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SUBORDINATION RESULTS ASSOCIATED WITH HADAMARD PRODUCT

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ABSTRACT. In the present investigation, we consider an unified class of functions of complex order using Hadamard's convolution. We obtain a necessary and sufficient condition for functions to be in these classes.

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1. INTRODUCTION

Let $\mathcal{A}$ be the class of all analytic functions

$$f(z) = z + a_2z^2 + a_3z^2 + \cdots \tag{1.1}$$

in the open unit disk $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. A function $f \in \mathcal{A}$ is subordinate to an univalent function $g \in \mathcal{A}$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$. Let Ω be the family of analytic functions $\omega(z)$ in the unit disc Δ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(\omega(z))$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. The class $S^*(\phi)$, introduced and studied by Ma and Minda [7], consists of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \quad (z \in \Delta).$$

Recently, Ravichandran et al. [10] defined classes related to the class of starlike functions of complex order defined as

Definition 1.1. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Following the work of Ma and Minda [7], Shanmugam and Sivasubramanian [16] obtained Fekete-Szegö inequality for the more general class $M_\alpha(\phi)$, defined by

$$\frac{\alpha z^2 f''(z) + zf'(z)}{(1-\alpha)f(z) + \alpha zf'(z)} \prec \phi(z),$$

where $\phi(z)$ satisfies the condition mentioned in Definition 1.1.

Let g be a fixed function. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad \in \mathcal{A},$$

the Hadamard product(or convolution product) is given by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

For various choices of $g(z)$ we get different operators and are listed below.

(1) For $g(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}, \ldots, (\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}, \ldots, (\beta_s)_{n-1}(1)_{n-1}} z^n$, we get the Dziok–Srivastava operator $H_{q,s}(\alpha)f(z)$ introduced by Dziok and Srivastava [5].

(2) For $g(z) = \phi(a,c,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^n$, we get the Carlson-Shaffer operator $L(a,c)f(z)$ introduced by Carlson-Shaffer [1].

(3) For $g(z) = \frac{z}{(1-z)^{\lambda+1}}$, we get the Ruschweyh operator $D^{\lambda} f(z)$ introduced by Ruschweyh[13].

(4) For $g(z) = z + \sum_{n=2}^{\infty} n^m z^n \quad (\, m \geq 0)$, we get the Sălăgean operator $D^m f(z)$ introduced by Sălăgean [15].

(5) For $g(z) = z + \sum_{n=2}^{\infty} \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n \quad (\, \lambda \geq 0;\ k \in \mathbb{Z})$, we get the multiplier transformation $I(\lambda, k)$ introduced by Cho and Srivastava[3].

(6) For $g(z) = z + \sum_{n=2}^{\infty} n \left(\frac{n+\lambda}{1+\lambda}\right)^k z^n \quad (\, \lambda \geq 0;\ k \in \mathbb{Z})$, the multiplier transformation $I(\lambda, k)$ introduced by Cho and Kim [2]

In this paper, we introduce a more general class of complex order $M[g,b,\alpha](\phi)$ which we define below.

Definition 1.2. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ with $\phi(0) = 1$, $\phi'(0) > 0$ which maps the unit disk Δ onto a region starlike with respect to 1 which is symmetric with respect to the real axis. Then the class $M[g,b,\alpha](\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\Psi(g,\alpha) - 1\right) \prec \phi(z), \quad (\alpha \geq 0)$$

where,

$$\Psi(g,\alpha) := \frac{\alpha z^2 (f*g)''(z) + z(f*g)'(z)}{(1-\alpha)(f*g)(z) + \alpha z (f*g)'(z)}.$$

Clearly,

$$\begin{aligned} M\left[\frac{z}{1-z}, b, 0\right](\phi) &\equiv S_b^*(\phi), \\ M\left[\frac{z}{1-z}, b, 1\right](\phi) &\equiv M\left[\frac{z}{(1-z)^2}, b, 0\right](\phi) \equiv C_b(\phi), \end{aligned}$$

and $M[\frac{z}{(1-z)^2}, b, 1](\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{z^2 f'''(z) + 2z f''(z)}{z f''(z) + f'(z)}\right) \prec \phi(z).$$

Motivated essentially by the aforementioned works, we obtain certain necessary and sufficient conditions for the unified class of functions $M[g,b,\alpha](\phi)$ which we have defined. The Motivation of this paper is to generalize the results obtained by Srivastava and Lashin [17].

