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ON THE INEQUALITY $a^{2a} + b^{2b} + c^{2c} \geq a^{2b} + b^{2c} + c^{2a}$

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ABSTRACT. In this paper we give a complete proof of $a^{2a} + b^{2b} + c^{2c} \geq a^{2b} + b^{2c} + c^{2a}$ for all positive real numbers a, b and c. Furthermore, we present another way to prove the statement for $c = 1/2 < b < a < 1$.

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1. INTRODUCTION

In [1, 2] several results about inequalities for power-exponential functions were introduced and proved. In particular, in [1], the following inequality

$$a^{2a} + b^{2b} + c^{2c} \geq a^{2b} + b^{2c} + c^{2a}, \tag{1.1}$$

for $a, b, c \in \mathbb{R}^+$ was presented as a conjecture. Thus, the goal of the present paper is to give a proof of (1.1). More precisely, we analyze (1.1) and prove the following theorem:

Theorem 1.1. *The inequality* (1.1) *holds for all positive real numbers* a, b *and* c.

The proof of Theorem 1.1 is self-contained. However, for completeness of the discussion given below, we recall a result of [1], which play an important role in the elaboration elaboration of the arguments to prove Theorem 1.1.

Theorem 1.2. *The inequality* $a^{2a} + b^{2b} \geq a^{2b} + b^{2a}$, *holds for all positive real numbers* a *and* b.

The paper is organized as follows. In section 2 we introduce the notation and a preliminary result. In section 3 we present a complete proof of Theorem 1.1. In section 4 we prove the Theorem 1.1 for $c = 1/2 < b < a < 1$, in a different way.

2. PRELIMINARIES AND NOTATION

Let us begin by introducing and proving a useful proposition.

Proposition 2.1. *Consider* $s \in \mathbb{R}^+$ *with* $s \neq 1$ *and* $f, g : \mathbb{R}^+ \to \mathbb{R}$ *defined as follows*

$$f(x) = x^s - x - y^s + y, \quad \textit{and} \quad g(x) = \begin{cases} e^{-\ln(x)/(x-1)}, & x \notin \{0,1\}, \\ e^{-1}, & x = 1. \\ 0, & x = 0. \end{cases}$$

Then, the following properties are satisfied

(i) $f(y) = 0$ *and* $f(0) = f(1) = -y^s + y$.
(ii) *If* $s > 1$, f *is strictly increasing on* $]g(s), \infty[$ *and strictly decreasing on* $]0, g(s)[$.
(iii) *If* $s \in]0, 1[$, f *is strictly decreasing on* $]g(s), \infty[$ *and strictly increasing on* $]0, g(s)[$.
(iv) g *is continuous on* $\mathbb{R}^+ \cup \{0\}$ *and strictly increasing on* $\mathbb{R}^+$. *Furthermore* $y = 1$ *is a horizontal asymptote of* $y = g(x)$.

Proof. (i). We use the definition of f. Then $f(y) = y^s - y - y^s + y = 0$ and $f(0) = f(1) = -y^s + y$.

(ii)-(iii). Differentiating f, we have that $f'(x) = sx^{s-1} - 1$ and we deduce the enunciated property, since $f'(x) = 0$ for $x = e^{-\ln(s)/(s-1)} = g(s)$ and

$$f'(x) \begin{cases} \text{is positive on }]g(s), \infty[, \text{ for } s \in]1, \infty[; \\ \text{is negative on }]0, g(s)[, \text{ for } s \in]1, \infty[; \\ \text{is negative on }]g(s), \infty[, \text{ for } s \in]0, 1[\text{ and} \\ \text{is positive on }]0, g(s)[, \text{ for } s \in]0, 1[. \end{cases}$$

(iv). We obtain the continuity of g by construction and the monotonic behavior as a consequence of

$$g'(x) = \frac{-x + 1 + x\ln(x)}{x(x-1)^2} g(x) > 0, \text{ for } x \neq 1.$$

We notice that $g(x) \to 1$ when $x \to +\infty$. Thus $y = 1$ is a horizontal asymptote of $y = g(x)$. ∎

If a, b, c are three positive real numbers, we have the following six cases

(a1) $a>\max\{b,c\}>0$, (a3) $c>\max\{a,b\}>0$, (a5) $a=b\neq c$,
(a2) $b>\max\{a,c\}>0$, (a4) $a=b=c$, (a6) $a\neq b=c$.

