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FEJÉR-TYPE INEQUALITIES

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ABSTRACT. The aim of this paper is to present some new Fejér-type results for convex functions. Improvements of Young's inequality (the arithmetic-geometric mean inequality) and other applications to special means are pointed as well.

Key words and phrases: Fejér's inequality, convex function, Hermite-Hadamard inequality, Young's inequality.

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1. PRELIMINARIES

We start from an important result related to the convex functions due to Ch. Hermite [9] and J. Hadamard [8] which asserts that for every continuous convex function $f : [a, b] \to \mathbb{R}$ the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Fejér [6] established the following well-known weighted generalization:

Proposition 1.1. *If $f : [a, b] \to \mathbb{R}$ is continuous and convex and if $g : [a, b] \to \mathbb{R}_+$ is integrable and symmetric about $\frac{a+b}{2}$ (i.e. $g(x) = g(a + b - x)$), then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\mathrm{d}x \leq \int_a^b f(x)g(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\mathrm{d}x. \tag{1.2}$$

Before stating the results we recall some useful facts from the literature. S. S. Dragomir, P. Cerone and A. Sofo present in [3, 4] the following estimates of the precision in the Hermite-Hadamard inequality:

Proposition 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq f'' \leq M$. Then*

$$m\frac{(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - f\left(\frac{a+b}{2}\right) \leq M\frac{(b-a)^2}{24} \tag{1.3}$$

and

$$m\frac{(b-a)^2}{12} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq M\frac{(b-a)^2}{12}. \tag{1.4}$$

These inequalities follow from the Hermite-Hadamard inequality, for the convex functions $f(x) - m\frac{x^2}{2}$ and $M\frac{x^2}{2} - f(x)$.

Motivated by the above results, the purpose of this paper is to discuss further inequalities of Fejér type.

2. FEJÉR TYPE INEQUALITIES FOR CONVEX FUNCTIONS

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq f'' \leq M$. Then*

$$m\frac{\lambda(1-\lambda)}{2}(a-b)^2 \leq \lambda f(a) + (1-\lambda)f(b) - f(\lambda a + (1-\lambda)b) \leq M\frac{\lambda(1-\lambda)}{2}(a-b)^2, \tag{2.1}$$

for all $\lambda \in [0, 1]$.

Proof. We consider the function $g : [0, 1] \to \mathbb{R}$, defined by

$$g(\lambda) = \lambda f(a) + (1-\lambda)f(b) - f(\lambda a + (1-\lambda)b) - m\frac{\lambda(1-\lambda)}{2}(a-b)^2.$$

Since

$$g''(\lambda) = (a-b)^2[m - f''(\lambda a + (1-\lambda)b)] \leq 0,$$

the function g is concave. But $g(0) = g(1) = 0$, which implies that

$$0 = (1-\lambda)g(0) + \lambda g(1) \leq g((1-\lambda)\cdot 0 + \lambda \cdot 1) = g(\lambda),$$

for all $\lambda \in [0,1]$. Therefore, we obtain the first part of inequality (2.1).
To see that the later inequality holds, our next step is to take the convex function $h : [0,1] \to \mathbb{R}$, defined by

$$h(\lambda) = \lambda f(a) + (1-\lambda) f(b) - f(\lambda a + (1-\lambda) b) - M \frac{\lambda(1-\lambda)}{2}(a-b)^2.$$

Since $h(0) = h(1) = 0$,

$$0 = (1-\lambda) h(0) + \lambda h(1) \geq h((1-\lambda)\cdot 0 + \lambda \cdot 1) = h(\lambda),$$

for all $\lambda \in [0,1]$. The assertion is now clear. ∎

For a slight generalization and alternative proof of Theorem 2.1 the reader is referred to [7, Theorem 4.2].

Remark 2.1. By integrating each term of the inequality (2.1) on $[0,1]$ with respect to the variable λ we recover the inequality (1.4).

Corollary 2.2. *Preserving the notation of Theorem 2.1, the following inequalities hold:*

$$m\frac{(1-2\lambda)^2}{8}(a-b)^2 \leq \frac{f(\lambda a + (1-\lambda) b) + f((1-\lambda) a + \lambda b)}{2} - f\left(\frac{a+b}{2}\right) \tag{2.2}$$

$$\leq M\frac{(1-2\lambda)^2}{8}(a-b)^2 \tag{2.3}$$

for all $\lambda \in [0,1]$.

