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SOLVING FRACTIONAL TRANSPORT EQUATION VIA WALSH FUNCTION

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ABSTRACT. A method for the solution of fractional transport equation in three-dimensional case by using Walsh function is presented. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem.

Key words and phrases: Linear transport equation, Isotropic scattering, Walsh function.

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1. INTRODUCTION

The spectral methods established an analytical formulation whose basic goals is to find exact solution for approximations of the fractional transport equation, this technique by using the fractional derivatives greatly simplifying the problem and making it computational plausible on the other hand that permit us to solve some of the particular cases and then we can check that the solution is close to the dynamics of some anomalous processes, the other aspect of this technique, it allows us to establish a fractional derivative which performs the same mapping of a given linear operator, it becomes to use the Riemann-Liouville definition for fractional derivatives and considerate the ordinary model and look that in the limit of some situations where the ordinary model do not work fine it is necessary to introduce such fractional operators in the model so we solve the problem.

In ordinary cases several approaches have been suggested among them, the method proposed by Chandrasekhar [7] solves analytically the discrete equations, (S_N equations), the SGF method [3, 4], is a numerical nodal method that generates numerical solution for the S_N equations in slab geometry that is completely free of spatial truncation error. The LTS_N method [28] solve analytically the S_N equations employing the Laplace Transform technique in the spatial variable (finite domain). Recently, following the idea encompassed by the LTS_N method, we have derived a generic method, prevailing the analyticity, for solving one-dimensional approximation that transform the transport equation into a set of differential equations.

In our recent work, we have presented a new approximation where the one dimensional fractional integro-differential equation is converted into a system of fractional differential equation (**FDE**), where we are using the Chebyshev polynomials [21].

There are three classes of set of orthogonal functions which are widely used. The first includes sets of piecewise constant basis functions (e.g. Walsh, block-pulse, etc.). The second consists of sets of orthogonal polynomials (e.g. Laguerre, Legendre, Chebyshev, etc.). The third is the widely used sets of sine-cosine functions in Fourier series.

Briefly speaking, in this paper the Walsh function is used for solving the three-dimensional case of fractional transport equation. This method is based on expansion of the angular flux in a truncated series of Walsh function in the angular variable. By replacing this development in the transport equation, this will result a first-order fractional linear differential system is solved for the spatial function the convergence of a solution defined for all the spatial variables obtained in the context of the discrete-ordinates approximation for the isotropic case.

The paper has been organized as follows. Section 2 contains a definition and some properties of Walsh function, in Section 3 we enlist some basic results and definitions of fractional derivatives, Section 4 describes how to convert a fractional transport equation into a first-order fractional linear differential equation system by using Chebyshev polynomials, and in Section 5 we report the convergence of the spectral solution, finally we give an specific application of this method in Section 6**.**

2. WALSH FUNCTION

The Walsh functions have many properties similar to those of the trigonometric functions. For example they form a complete, total collection of functions with respect to the space of square Lebesgue integrable functions. However, they are simpler in structure to the trigonometric functions because they take only the values 1 and -1. They may be expressed as linear combinations of the Haar functions [13], so many proofs about the Haar functions carry over to the Walsh system easily. Moreover, the Walsh functions are Haar wavelet packets; see [29]. For a good account of the properties of the Haar wavelets and other wavelets. We use the ordering of the Walsh functions due to Paley [15]. Any function $f \in L^2\,[0,1]$ can be expanded as a series of Walsh functions

$$f(x) = \sum_{i=0}^{\infty} c_i W_i(x) \quad \text{where } c_i = \int_0^1 f(x) W_i(x). \tag{2.1}$$

Fine [11] discovered an important property of the Walsh Fourier series: the $m = 2^n$ th partial sum of the Walsh series of a function f is piece-wise constant, equal to the L^1 mean of f, on each subinterval $[(i-1)/m, i/m]$. For this reason, Walsh series in applications are always truncated to $m = 2^n$ terms. In this case, the coefficients c_i of the Walsh (-Fourier) series are given by

$$c_i = \sum_{j=0}^{m-1} \frac{1}{m} W_{ij} f_j, \tag{2.2}$$

where f_j is the average value of the function $f(x)$ in the jth interval of width $1/m$ in the interval $(0,1)$, and W_{ij} is the value of the ith Walsh function in the jth subinterval. The order m Walsh matrix, $\mathcal{W}_m$, has elements W_{ij} .