We can separate each of this three first alternatives into two cases. For instance, the first alternative is equivalent to $a\geq 1$ and $a>\max\{b,c\}>0$ or $1>a\geq\max\{b,c\}>0$. Considering this observation and for clarity in the proof of Theorem 1.1, we introduce the following set notation

$$\begin{aligned}
\mathbb{R}^3_+ &= \{(x,y,z)\in\mathbb{R}^3 \quad / \quad x>0, \quad y>0 \quad \text{and} \quad z>0\}\\
\mathbb{E}_1 &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad x=y=z \text{ or } x=y\neq z \text{ or } x\neq y=z\Big\},\\
\mathbb{E}^+_a &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad a\geq 1 \text{ and } a>\max\{b,c\}\Big\},\\
\mathbb{E}^-_a &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad 1>a>\max\{b,c\}\Big\},\\
\mathbb{E}^+_b &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad b\geq 1 \text{ and } b>\max\{a,c\}\Big\},\\
\mathbb{E}^-_b &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad 1>b>\max\{a,c\}\Big\},\\
\mathbb{E}^+_c &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad c\geq 1 \text{ and } c>\max\{a,b\}\Big\} \quad \text{and}\\
\mathbb{E}^-_c &= \Big\{(x,y,z)\in\mathbb{R}^3_+ \quad / \quad 1>c>\max\{a,b\}\Big\}.
\end{aligned}$$

We note that the family $\Big\{\mathbb{E}_1,\mathbb{E}^+_a,\mathbb{E}^-_a,\mathbb{E}^+_b,\mathbb{E}^-_b,\mathbb{E}^+_c,\mathbb{E}^-_c\Big\}$ is a set partition of $\mathbb{R}^3_+$.

3. Proof of Theorem 1.1

We present the proof of Theorem 1.1 by analyzing three cases.

3.1. Case $(a,b,c)\in\mathbb{E}_1$. This special case is a direct consequence of Theorem 1.2.

3.2. Case $(a,b,c)\in\mathbb{E}^+_a\cup\mathbb{E}^+_b\cup\mathbb{E}^+_c$. If $(a,b,c)\in\mathbb{E}^+_a$, we apply the Theorem 1.2 and proposition 2.1 as follows. We select $x=a^{2b}, y=c^{2b}$ and $s=a/b$, the monotonic behavior and properties of function f, defined on proposition 2.1, implies that

$$a^{2a}+c^{2b}>a^{2b}+c^{2a}, \tag{3.1}$$

since $x>y$, $x>1$ and $s>1$. The corresponding proof of (3.1) needs the distinction of two cases: $y>1$ and $y<1$. If $y>1$, then $y\in\,]g(s),\infty[$, so f is strictly increasing and $x>y$ implies (3.1). For $y<1$, we note that $-y^s+y\geq 0$ since $s>1$ and $1\in]g(s),\infty[$, then the assumption $x>1$ implies that $f(x)>f(1)=-y^s+y\geq 0=f(y)$ and (3.1) is again true for this subcase. Moreover, by Theorem 1.2, we recall that the inequality

$$c^{2c}+b^{2b}>b^{2c}+c^{2b}, \tag{3.2}$$

holds, for all positive real numbers b and c. Adding (3.1) and (3.2) we deduce (1.1).

The proof for $(a,b,c)\in\mathbb{E}^+_b\cup\mathbb{E}^+_c$ is similar to the case $(a,b,c)\in\mathbb{E}^+_a$ and we omit the details. For $(a,b,c)\in\mathbb{E}^+_b$ we choose $x=b^{2c}, y=c^{2c}$ and $s=b/c$. If $(a,b,c)\in\mathbb{E}^+_c$ we select $x=c^{2a}, y=b^{2a}$ and $s=c/a$.

3.3. **Case** $(a,b,c) \in \mathbb{E}_a^- \cup \mathbb{E}_b^- \cup \mathbb{E}_c^-$**.** Let us assume that $(a,b,c) \in \mathbb{E}_a^-$. We distinguish two subcases: $c < b < a$ and $b < c < a$. In first time, if $c < b < a$, we apply the function f (see proposition 2.1) with $x = b^{2c}$, $y = c^{2c}$ and $s = a/c$ for $c < 1/2$ to prove

$$b^{2a} + c^{2c} > b^{2c} + c^{2a} \tag{3.3}$$

and with $x = c^{2a}$, $y = a^{2a}$ and $s = b/a$ for $c \geq 1/2$ to deduce (3.1). Indeed, if $c < 1/2$, we notice that $s > 1, c^{2c} > c$ and

$$c < a \quad \Rightarrow \quad 2c - a < c \quad \Rightarrow \quad c^a a^c > c^{2c} \quad \Rightarrow \quad c > e^{-c\ln(a/c)/(a-c)} = g(s), \tag{3.4}$$