Proof. According to Theorem 2.1 for $\lambda = \frac{1}{2}$ we obtain the following result, previously established in [5]:

$$\frac{m}{8}(b-a)^2 \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \leq \frac{M}{8}(b-a)^2. \tag{2.4}$$

We consider the above inequality (2.4) replacing $a \to \lambda a + (1-\lambda) b$ and $b \to (1-\lambda) a + \lambda b$ (the hypothesis $m \leq f'' \leq M$ is still working on the interval with these endpoints because it is contained by $[a,b]$) and we get the claimed result. ∎

Remark 2.2. Notice that by integrating all terms of (2.2) on $[0,1]$ with respect to λ we recover now the inequality (1.3).

Next we give some estimates of the Fejér inequalities (Proposition 1.1):

Theorem 2.3. *Let* $f : [a,b] \to \mathbb{R}$ *be a twice differentiable function such that there exist real constants* m *and* M *so that* $m \leq f'' \leq M$. *Assume* $g : [a,b] \to \mathbb{R}_+$ *is integrable and symmetric about* $\frac{a+b}{2}$. *Then the following inequalities hold:*

$$\frac{m}{2}\int_a^b (t-a)(b-t) g(t)\,\mathrm{d}t \leq \frac{f(a)+f(b)}{2}\int_a^b g(t)\,\mathrm{d}t - \int_a^b f(t) g(t)\,\mathrm{d}t \tag{2.5}$$

$$\leq \frac{M}{2}\int_a^b (t-a)(b-t) g(t)\,\mathrm{d}t \tag{2.6}$$

and

$$\frac{m}{8}\int_a^b (2t-a-b)^2 g(t)\,\mathrm{d}t \leq \int_a^b f(t) g(t)\,\mathrm{d}t - f\left(\frac{a+b}{2}\right)\int_a^b g(t)\,\mathrm{d}t \tag{2.7}$$

$$\leq \frac{M}{8}\int_a^b (2t-a-b)^2 g(t)\,\mathrm{d}t. \tag{2.8}$$

Proof. We multiply (2.1) by $g(\lambda a+(1-\lambda)b)$ and integrate the result on $[0,1]$ with respect to the variable λ. Using the change of the variable $\lambda a+(1-\lambda)b=t$ we get

$$\begin{aligned}
&\frac{m}{2}\int_a^b (t-a)(b-t)g(t)\,\mathrm{d}t\\
\leq\;& f(a)\int_a^b \frac{b-t}{b-a}g(t)\,\mathrm{d}t+f(b)\int_a^b \frac{t-a}{b-a}g(t)\,\mathrm{d}t-\int_a^b f(t)g(t)\,\mathrm{d}t\\
\leq\;& \frac{M}{2}\int_a^b (t-a)(b-t)g(t)\,dt. \qquad (2.9)
\end{aligned}$$

On the other hand, due to the symmetry property of g, for $t=a+b-x$, we also have

$$\begin{aligned}
&\frac{m}{2}\int_a^b (b-x)(x-a)g(x)\,dx\\
\leq\;& f(a)\int_a^b \frac{x-a}{b-a}g(x)\,dx+f(b)\int_a^b \frac{b-x}{b-a}g(x)\,dx-\int_a^b f(t)g(t)\,dt\\
\leq\;& \frac{M}{2}\int_a^b (b-x)(x-a)g(x)\,dx. \qquad (2.10)
\end{aligned}$$

Summing (2.9) and (2.10) we find (2.5).

In order to prove the remaining inequalities (2.7) we follow same steps as above, using (2.2) instead of (2.1). The computation is straightforward, taking into account the symmetry of g (applied now as $g(\lambda a+(1-\lambda)b)=g((1-\lambda)a+\lambda b)$). We omit the details.

This completes the proof. ∎

It is remarkable that (2.5) agrees, having an extended form, with [12, pp.53, Exercise 4].

