inequality 2.25.

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