The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 9, Issue 1, Article 13, pp. 1-9, 2012

ON THE THREE VARIABLE RECIPROCITY THEOREM AND ITS APPLICATIONS

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Received 17 November, 2009; accepted 18 July, 2011; published 20 March, 2012.

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ABSTRACT. In this paper we show how the three variable reciprocity theorem can be easily derived from the well known two variable reciprocity theorem of Ramanujan by parameter augmentation. Further we derive some q-gamma, q-beta and eta-function identities from the three variable reciprocity theorem.

Key words and phrases: Reciprocity theorem, Operator identity.

2000 *Mathematics Subject Classification.* 05A30, 33D15, 11B65.

ISSN (electronic): 1449-5910

1. INTRODUCTION

In his lost notebook [18, p.40] S. Ramanujan recorded the following two variable reciprocity theorem:

$$\rho(a,b)-\rho(b,a)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty} \tag{1.1}$$

where

$$\rho(a,b)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n},\quad a\neq -q^{-n}, n\in\mathbb{Z}^+\ \text{ and }\ |q|<1,$$

and as usual,

$$\begin{aligned}(a)_\infty &:= (a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1\\ (a)_n &:= (a;q)_n=\frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad n:\text{an integer}.\end{aligned}$$

For proofs of (1.1) one may refer the works of G. E. Andrews [3], D. D. Somashekara and S. N. Fathima [19], T. Kim, Somashekara and Fathima [15], B. C. Berndt, S. H. Chan, B. P. Yeap and A. J. Yee [9], C. Adiga and N. Anitha [1], P. S. Guruprasad and N. Pradeep [11] and S. Y. Kang [14].

Recently Kang [14] has given several generalizations of the reciprocity theorem (1.1). One of the generalizations of (1.1) is the following three-variable reciprocity theorem:

$$\rho_3(a,b,c)-\rho_3(b,a,c)=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c)_\infty(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-c/a)_\infty(-c/b)_\infty(-aq)_\infty(-bq)_\infty}, \tag{1.2}$$

where

$$\rho_3(a,b,c)=\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(c)_n(-1)^n q^{n(n+1)/2}a^n b^{-n}}{(-aq)_n(-c/b)_{n+1}},\ a\neq -q^{-n},\ c/b\neq -1,-q^{-n}$$

for $n\in\mathbb{Z}^+$ and $|q|<1$. Putting $c=0$ in (1.2) we obtain (1.1). Kang [14] has proved (1.2) using the well known ${}_1\psi_1$-summation formula [17, Ch.16] and Jackson's transformation formula [10, p.526]. Adiga and Guruprasad [2] have given a proof of (1.2) using the well known q-binomial theorem and the Gauss summation formula [4]. Somashekara and D. Mamta [21] have given a proof of (1.2) using the well known two variable reciprocity theorem (1.1), Jackson's transformation formula [11, p.526] and an identity obtained by them in [20].

The main objective of this paper is to derive the three variable reciprocity theorem using the two variable reciprocity theorem by parameter augmentation, the technique which was extensively used by Z. G. Liu in [16]. Further, we derive from the reciprocity theorem (1.2) some q-gamma , q-beta and eta-function identities, which complement the work of S. Bhargava and C. Adiga [6], Bhargava and Somashekara [7], Bhargava, Somashekara and Fathima [8], K. R. Vasuki [23]. For this purpose we employ the following definitions and some simple related results.

The q-difference operator and the q-shift operator ζ are defined by [16]

$$D_q f(a)=\frac{1}{a}(f(a)-f(aq))\quad and\quad \zeta\{f(a)\}=f(aq),$$

respectively. An operator θ is defined by

$$\theta=\zeta^{-1}D_q.$$

The operator $E(b\theta)$ is defined as

$$E(b\theta) = \sum_{n=0}^{\infty} \frac{(b\theta)^n q^{n(n-1)/2}}{(q;q)_n}\cdot$$

Then we have the following operator identities [16, Theorem-1]:

$$\begin{aligned} E(b\theta)\{(at;q)_\infty\} &= (at,bt;q)_\infty \\ E(b\theta)\{(as,at;q)_\infty\} &= \frac{(as,at,bs,bt;q)_\infty}{(abst/q;q)_\infty}, \quad \left|\frac{abst}{q}\right| < 1, \end{aligned}$$

where as usual,

$$(a_1,a_2,...,a_k;q)_n = (a_1;q)_n(a_2;q)_n...(a_k;q)_n,$$

where n is an integer or infinity.