Let $f(x)$ have a Walsh series with coefficients c_i and its integral from 0 to x have a Walsh series with coefficients b_i: $\int_0^x f(t)dt = \sum_{i=0}^{\infty} b_i W_i(x)$. If we truncate to $m = 2^n$ terms and use the obvious vector notation, then integration is performed by matrix multiplication $\mathbf{b} = P_m^T \mathbf{c}$ where

$$P_m^T = \begin{bmatrix} P_{m/2} & \frac{1}{2m} I_{m/2} \\ -\frac{1}{2m} I_{m/2} & O_{m/2} \end{bmatrix}, P_2^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix}, \tag{2.3}$$

and I_m is the unit matrix, O_m is the zero matrix (of order m), see [9].

Before to start to study the three-dimensional spectral solution of a fractional transport equation, we give some preliminaries.

3. PRELIMINARIES

We enlist some definitions and basic results [26, 22, 23].

Definition 3.1. A real function $f(x), x > 0$ is said to be in the space $C_{\alpha,\alpha\in R}$ if there exists a real number $p(> \alpha)$, such that $f(x) = x^p f_1(x)$ where $f_1(x) = C[0,\infty)$. Clearly $C_\alpha \subset C_\beta$ if $\beta \leq \alpha$.

Definition 3.2. A function $f(x), x > 0$ is said to be in space C_α^m, $m \in N \cup \{0\}$, if $f_{(m)} \in C_\alpha$.

Definition 3.3. The (left sided) Riemann-Liouville fractional integral of order $\mu > 0$, of a function $f \in C_\alpha, \alpha \geq 1$ is defined as:

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau,\ \mu > 0, t > 0, \tag{3.1}$$
$$I^0 f(t) = f(t),$$

Definition 3.4. The (left sided) Riemann-Liouville fractional derivative of f, $f \in C_{-1}^m$, $m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D^\mu f(t) = \frac{d^m}{dt^m} I^{m-\mu} f(t),\ m-1 < \mu \leq m,\ m \in N. \tag{3.2}$$

Definition 3.5. The (left sided) Caputo fractional derivative of f, $f \in C_{-1}^m$, $m \in N \cup \{0\}$ of order $\alpha > 0$, is defined as:

$$D_c^\mu f(t) = \begin{cases} \left[I^{m-\mu} f^{(m)}(t)\right] & , m-1 < \mu \leq m,\ m \in N, \\ \frac{d^m}{dt^m} f(t) & \mu = m. \end{cases} \tag{3.3}$$

Note that

(i) $I^{\mu}t^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\mu+1)}t^{\gamma+\mu}, \qquad \mu>0, \gamma>-1, t>0.$
(ii) $I^{\mu}D_c^{\mu}f(t)=f(t)-\sum_{k=0}^{m-1}f^{(k)}(0_+)\frac{t^k}{k!}, m-1<\mu\leq m, m\in N.$
(iii) $D_c^{\mu}f(t)=D^{\mu}\left(f(t)-\sum_{k=0}^{m-1}f^{(k)}(0_+)\frac{t^k}{k!}\right), m-1<\mu\leq m, m\in N.$
(iv) $D^{\beta}I^{\alpha}f(t)=\begin{cases} I^{\alpha-\beta}f(t) \text{ if } \alpha>\beta, \\ f(t) \qquad \text{ if } \alpha=\beta, \\ D^{\beta-\alpha}f(t) \text{ if } \alpha<\beta, \end{cases}$
(v) $D_c^{\alpha}D^m f(t)=D^{\alpha+m}f(t), m=0,1,2,...,n-1<\alpha<n.$

4. THE THREE-DIMENSIONAL SPECTRAL SOLUTION

Consider the three-dimensional linear, steady state, transport equation given by

$$\mu\frac{\partial^{\gamma}}{\partial x^{\gamma}}\Psi(\mathbf{x},\mu,\theta)+\sqrt{1-\mu^2}\left(\cos\theta\frac{\partial^{\gamma}}{\partial y^{\gamma}}\Psi(\mathbf{x},\mu,\theta)+\sin\theta\frac{\partial^{\gamma}}{\partial z^{\gamma}}\Psi(\mathbf{x},\mu,\theta)\right)$$

$$+\sigma_t\Psi(\mathbf{x},\mu,\theta)=\int_{-1}^{1}\int_0^{2\pi}\sigma_s(\mu',\theta'\to\mu,\theta)\Psi(\mathbf{x},\mu',\theta')d\theta' d\mu'+S(\mathbf{x},\mu,\theta) \tag{4.1}$$