i.e. $x > y > c > g(s)$, then the strictly increasing behavior of f on $]g(s), \infty[$, implies the inequality (3.5); else if $c > 1/2$, we have that $s < 1$ and

$$c < b < a < 1 \quad \Rightarrow \quad c^{2a} b < c^{2b} a \quad \Rightarrow \quad c^{2a} > e^{-b\ln(b/a)/(b-a)} = g(s), \tag{3.5}$$

i.e. $y > x > g(s)$, then the strictly decreasing behavior of f on $]g(s), \infty[$, implies the inequality (3.1). We observe that, by Theorem 1.2, the inequalities $a^{2a} + b^{2b} > a^{2b} + b^{2a}$ and $b^{2b} + c^{2c} > b^{2c} + c^{2b}$ holds, for all positive real numbers a, b and c. Adding $a^{2a} + b^{2b} > a^{2b} + b^{2a}$ with (3.5) and $b^{2b} + c^{2c} > b^{2c} + c^{2b}$ with (3.1), we obtain (1.1) for $c < 1/2$ and $c \geq 1/2$, respectively. Thus, the inequality (1.1) holds for $0 < c < b < a < 1$, as desired. Secondarily, if $b < c < a$, we proceed in a similar form, selecting $x = a^{2b}, y = b^{2b}$ and $s = c/b$ for $b \geq 1/2$, to prove (3.1) and $x = a^{2b}, y = c^{2b}$ and $s = a/b$ for $b \geq 1/2$, to deduce

$$a^{2c} + b^{2b} > a^{2b} + b^{2c}. \tag{3.6}$$

In the case of $b < 1/2$, if proceed in the same manner to (3.6), we get that $b > g(s)$, then $x > y > b > g(s)$ and $s > 1$ implies the strictly increasing behavior of f on $]g(s), \infty[$, which leads to (3.1). Adding the inequality (3.1) with (3.2) we obtain (1.1). Meanwhile, for $b \geq 1/2$ we note the function $m : [b, 1[\to \mathbb{R}$ defined by $m(x) = xb^{2x} - b^{2b+1}$ satisfies the following properties

$$m(b) = 0, \quad m(1) = b(b - b^{2b}) > 0,$$
$$m \text{ has a unique maximum on } [b,1[\text{ at } x_{\max} = \frac{-1}{2\ln b} \geq \frac{1}{2\ln 2} \approx 0.61,$$

then $m(x) \geq 0$ for $x \in [b, 1[$. In particular for $x = c$, we have that $m(c) = cb^{2c} - b^{2b+1} \geq 0$ which implies $b^{2b} \geq g(s)$ and we follow (3.6) by application of f.

For $(a,b,c) \in \mathbb{E}_b^- \cup \mathbb{E}_c^-$ we can follow line by line the proof of $(a,b,c) \in \mathbb{E}_a^-$. However, we can obtain a direct proof by apply the result obtained for $(a,b,c) \in \mathbb{E}_a^-$ by interchanging the role of variables. For instance, if $(a,b,c) \in \mathbb{E}_b^-$ then $(b,a,c) \in \mathbb{E}_a^-$ which implies (1.1).

4. AN ADDITIONAL REMARK

In this section we present another proof for $c = 1/2 < b < a < 1$. We define $h :]1/2, a[\to \mathbb{R}$ as follows

$$h(x) = a^{2a} + x^{2x} + \frac{1}{2} - a^{2x} - x - \left(\frac{1}{2}\right)^{2a}, \quad x \in [1/2, a].$$

Thus the proof of (1.1) is reduced to prove $h(x) \geq 0$ for $x \in [1/2, a]$. An application of Teorem 1.2 implies the following estimate

$$\begin{aligned} h(x) &= \left[a^{2a} + x^{2x} - a^{2x} - x^{2a}\right] + x^{2a} + \frac{1}{2} - x - \left(\frac{1}{2}\right)^{2a} \\ &\geq x^{2a} + \frac{1}{2} - x - \left(\frac{1}{2}\right)^{2a} := h_1(x), \quad x \in [1/2, a]. \end{aligned}$$

We note that h_1 is an increasing function on $[1/2, a]$. In fact, because x^{2a-1} is strictly increasing and 4^a is convex on $[1/2, 1]$, we have

$$h_1'(x) = 2ax^{2a-1} - 1 > 2a\left(\frac{1}{2}\right)^{2a-1} - 1 = \frac{4a - 4^a}{4^a} > 0, \quad x \in [1/2, a].$$

Hence, $h(x) \geq h_1(x) > h_1(1/2) = 0$, for all $x \in [1/2, a]$, and the inequality (1.1) holds for $c = 1/2 < b < a < 1$.

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