The q-gamma function $\Gamma_q(x)$, a q-analogue of Euler's gamma function, was introduced by J. Thomae [22] and later by Jackson [13] as,

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \tag{1.3}$$

where q is a fixed real number $0<q<1$. Heine [12] gave an equivalent definition, but without the factor $(1-q)^{1-x}$. When $x=n+1$ with n a nonnegative integer, this definition (1.3) reduces to

$$\Gamma_q(n+1) = 1(1+q)(1+q+q^2)...(1+q+...+q^{n-1}), \tag{1.4}$$

which clearly approaches $n!$ as $q\to 1^-$. Hence $\Gamma_q(n+1)$ tends to $\Gamma(n+1)=n!$, as $q\to 1^-$. The definition of $\Gamma_q(x)$ can be extended to $|q|<1$ by using the principal values of q^x and $(1-q)^{1-x}$ in (1.3). Clearly,

$$\begin{aligned} \Gamma_q(x+1) &= \frac{(q;q)_\infty}{(q^{x+1};q)_\infty}(1-q)^{-x} \\ &= \prod_{n=1}^{\infty} \frac{(1-q^n)(1-q^{n+1})^x}{(1-q^{n+x})(1-q^n)^x}\cdot \end{aligned}$$

Hence

$$\begin{aligned} \lim_{q\to 1^-} \Gamma_q(x+1) &= \prod_{n=1}^{\infty} \frac{n}{n+x}\left(\frac{n+1}{n}\right)^x \\ &= x\left[x^{-1}\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^x\right] \\ &= \Gamma(x+1)\cdot \end{aligned}$$

Thus $\Gamma_q(x)\to\Gamma(x)$, the ordinary gamma function, as $q\to 1$. Askey [5] has obtained q-analogue of several classical results involving gamma function. For instance, he defined

$$B_q(x,y) = (1-q)\sum_{n=0}^{\infty} q^{nx}\frac{(q^{n+1})_\infty}{(q^{n+y})_\infty}, \tag{1.5}$$

and proved that

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}\cdot \tag{1.6}$$

The Dedekind eta-function is defined by

$$\begin{aligned} \eta(\tau) &:= e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}) \\ &= q^{1/24}(q;q)_\infty, \end{aligned} \tag{1.7}$$

where $q=e^{2\pi i\tau}$.

2. A PROOF OF (1.2)

Replacing a by $-ac$ and b by $-ad$ in (1.1), and multiplying the resulting identity throughout by $acd/(1-ac)(1-ad)$ we obtain

$$d\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(ad)_{n+1}}(d/c)^n - c\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(ac)_{n+1}}(c/d)^n = d\frac{(q,qd/c,c/d;q)_\infty}{(ac,ad;q)_\infty}.$$

This can be written as

$$d\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}(d/c)^n(adq^{n+1},ac;q)_\infty -$$

$$c\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}(c/d)^n(acq^{n+1},ad;q)_\infty = d(q,qd/c,c/d;q)_\infty.$$

On applying $E(b\theta)$ to both sides with respect to the variable a we obtain

$$d\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}(d/c)^n\frac{(adq^{n+1},ac,bdq^{n+1},bc;q)_\infty}{(abcdq^n;q)_\infty}$$

$$-c\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}(c/d)^n\frac{(acq^{n+1},ad,bcq^{n+1},bd;q)_\infty}{(abcdq^n)_\infty} = d(q,qd/c,c/d;q)_\infty.$$