where we assume that the spatial variable $\mathbf{x}:=(x,y,z)$ varies in the cubic domain $\Omega := \{(x,y,z): -1\leq x,y,z\leq 1\}$, $0<\gamma\leq 1$ and $\Psi(\mathbf{x},\mu,\theta):=\Psi(x,y,z,\mu,\theta)$ is the angular flux in the direction defined by $\mu\in[-1,1]$ and $\theta\in[0,2\pi]$. σ_t and σ_s denote the total and the differential cross section, respectively, $\sigma_s(\mu',\phi'\to\mu,\phi)$ describes the scattering from an assumed pre-collision angular coordinates (μ',θ') to a post-collision coordinates (μ,θ) and S is the source term. See [14] for further details.

Note that, in the case of one-speed neutron transport equation; taking the angular variable in a disc, this problem would corresponds to a three dimensional case with all functions being constant in the azimuthal direction of the z variable. In this way the actual spatial domain may be assumed to be a cylinder with the cross-section Ω and the axial symmetry in z. Then D will correspond to the projection of the points on the unit sphere (the "speed") onto the unit disc (which coincides with D.).

Given the functions $f_1(y,z,\mu,\phi)$, $f_2(x,z,\mu,\phi)$ and $f_3(x,y,\mu,\phi)$ describing the incident flux, we seek for a solution of (4.1) subject to the following boundary conditions:

For the boundary terms in x; for $0\leq\theta\leq 2\pi$, let

$$\Psi(x=\pm1,y,z,\mu,\theta)=\begin{cases} f_1(y,z,\mu,\theta), \ x=-1, & 0<\mu\leq 1, \\ 0, \ x=1, & -1\leq\mu<0. \end{cases} \tag{4.2}$$

For the boundary terms in y and for $-1\leq\mu<1$,

$$\Psi(x,y=\pm1,z,\mu,\theta)=\begin{cases} f_2(x,z,\mu,\theta), \ y=-1, & 0<\cos\theta\leq 1, \\ 0, \ y=1, & -1\leq\cos\theta<0. \end{cases} \tag{4.3}$$

Finally, for the boundary terms in z; for $-1\leq\mu<1$,

$$\Psi(x,y,z=\pm1,\mu,\theta)=\begin{cases} f_3(x,y,\mu,\theta), \ z=-1, & 0\leq\theta<\pi, \\ 0, \ z=1, & \pi<\theta\leq 2\pi. \end{cases} \tag{4.4}$$

Theorem 4.1. *Consider the fractional integro-differential equation (4.1) under the boundary conditions (4.2), (4.3) and (4.4), then the function $\Psi(x, y, z, \mu, \theta)$ satisfy the following first-order fractional linear differential equation system for the spatial component $\Psi_{i,j}(x, \mu, \theta)$*

$$\mu\frac{\partial^\gamma \Psi_{i,j}}{\partial x^\gamma}(x,\mu,\theta) + \sigma_t \Psi_{i,j}(x,\mu,\theta) = G_{i,j}(x;\mu,\theta)$$
$$\int_{-1}^{1}\int_{-1}^{1} \sigma_s(\mu',\theta' \to \mu,\theta)\Psi_{i,j}(x,\mu',\theta')d\theta' d\mu'$$

with the boundary conditions

$$\Psi_{i,j}(-1,\mu,\eta) = f_1^{i,j}(\mu,\theta),$$

where

$$f_1^{i,j}(\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}} f_1(y,z,\mu,\theta)dzdy,$$
$$\Psi_{i,j}(1,-\mu,\theta) = 0,$$

and

$$G_{i,j}(x;\mu,\theta) = S_{i,j}(x,\mu,\theta) -$$
$$\sqrt{1-\mu^2}\left[\cos\theta \sum_{k=i+1}^{I} A_i^k \Psi_{k,j}(x,\mu,\theta) + \sin\theta \sum_{l=j+1}^{J} B_j^l \Psi_{i,l}(x,\mu,\theta)\right],$$

with

$$S_{i,j}(x,\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1} \frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}} S(\mathbf{x},\mu,\theta)dzdy,$$
$$A_i^k = \frac{2}{\pi}\int_{-1}^{1} \frac{d}{dy}(T_k(y))\frac{T_i(y)}{\sqrt{1-y^2}}dy$$
$$B_j^l = \frac{2}{\pi}\int_{-1}^{1} \frac{d}{dy}(R_l(y))\frac{R_j(z)}{\sqrt{1-z^2}}dz.$$