Our results includes several known results. To see this, let $M\left[\frac{z}{1-z}, b, 0\right](A,B) \equiv S^*(A,B,b)$ and $M\left[\frac{z}{1-z}, b, 0\right](A,B) \equiv C(A,B,b)$ ($b \neq 0$, complex) denote the classes $S_b^*(\phi)$ and $C_b(\phi)$ respectively when

$$\phi(z) = \frac{1+Az}{1+Bz} \quad (-1 \leq B < A \leq 1).$$

The class $S^*(A,B,b)$ and therefore the class $S_b^*(\phi)$ specialize to several well-known classes of univalent functions for suitable choices of A, B and b. The class $S^*(A,B,1)$ is denoted by $S^*(A,B)$.

Some of these classes are listed below where $ST(b)$ denotes $1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)$.

(1) $S^*(1,-1,1)$ is the class S^* of starlike functions [6, 4, 9].
(2) $S^*(1,-1,b)$ is the class of starlike functions of complex order introduced by Wiatrowski [18]. We denote this class by S_b^*.
(3) $S^*(1,-1,1-\beta)$, $0\le\beta<1,$ is the class $S^*(\beta)$ of starlike functions of order β. This class was introduced by Robertson [11].
(4) $S^*(1,0,b)$ is the set defined by $|ST(b)-1|<1$.
(5) $S^*(\beta,0,b)$ is the set defined by $|ST(b)-1|<\beta, 0\le\beta<1$.
(6) $S^*(\beta,-\beta,b)$ is the set defined by $\left|\frac{ST(b)-1}{ST(b)+1}\right|<\beta, 0\le\beta<1$.

To prove our main result, we need the following results.

The following result follows from a result of Ruscheweyh [13] for functions in the class $S^*(\phi)$ (see Ruscheweyh [14, Theorem 2.37, pages 86–88]).

Lemma 1.1. [10] *Let ϕ be a convex function defined on Δ, $\phi(0)=1$. Define $F(z)$ by*

$$F(z)=z\exp\left(\int_0^z\frac{\phi(x)-1}{x}dx\right). \tag{1.2}$$

Let $q(z)=1+c_1z+\cdots$ be analytic in Δ. Then

$$1+\frac{zq'(z)}{q(z)}\prec\phi(z) \tag{1.3}$$

if and only if for all $|s|\le 1$ and $|t|\le 1$, we have

$$\frac{q(tz)}{q(sz)}\prec\frac{sF(tz)}{tF(sz)}. \tag{1.4}$$

Lemma 1.2. [8, Corollary 3.4h.1, p.135] *Let $q(z)$ be univalent in Δ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $\frac{zq'(z)}{\varphi(q(z))}$ is starlike, and*

$$zp'(z)\varphi(p(z))\prec zq'(z)\varphi(q(z)),$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

2. MAIN RESULTS

By making use of Lemma 1.1, we have the following:

Theorem 2.1. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f\in M[g,b,\alpha](\phi)$ if and only if for all $|s|\le 1$ and $|t|\le 1$, we have*

$$\left(\frac{s\left[(1-\alpha)\ (f*g)(tz)+\alpha z(f*g)'(tz)\right]}{t\left[(1-\alpha)\ (f*g)(sz)+\alpha z(f*g)'(sz)\right]}\right)^{1/b}\prec\frac{sF(tz)}{tF(sz)}. \tag{2.1}$$

Proof. Define the function $p(z)$ by

$$p(z):=\left(\frac{(1-\alpha)\ (f*g)(z)+\alpha z(f*g)'(z)}{z}\right)^{1/b}. \tag{2.2}$$

By taking logarithmic derivative of $p(z)$ given by (2.2), we get

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left\{\frac{\alpha z^2(f*g)''(z)+z(f*g)'(z)}{(1-\alpha)(f*g)(z)+\alpha z(f*g)'(z)} - 1\right\}. \tag{2.3}$$