Multiplying throughout by $(abcd)_\infty/(ad)_\infty(ac)_\infty(bd)_\infty(bc)_\infty$ we obtain

$$\begin{aligned} d\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(abcd)_n}{(ad)_{n+1}(bd)_{n+1}}(d/c)^n &- c\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}(abcd)_n}{(ac)_{n+1}(bc)_{n+1}}(c/d)^n \\ &= d\frac{(q,qd/c,c/d,abcd;q)_\infty}{(ac,ad,bd,bc;q)_\infty}. \end{aligned}$$

Multiplying the above equation by $(1-ac)(1-ad)/adc$ and then replacing a by $-a$, b by $-c/b$, c by 1 and d by b/a we obtain (1.2)

3. q-GAMMA AND q-BETA FUNCTION IDENTITIES

In this section we deduce some interesting identities involving q-gamma and q-beta functions. If $0<q<1$ and $0<x<1$ then

$$\frac{\Gamma_q(1+x)}{\Gamma_q(1-x)} = \tag{3.1}$$

$$\left[\frac{(1-q^{2x-1})}{(1-q)^{2x}}\sum_{n=0}^{\infty}\frac{(q^{2x})_n(-1)^nq^{\frac{n(n+1)}{2}-nx}}{(q^x)_n(q)_{n+1}} - \frac{q^x(1-q^{x-1})}{(1-q)^{2x}}\sum_{n=0}^{\infty}\frac{(-1)^nq^{\frac{n(n+1)}{2}+nx}}{(q^{x+1})_{n+1}}\right].$$

Proof. Putting $a=-q^{x-1}$, $b=-q^{2x-1}$, $c=q^{2x}$ in (1.2) we obtain

$$(1-q^{1-2x})\sum_{n=0}^{\infty}\frac{(q^{2x})_n(-1)^n q^{\frac{n(n+1)}{2}-nx}}{(q^x)_n(q)_{n+1}} - (1-q^{1-x})\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}+nx}}{(q^{x+1})_{n+1}}$$
$$=(-q^{1-2x}+q^{1-x})\frac{(q^{1-x})_\infty(q^{1+x})_\infty}{(q^{1+x})_\infty(q^x)_\infty}.$$

Multiplying throughout by $-q^{2x-1}$ we obtain

$$(1-q^{2x-1})\sum_{n=0}^{\infty}\frac{(q^{2x})_n(-1)^n q^{\frac{n(n+1)}{2}-nx}}{(q^x)_n(q)_{n+1}} - (-q^{2x-1}+q^x)\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}+nx}}{(q^{x+1})_{n+1}}$$
$$=\frac{(q^{1-x})_\infty}{(q^{1+x})_\infty}.$$

On using (1.3) and simplifying we obtain (3.1). ∎

If $0<q<1$ and $0<x<1$ then

$$\frac{\Gamma_q(2x)\Gamma_q(1-x)}{\Gamma_q(x)}=\frac{(-q^{1+x})_\infty(-q)_\infty}{(-q^{1+2x})_\infty(-q^{-2x})_\infty}\times \tag{3.2}$$
$$\left[2(1-q^{-x})\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}}}{(-q^{1+x})_n(q^{2x})_{n+1}}+(q^{-x}+q^{-2x})\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}-2nx}}{(q^{1-x})_n(-q)_n}\right].$$

Proof. Putting $a=q^x$, $b=-q^{-x}$ and $c=q^x$ in (1.2) we obtain, on simplification,

$$(1-q^x)\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}}}{(-q_n^{1+x})(q^{2x})_{n+1}} + (q^{-x}+q^{-2x})\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}-2nx}}{2(q^{1+x})_n(-q)_n}$$
$$=\frac{(q^x)_\infty(-q^{1+2x})_\infty(-q^{-2x})_\infty(q)_\infty}{2(-q)_\infty(q^{2x})_\infty(-q^{1+x})_\infty(q^{1-x})_\infty}.$$