Proof. Expanding the angular flux $\Psi(x, y, z, \mu, \phi)$ in a truncated series of Chebyshev polynomials $T_i(y)$and $R_j(z)$ leads to

$$\Psi(x,y,z,\mu,\theta) = \sum_{i=0}^{I}\sum_{j=0}^{J} \Psi_{i,j}(x,\mu,\theta)T_i(y)R_j(z). \tag{4.5}$$

We insert $\Psi(x, y, z, \mu, \theta)$ given by (4.5) into the boundary condition in (4.3), for $y = \pm 1$. Multiplying the resulting expressions by $\frac{R_j(z)}{\sqrt{1-z^2}}$ and integrating over z, we get the components $\Psi_{0,j}(x, \mu, \theta)$ for $j = 0, ...J$:

$$\Psi_{0,j}(x,\mu,\theta) = f_2^j(x,\mu,\theta) - \sum_{i=1}^{I}(-1)^j \Psi_{i,j}(x,\mu,\theta); \quad 0 < \cos\theta \le 1, \tag{4.6}$$

and

$$\Psi_{0,j}(x,\mu,\theta) = -\sum_{i=1}^{I} \Psi_{i,j}(x,\mu,\theta); \quad -1 \le \cos\theta < 0. \tag{4.7}$$

Similarly, we substitute $\Psi(x, y, z, \mu, \theta)$ from (4.5) into the boundary conditions for $z = \pm 1$, multiply the resulting expression by $\frac{T_i(y)}{\sqrt{1-y^2}}$, $i = 0, ...I$ and integrating over y, to define the components $\Psi_{i,0}(x, \mu, \theta)$: For $-1 \le x \le 1$, $-1 < \mu < 1$,

$$\Psi_{i,0}(x,\mu,\theta) = f_3^i(x,\mu,\theta) - \sum_{j=1}^{J}(-1)^j \Psi_{i,j}(x,\mu,\theta); \quad 0 \le \theta < \pi,$$

$$\Psi_{i,0}(x,\mu,\theta) = -\sum_{j=1}^{J} \Psi_{i,j}(x,\mu,\theta); \quad \pi < \theta \le 2\pi,$$

where

$$f_2^{\beta}(x,\mu,\theta) = \frac{2-\delta_{0,j}}{\pi}\int_{-1}^{1} f_2(x,z,\mu,\theta)\frac{R_j(z)}{\sqrt{1-z^2}}dz$$

$$f_3^{i}(x,\mu,\theta) = \frac{2-\delta_{i,0}}{\pi}\int_{-1}^{1} f_3(x,y,\mu,\theta)\frac{T_i(y)}{\sqrt{1-y^2}}dy.$$

To determine the components $\Psi_{i,j}(x,\mu,\theta)$, $i = 1,...I$, and $j = 1,...J$, we substitute $\Psi(x,\mu,\theta)$, from (4.3) into (4.1) and the boundary conditions for $x = \pm 1$. Multiplying the resulting expressions by $\frac{T_i(y)}{\sqrt{1-y^2}} \times \frac{R_j(z)}{\sqrt{1-z^2}}$, and integrating over y and z we obtain $I \times J$ one-dimensional transport problems, viz

$$\mu\frac{\partial^{\gamma}\Psi_{i,j}}{\partial x^{\gamma}}(x,\mu,\theta) + \sigma_t \Psi_{i,j}(x,\mu,\theta) = G_{i,j}(x;\mu,\theta)$$

$$\int_{-1}^{1}\int_{-1}^{1} \sigma_s(\mu^{'},\theta^{'} \to \mu,\theta)\Psi_{i,j}(x,\mu^{'},\theta^{'})d\theta^{'}d\mu^{'} \tag{4.8}$$

with the boundary conditions

$$\Psi_{i,j}(-1,\mu,\eta) = f_1^{i,j}(\mu,\theta), \tag{4.9}$$

where

$$f_1^{i,j}(\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}}f_1(y,z,\mu,\theta)dzdy, \tag{4.10}$$