The result now follows from Lemma 1.1. ∎

Letting $g(z) = \dfrac{z}{1-z}$ and $\alpha = 0$ in Theorem 2.1, we obtain

Corollary 2.2. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in S_b^*(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$\left(\frac{sf(tz)}{tf(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}. \tag{2.4}$$

Letting $g(z) = \dfrac{z}{1-z}$ and $\alpha = 1$ in Theorem 2.1, we obtain

Corollary 2.3. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. The function $f \in C_b(\phi)$ if and only if for all $|s| \le 1$ and $|t| \le 1$, we have*

$$\left(\frac{f'(tz)}{f'(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}.$$

As an immediate consequence of the above Corollary 2.3, we have

Corollary 2.4. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. If $f \in S_b^*(\phi)$, then we have*

$$\frac{f(z)}{z} \prec \left(\frac{F(z)}{z}\right)^b. \tag{2.5}$$

Letting $g(z) = \dfrac{z}{(1-z)^2}$ and $\alpha = 1$ in Theorem 2.1, we have

Corollary 2.5. *Let $\phi(z)$ and $F(z)$ be as in Lemma 1.1. Then*

$$1 + \frac{1}{b}\left(\frac{z^2 f'''(z) + 2zf''(z)}{zf''(z) + f'(z)}\right) \prec \phi(z). \tag{2.6}$$

if and only if for all $|s| \le 1$ and $|t| \le 1$, we have

$$\left(\frac{(zf')'(tz)}{(zf')'(sz)}\right)^{\frac{1}{b}} \prec \frac{sF(tz)}{tF(sz)}.$$

Theorem 2.6. *Let ϕ starlike with respect to 1 and $F(z)$ is given by (1.2) be starlike. If $f \in M[g, b, \alpha](\phi)$, then we have*

$$\frac{(1-\alpha)\ (f*g)(z) + \alpha z(f*g)'(z)}{z} \prec \left(\frac{F(z)}{z}\right)^b. \tag{2.7}$$

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \left(\frac{(1-\alpha)\ (f*g)(z) + \alpha z(f*g)'(z)}{z}\right)^{1/b}, \quad q(z) := \left(\frac{F(z)}{z}\right).$$

Then a computation yields

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left(\Psi(g,\alpha) - 1\right)$$

where

$$\Psi(g,\alpha)=\left\{\frac{\alpha z^2(f*g)''(z)+z(f*g)'(z)}{(1-\alpha)(f*g)(z)+\alpha z(f*g)'(z)}-1\right\}$$

and

$$\frac{zq'(z)}{q(z)}=\left(\frac{zF'(z)}{F(z)}-1\right)=\phi(z)-1.$$

Since $f\in M[g,b,\alpha](\phi)$, we have

$$\frac{zp'(z)}{p(z)}=\frac{1}{b}\left(\Psi(g,\alpha)-1\right)\prec\phi(z)-1=\frac{zq'(z)}{q(z)}.$$

The result now follows by an application of Lemma 1.2. ∎

Letting $g(z)=\dfrac{z}{(1-z)^2}$ and $\alpha=1$ in Theorem 2.6, we have

Corollary 2.7. *Let ϕ starlike with respect to 1 and $F(z)$ is given by (1.2) be starlike. If*

$$1+\frac{1}{b}\left(\frac{z^2f'''(z)+2zf''(z)}{zf''(z)+f'(z)}\right)\prec\phi(z). \tag{2.8}$$

then we have

$$zf''(z)+f'(z)\prec\left(\frac{F(z)}{z}\right)^b. \tag{2.9}$$

By taking $\phi(z)=\dfrac{1+z}{1-z}$, $g(z)=\dfrac{z}{1-z}$ and $\alpha=0$ in Theorem 2.6, we get the following result of Srivastava and Lashin [17]:

Corollary 2.8. *If $f\in S_b^*$, then*

$$\frac{f(z)}{z}\prec\frac{1}{(1-z)^{2b}}.$$

By taking $\phi(z)=\dfrac{1+z}{1-z}$, $g(z)=\dfrac{z}{1-z}$ and $\alpha=1$ in Theorem 2.6, we get another result of Srivastava and Lashin [17]:

Corollary 2.9. *If $f\in C_b$, then*

$$f'(z)\prec\frac{1}{(1-z)^{2b}}.$$

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