On using (1.3) and simplifying we obtain (3.2). ∎

If $0<q<1$, $0<y<x<1$ then

$$\frac{\Gamma_q(x)\Gamma_q(y)}{B_q(1-x+y,x-y)}=\frac{(-q^{1-x})_\infty(-q^{1-y})_\infty(1-q)^{1-x-y}}{(q^2;q^2)_\infty}\times \tag{3.3}$$
$$\left[(1+q^{-y})\sum_{n=0}^{\infty}\frac{(-q)_{n-1}(-1)^n q^{\frac{n(n+1)}{2}-nx+ny}}{(-q^{1-x})_n(q^y)_{n+1}}\right.$$
$$\left.-(q^{-y}+q^{x-y})\sum_{n=0}^{\infty}\frac{(-q)_{n-1}(-1)^n q^{\frac{n(n+1)}{2}+nx-ny}}{(-q^{1-y})_n(q^x)_{n+1}}\right].$$

Proof. Putting $a=q^{-x}, b=q^{-y}$ and $c=-1$ in (1.2) we obtain, on simplification,

$$(1-q^{-y})\sum_{n=0}^{\infty}\frac{(-q)_{n-1}(-1)^n q^{\frac{n(n+1)}{2}-nx+ny}}{(-q^{1-x})_n(q^y)_{n+1}}$$
$$-(q^{-y}+q^{x-y})\sum_{n=0}^{\infty}\frac{(-q)_{n-1}(-1)^n q^{\frac{n(n+1)}{2}+nx-ny}}{(-q^{1-y})_n(q^x)_{n+1}}=\frac{(q^2;q^2)_\infty(q^{1-x+y})_\infty(q^{x-y})_\infty}{(q^x)_\infty(q^y)_\infty(-q^{1-x})_\infty(-q^{1-y})_\infty}.$$

On using (1.3) and (1.6) and simplifying we obtain (3.3). ∎

If $0 < q < 1$ and $x = 2, 3, ...$, then

$$\frac{\Gamma_q(x-1)}{\Gamma_q(x)} = \frac{(-q^x)_\infty}{2(-q)_\infty} \times \left[(1-q)\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n-1)}{2}}}{(-q)_n(q^{x-1})_{n+1}} - 2q\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+3)}{2}}}{(q^2)_n(-q^x)_{n+1}}\right]. \tag{3.4}$$

Proof. Putting $a = 1$, $b = -q$ and $c = q^x$ in (1.2) we obtain, on simplification,

$$(1-q)\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n-1)}{2}}}{(-q)_n(q^{x-1})_{n+1}} + 2q\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+3)}{2}}}{(q^2)_n(-q^x)_{n+1}} = \frac{2(1-q)(q^x)_\infty(-q)_\infty}{(-q^x)_\infty(q^{x-1})_\infty}.$$

On using (1.3) and simplifying we obtain (3.4). ∎

If $0 < q < 1$ and $x = 2, 3, ...$, then

$$\frac{\Gamma_q(x-1)}{\Gamma_q(x)} = \frac{(-q^{x-1})_\infty}{2(-q)_\infty} \times \left[\frac{(1-q)}{(1+q)}\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}}}{(q^2)_n(q^{x-1})_{n+1}} + \sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}}}{(q^2)_n(-q^{x-1})_{n+1}}\right]. \tag{3.5}$$

Proof. Putting $a = q$, $b = -q$ and $c = q^x$ in (1.2) we obtain, on simplification,

$$\begin{aligned}(1-q)\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}}}{(q^2)_n(q^{x-1})_{n+1}} \quad &+ \quad (1+q)\sum_{n=0}^{\infty}\frac{(q^x)_n q^{\frac{n(n+1)}{2}}}{(q^2)_n(-q^{x-1})_{n+1}} \\ &= \frac{2(q^x)_\infty(-q)^2_\infty(q)_\infty}{(-q^{x-1})_\infty(q^{x-1})_\infty(-q^2)_\infty(q^2)_\infty} \\ &= \frac{2(1-q)(q^x)_\infty(-q)_\infty(1+q)}{(-q^{x-1})_\infty(q^{x-1})_\infty}.\end{aligned}$$