and

$$\Psi_{i,j}(1,-\mu,\theta) = 0,$$

for $0 < \mu \le 1$, and $0 \le \theta \le 2\pi$. Finally

$$G_{i,j}(x;\mu,\theta) = S_{i,j}(x,\mu,\theta) -$$

$$\sqrt{1-\mu^2}\left[\cos\theta \sum_{k=i+1}^{I} A_i^k \Psi_{k,j}(x,\mu,\theta) + \sin\theta \sum_{l=j+1}^{J} B_j^l \Psi_{i,l}(x,\mu,\theta)\right], \tag{4.11}$$

with

$$S_{i,j}(x,\mu,\theta) = \frac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{T_i(y)R_j(z)}{\sqrt{(1-y^2)(1-z^2)}}S(\mathbf{x},\mu,\theta)dzdy, \tag{4.12}$$

$$A_i^k = \frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(T_k(y))\frac{T_i(y)}{\sqrt{1-y^2}}dy \tag{4.13}$$

$$B_j^l = \frac{2}{\pi}\int_{-1}^{1}\frac{d}{dy}(R_l(y))\frac{R_j(z)}{\sqrt{1-z^2}}dz. \tag{4.14}$$

Now, starting from the solution of the problem given by equations (4.8)-(4.14) for $\Psi_{I,J}(x,\mu,\theta)$, we then solve the problems for the other components, in the decreasing order in i and j. Recall that $\sum_{i=I+1}^{I} ... = \sum_{j=J+1}^{J} \equiv 0$. Hence, solving $I \times J$ one-dimensional problems, the angular flux $\Psi(\mathbf{x}, \mu, \theta)$ is now completely determined through (4.5). ∎

Remark 4.1. If we have to deal with different type of boundary conditions, we have to keep in mind that the first components $\Psi_{i,0}(x, \mu, \theta)$ and $\Psi_{0,j}(x, \mu, \theta)$ for $i = 1, ..., I$ and $j = 1, ..., J$ will satisfy one-dimensional transport problems subject to the same of boundary conditions of the original problem in the variable x.

5. ANALYSIS

Now we would like to solve the first fractional order linear differential equation system with isotropic scattering, i.e., $\sigma_s(\mu', \phi' \rightarrow \mu, \phi) \equiv \sigma_s =$ constant by using the Walch function. Assuming isotropic scattering, the equation (4.8) is written as

$$\mu \frac{\partial^\gamma \Psi_{i,j}}{\partial x^\gamma}(x, \mu, \theta) + \sigma_t \Psi_{i,j}(x, \mu, \theta) = G_{i,j}(x; \mu, \theta)$$

$$\sigma \int_{-1}^{1} \int_{0}^{2\pi} \Psi_{i,j}(x, \mu', \theta') d\theta' d\mu' \tag{5.1}$$

for $\mathbf{x} \in \Omega := \{(x, y){:}0 \le x \le 1, -1 \le y \le 1\}, 0 < \gamma \le 1, \mu \in [-1, 1]$ and $\theta \in [0, 2\pi]$.

Subject to the following boundary conditions (4.9).

Theorem 5.1. *Consider the fractional integro-differential equation (5.1) under the boundary conditions (4.9), then the function $\Psi_{i,j}(x, \mu, \theta)$ satisfy the following fractional linear system of algebraic equations*

$$\sum_{n=0}^{N} D_{n,m} \overline{Z}_{n,i,j}^{(\gamma)}(x, \theta) - \sigma_s \sum_{n=0}^{N} \overline{Y}_{n,i,j}(x, \theta) + \sigma_t \overline{Y}_{n,i,j}(x, \theta) =$$

$$\int_{-1}^{1} \overline{G}_{i,j}(x, \mu, \theta) W_n^e(\mu) d\mu + \sum_{n=0}^{N} D_{n,m} \overline{Z}_{n,i,j}(x, \theta).$$

$$\sum_{n=0}^{N} D_{n,m} \overline{Y}_{n,i,j}^{(\gamma)}(x, \theta) - \sigma_s \sum_{n=0}^{N} \overline{Z}_{n,k}(x, \theta) + \sigma_t \overline{Z}_{n,i,j}(x, \theta) =$$

$$\int_{-1}^{1} \overline{G}_{i,j}(x, \mu, \theta) W_n^o(\mu) d\mu + \sum_{n=0}^{N} D_{n,m} \overline{Y}_{n,i,j}(x, \theta).$$

Proof. For this problem we expand the angular flux in terms of the Walsh function in the angular variable with its domain extended into the interval $[-1, 1]$. To this end, the Walsh function $W_n(\mu)$ are extended in an even and odd fashion as follows [6]:

$$W_n^e(\mu) = \begin{cases} W_n(\mu), & \text{if } \mu \ge 0 \\ W_n(-\mu), & \text{if } \mu < 0 \end{cases}, \tag{5.2}$$

$$W_n^o(\mu) = \begin{cases} W_n(\mu), & \text{if } \mu \ge 0 \\ -W_n(-\mu), & \text{if } \mu < 0 \end{cases}, \tag{5.3}$$

for $n = 0, 1, ..., N$. The important feature of this procedure relies on the fact that a function $f(\mu)$ defined in the interval $[-1, 1]$ can be expanded in terms of these extended functions in the manner:

$$f(\mu) = \sum_{n=0}^{\infty} [a_n W_n^e(\mu) + b_n W_n^o(\mu)], \tag{5.4}$$

where the coefficients a_n and b_n are determined as:

$$a_n = \frac{1}{2}\int_{-1}^{1} f(\mu)W_n^e(\mu)d\mu, \tag{5.5}$$

$$b_n = \frac{1}{2}\int_{-1}^{1} f(\mu)W_n^o(\mu)d\mu, \tag{5.6}$$

So, in order to use the Walsh function for the solution of the problem (5.1), the angular flux is approximated by the truncated expansion:

$$\Psi_{i,j}(x,\mu,\theta) = \sum_{n=0}^{N}\left[Y_{n,i,j}(x,\theta)W_n^e(\mu) + Z_{n,i,j}(x,\theta)W_n^o(\mu)\right] \tag{5.7}$$

Inserting this expansion into the linear transport equation (5.1), it turns out:

$$\begin{gathered}\sum_{n=0}^{N}\left[\left\{\mu\frac{\partial^\gamma Y_{n,i,j}}{\partial x^\gamma}(x,\theta) + \sigma_t Y_{n,i,j}(x,\theta)\right\}W_n^e(\mu) + \right.\\ \left.\left\{\mu\frac{\partial^\gamma Z_{n,i,j}}{\partial x^\gamma}(x,\theta) + \sigma_t Z_{n,i,j}(x,\theta)\right\}W_n^o(\mu)\right] = \\ \sum_{n=0}^{N}\sigma_s\left[\int_{-1}^{1}\int_{0}^{2\pi} Y_{n,i,j}(x,\theta')W_n^e(\mu')d\theta' d\mu' + \right.\\ \left.\int_{-1}^{1}\int_{0}^{2\pi} Y_{n,i,j}(x,\theta')W_n^o(\mu')d\theta' d\mu'\right] + G_{i,j}(x,\mu,\theta)\end{gathered} \tag{5.8}$$

Multiplying equation (5.8) by W_m^e , $m = 0.,...,N$ and integrating over the interval $[-1,1]$, results:

$$\begin{gathered}\sum_{n=0}^{N}\left[\frac{\partial^\gamma Z_{n,i,j}}{\partial x^\gamma}(x,\theta)\int_{-1}^{1}\mu W_n^o(\mu)W_n^e(\mu)d\mu\right.\\ \left.+\sigma_t Y_{n,i,j}(x,\theta)\int_{-1}^{1}W_n^e(\mu)W_m^e(\mu)d\mu\right] = \\ \sum_{n=0}^{N}\sigma_s\left[\int_{0}^{2\pi} Y_{n,i,j}(x,\theta')d\theta'\int_{-1}^{1}W_n^o(\mu')W_n^o(\mu')d\mu'\right] + \\ \int_{-1}^{1}G_{i,j}(x,\mu,\theta)W_n^e(\mu)d\mu\end{gathered} \tag{5.9}$$

Similarly, multiplying equation (5.8) by W_m^0 , $m = 0.,...,N$ and integrating yields:

$$\begin{gathered}\sum_{n=0}^{N}\left[\frac{\partial^\gamma Y_{n,i,j}}{\partial x^\gamma}(x,\theta)\int_{-1}^{1}\mu W_n^o(\mu)W_n^e(\mu)d\mu\right.\\ \left.+\sigma_t Z_{n,i,j}(x,\theta)\int_{-1}^{1}W_n^0(\mu)W_m^0(\mu)d\mu\right] = \\ \sum_{n=0}^{N}\sigma_s\left[\int_{0}^{2\pi} Z_{n,i,j}(x,\theta')d\theta'\int_{-1}^{1}W_n^o(\mu')W_n^o(\mu')d\mu'\right] +\end{gathered} \tag{5.10}$$