On using (1.3) and simplifying we obtain (3.5). ∎

If $0 < q < 1$ and $0 < y < x < 1$ then,

$$\frac{\Gamma_q(x)\Gamma_q(1-y)B_q(x,1-x)}{B_q(x-y,x)B_q(x-y,1-x+y)} = \frac{(1-q^{-y})}{(1-q)^{x-y}}\sum_{n=0}^{\infty}\frac{(q^{x-y})_n(-1)^n q^{\frac{n(n+1)}{2}-nx+ny}}{(q^{1-x})_n(q^x)_{n+1}} + \frac{(q^{-y}-q^{x-y})}{(1-q)^{x-y}} \times \sum_{n=0}^{\infty}\frac{(q^{x-y})_n(-1)^n q^{\frac{n(n+1)}{2}+nx-ny}}{(q^{1-y})_n(q^{2x-y})_{n+1}}. \tag{3.6}$$

Proof. Putting $a=-q^{-x}$, $b=-q^{-y}$, and $c=q^{x-y}$ in (1.2) we obtain, on simplification,

$$(1-q^{-y})\sum_{n=0}^{\infty}\frac{(q^{x-y})_n(-1)^n q^{\frac{n(n+1)}{2}-nx+ny}}{(q^{1-x})_n(q^x)_{n+1}}+(q^{-y}-q^{x-y})\times$$

$$\sum_{n=0}^{\infty}\frac{(q^{x-y})_n(-1)^n q^{\frac{n(n+1)}{2}+nx-ny}}{(q^{1-y})_n(q^{2x-y})_{n+1}}=\frac{(q^{x-y})_\infty(q^{1-x+y})_\infty(q^{x-y})_\infty(q)_\infty}{(q^{2x-y})_\infty(q^x)_\infty(q^{1-x})_\infty)(q^{1-y})_\infty}.$$

On using (1.3), (1.6) and simplifying we obtain (3.6). ∎

4. SOME ETA-FUNCTION IDENTITIES

This section will be devoted to obtain some eta function identities.

$$\frac{\eta(2\tau)}{\eta^2(\tau)}=\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(-q)_n}{(q)_n^2}q^{\frac{n(n-1)}{2}}\right]. \tag{4.1}$$

Proof. Putting $a=-q$, $b=q$ and $c=-q^2$ in (1.2) we obtain

$$\frac{2(-q)_\infty}{(q^2)_\infty}=(1+q)\sum_{n=0}^{\infty}\frac{(-q^2)_n q^{\frac{n(n+1)}{2}}}{(q^2)_n(q)_{n+1}}+(1-q)\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(-q)_{n+1}}.$$

Since

$$\sum_{n=0}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(-q)_{n+1}}=1,$$

on using (1.7) and simplifying we obtain (4.1). ∎

$$\frac{\eta^5(4\tau)}{\eta^4(2\tau)}=\frac{1}{2}\left[\frac{1}{(1+q)q^{1/2}}\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q^2)_n(q;q^2)_{n+1}}\right.$$
$$\left.-\frac{(1-q)\,q^{1/2}}{(1+q)^2}\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)}}{(-q^3;q^2)_n(-q^3;q^2)_{n+1}}\right]. \tag{4.2}$$

Proof. Putting $a=-q^{-1/2}$, $b=q^{1/2}$ and $c=-q$ in (1.2) and then replacing q by q^2 we obtain

$$\left(\frac{1+q^2}{q}\right)\frac{(-q^2;q^2)_\infty(-1;q^2)_\infty(-q^4;q^2)_\infty(q^2;q^2)_\infty}{(-q^3;q^2)_\infty(q;q^2)_\infty^2(-q^3;q^2)_\infty}$$
$$=\left(\frac{1+q}{q}\right)\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n\,q^{n(n+1)}}{(q;q^2)_n(q;q^2)_{n+1}}-(1-q)\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\,q^{n(n+1)}}{(-q^3;q^2)_n(-q^3;q^2)_{n+1}}.$$

On simplifying and using (1.7) we obtain (4.2). ∎

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