$$\int_{-1}^{1} G_{i,j}(x,\mu,\theta)W_n^0(\mu)d\mu$$

The integrals appearing in equations (5.9) and (5.10) are known and are given [6] as

$$D_{n,m} = \frac{1}{2}\int_{-1}^{1} \mu W_n^o(\mu)W_m^e(\mu)d\mu = \int_0^1 \mu W_{(n+m)\bmod 2}(\mu) \tag{5.11}$$

or

$$D_{n,m} = \begin{cases} 1/2 & \text{if } n = m \\ -2^{-(k+2)} & \text{, if } (n+m) \bmod 2 = 2^k,\ k \text{ natural} \\ 0 & \text{at another case} \end{cases} \tag{5.12}$$

where the notation $(n+m) \bmod 2$ denotes the $\bmod 2$ sum of the binary digits n and m [8] we obtain an algebraic linear system of equations

$$\sum_{n=0}^{N} D_{n,m}\overline{Z}_{n,i,j}^{(\gamma)}(x,\theta) + (\sigma_t - \sigma_s)\sum_{n=0}^{N}\overline{Y}_{n,i,j}(x,\theta) =$$

$$\int_{-1}^{1}\overline{G}_{i,j}(x,\mu,\theta)W_n^e(\mu)d\mu + \sum_{n=0}^{N} D_{n,m}\overline{Z}_{n,i,j}(x,\theta) \tag{5.13}$$

$$\sum_{n=0}^{N} D_{n,m}\overline{Y}_{n,i,j}^{(\gamma)}(x,\theta) + (\sigma_t - \sigma_s)\sum_{n=0}^{N}\overline{Z}_{n,i,j}(x,\theta) =$$

$$\int_{-1}^{1}\overline{G}_{i,j}(x,\mu,\theta)W_n^o(\mu)d\mu + \sum_{n=0}^{N} D_{n,m}\overline{Y}_{n,i,j}(x,\theta) \tag{5.14}$$

∎

6. SPECIFIC APPLICATION OF THE METHOD

Consider now the discrete ordinates (S_N) approximation of the equation (4.8) for $m = 1, ..., M$

$$\mu_m \frac{\partial^\gamma \Psi_{\alpha,\beta}}{\partial x^\gamma}(x,\mu_m,\phi_m) + \sigma_t\Psi_{\alpha,\beta}(x,\mu_m,\phi_m) = \sum_{n=1}^{M}\omega_n\Psi_{\alpha,\beta}(x,\mu_m,\phi_m) + G_{\alpha,\beta}(x,\mu_m,\phi_m) \tag{6.1}$$

where $0 < \gamma \leq 1$.

Theorem 6.1. *Consider the integro-differential equation (6.1), then the function $\Psi_{\alpha,\beta}(x,\mu,\theta)$ satisfy the following differential equation*

$$\mathcal{A}_k\frac{\partial^\gamma W_k}{\partial x^\gamma} + \mathcal{D}_k W_k = \mathcal{B}$$

Proof. expand $\Psi_{\alpha,\beta}(x,\mu_m,\phi_m)$ in a truncated series of Walsh functions i.e.

$$\Psi_{\alpha,\beta}(x,\mu_m,\phi_m) = \sum_{k=0}^{K} C_k(\mu_m,\phi_m)W_k(x) \tag{6.2}$$

where

$$C_k(\mu_m,\phi_m) = \int_0^1 \Psi_{\alpha,\beta}(x,\mu_m,\phi_m)W_k(x)$$

Inserting the equation (6.2) in equation (6.1) to get

$$\mu_m \frac{\partial^\gamma}{\partial x^\gamma} \sum_{k=0}^{K} C_k(\mu_m, \phi_m) W_k(x) + \sigma_t \sum_{k=0}^{K} C_k(\mu_m, \phi_m) W_k(x)$$

$$= \sum_{n=1}^{M} \omega_n \sum_{k=0}^{K} C_k(\mu_m, \phi_m) W_k(x) + G_{\alpha,\beta}(x, \mu_m, \phi_m) \tag{6.3}$$

with

$$G_{\alpha,\beta}(x, \mu_m, \phi_m) = S_{\alpha,\beta}(x, \mu_m, \phi_m) - \sqrt{1-\mu^2}$$

$$\times \left[\cos\phi \sum_{\alpha=i+1}^{I} A_i^\alpha \sum_{k=0}^{K} C_k(\mu_m, \phi_m) W_k(x) + \right. \tag{6.4}$$

$$\left. \sin\phi \sum_{\beta=j+1}^{J} B_j^\beta \sum_{k=0}^{K} C_k(\mu_m, \phi_m) W_k(x) \right]$$

then the matrix form of the equation (6.3) yields the following differential equation

$$\mathcal{A}_k \frac{\partial^\gamma W_k}{\partial x^\gamma} + \mathcal{D}_k W_k = \mathcal{B} \tag{6.5}$$

where

$$\mathcal{A}_k = \mu_m \sum_{k=0}^{K} C_k(\mu_m, \phi_m) \tag{6.6}$$

$$\mathcal{D}_k = \sum_{K=0}^{M} C_k(\mu_m, \phi_m) \left[\sigma_t - \sum_{n=1}^{M} \omega_n + \sqrt{1-\mu^2} \left[\cos\phi \sum_{\alpha=i+1}^{I} A_i^\alpha + \sin\phi \sum_{\beta=j+1}^{J} B_j^\beta \right] \right] \tag{6.7}$$

and

$$\mathcal{B} = S_{\alpha,\beta}(x, \mu_m, \phi_m) \tag{6.8}$$

the solution of differential equation for the vector W_k is thus constructed as follows [25]

$$W_k(x) = e_\gamma^{-\mathcal{A}^{-1}\mathcal{D}x} W_k(0) - \int_0^x e_\gamma^{-\mathcal{A}^{-1}\mathcal{D}(x-\xi)} \mathcal{A}^{-1} \mathcal{B}(\xi) d\xi \tag{6.9}$$

equation (6.9) depend on vector $W_k(0)$. Having established an analytical formulation for the exponential appearing in equation (6.9), the unknown components of vector $W_k(0)$ for the boundary problem (4.1) can be readily obtained applying the boundary conditions (4.2), (4.3) and (4.4).

An analytical formulation for the exponential of matrix D, appearing in equation (6.9), is given by [25]

$$e_\gamma^{-\mathcal{A}^{-1}\mathcal{D}x} = x^{\beta-1} \sum_{k=0}^{\infty} (\mathcal{A}^{-1}\mathcal{D})^k \frac{x^{k\gamma}}{\Gamma\left[(k+1)\gamma\right]}$$

Moreover, it is easy to see that the function $e_\gamma^{-\mathcal{A}^{-1}\mathcal{D}}$ satisfies the following properties [25]:

(i) If $\| \mathcal{A}^{-1}\mathcal{D} \| = \max_{i,j} \mid \mathcal{A}_{i,j}\mathcal{D}_{i,j} \mid$, where $\mathcal{A}_{i,j}$ and $\mathcal{D}_{i,j}$ are the components of matrix $\mathcal{A}$ and $\mathcal{D}$ respectively then

$\| e_\gamma^{\mathcal{A}^{-1}\mathcal{D}} \| \leq \sum_{k=0}^{\infty} \| \mathcal{A}^{-1}\mathcal{D} \|^k \frac{x^{(k+1)\gamma-1}}{\Gamma[(k+1)\gamma]}$ $(x > 0)$,

(ii) $e_\gamma^{\mathcal{A}^{-1}\mathcal{D}} e_\gamma^{Kx} \neq e_\gamma^{(\mathcal{A}^{-1}\mathcal{D}+K)x}$ $(\gamma \neq 1)$,

(iii) $D^{\gamma}e_{\gamma}^{\mathcal{A}^{-1}\mathcal{D}x} = (\mathcal{A}^{-1}\mathcal{D})e_{\gamma}^{\mathcal{A}^{-1}\mathcal{D}x}$,
where $\mathcal{A}, \mathcal{D}, K \in M_n(R)$ and $\gamma \in (0, 1]$. ∎

7. CONCLUSION

We have discussed a Walsh function for solving the fractional transport equation in three-dimensional case this method represent very interesting new ideas for studying the convergence of many numerical methods and can be extended easily to general linear transport problems should be general enough to consider higher spatial dimensions in a way similar to that presented in this paper, although we have not investigated this idea thoroughly. We will be considering more complicated geometries in future studies, during which we will ascertain this method's usefulness for larger spatial dimensional problems. In this context we expect to determine the unknown order of the fractional derivative comparing the kernel of the integral equation with the one of the Riemann-Liouville definition of fractional derivative. Our attention is focus in this direction.

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