Remark 2.3. If $g:[a,b]\to[0,1]$ then the function $h(x)=1-g(x)$ satisfies the same symmetry and positivity conditions and Theorem 2.3 also applies. That yields the following estimates of the precision in (2.5):

$$\begin{aligned}
&(b-a)\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\;-m\frac{(b-a)^2}{12}\right)\\
(2.11)\geq\;& \frac{f(a)+f(b)}{2}\int_a^b g(t)\,\mathrm{d}t-\int_a^b f(t)g(t)\,\mathrm{d}t\;-\;\frac{m}{2}\int_a^b (t-a)(b-t)g(t)\,\mathrm{d}t\geq 0
\end{aligned}$$

and

$$\begin{aligned}
&(b-a)\left(M\frac{(b-a)^2}{12}-\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right)\\
(2.12)\geq\;& \frac{M}{2}\int_a^b (t-a)(b-t)g(t)\,\mathrm{d}t-\frac{f(a)+f(b)}{2}\int_a^b g(t)\,\mathrm{d}t+\int_a^b f(t)g(t)\,\mathrm{d}t\;\geq 0.
\end{aligned}$$

By a similar technique one can estimate (2.7).

Remark 2.4. For the particular case $g(x)=1$ if we apply Theorem 2.3 on the intervals $\left[a,\frac{a+b}{2}\right]$, $\left[\frac{a+b}{2},b\right]$ we get:

(2.13)
$$\frac{m(b-a)^2}{48}\leq\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right)-\frac{1}{b-a}\int_a^b f(t)g(t)\,\mathrm{d}t\leq\frac{M(b-a)^2}{48}$$

and

(2.14)
$$\frac{m\,(b-a)^2}{96} \le \frac{1}{b-a}\int_a^b f(t)g\,(t)\,\mathrm{d}t - \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right) \le \frac{M\,(b-a)^2}{96}.$$

The following theorem gives new Fejér-type inequalities.

Theorem 2.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable, convex function with $f'\ge 0$ and $g:[a,b]\to\mathbb{R}_+$ be continuous. Then the following statements hold.*

1) If g is monotonically decreasing then

(2.15)
$$\frac{f\,(a)+f\,(b)}{2}\int_a^b g\,(t)\,\mathrm{d}t - \int_a^b f(t)g\,(t)\,\mathrm{d}t \ \ge \frac{f\,(a)+f\,(x)}{2}\int_a^x g\,(t)\,\mathrm{d}t - \int_a^x f(t)g\,(t)\,\mathrm{d}t \ \ge 0;$$

2) If g is monotonically increasing then

(2.16)
$$\int_a^b f(t)g\,(t)\,\mathrm{d}t \ - f\left(\frac{a+b}{2}\right)\int_a^b g\,(t)\,\mathrm{d}t \ge \int_a^x f(t)g\,(t)\,\mathrm{d}t \ - f\left(\frac{a+x}{2}\right)\int_a^x g\,(t)\,\mathrm{d}t \ge 0$$

for all $x\in(a,b)$.

Proof. 1) We consider the function $h_1:[a,b]\to\mathbb{R}$, defined by
$$h_1(x) = \frac{f\,(a)+f\,(x)}{2}\int_a^x g\,(t)\,\mathrm{d}t - \int_a^x f(t)g\,(t)\,\mathrm{d}t\,.$$
Its first derivative is
$$h_1'(x) = \frac{f'\,(x)}{2}\int_a^x g\,(t)\,\mathrm{d}t - \frac{f\,(x)-f\,(a)}{2}g\,(x)\,.$$
Using the mean value theorems there exist $c_1, c_2 \in [a,x]$ such that
$$h_1'(x) = \left(\frac{f'\,(x)}{2}g\,(c_1) - \frac{f'\,(c_2)}{2}g\,(x)\right)(x-a)\,.$$
Thus, by the convexity of f and to the monotonicity of g, we have $f'\,(x)\ge f'\,(c_2)\ge 0$ and $g\,(c_1)\ge g\,(x)\ge 0$, hence we conclude that h_1 is increasing on its domain and $h_1\,(b)\ge h_1\,(x)\ge h_1\,(a)=0$. Thus we have (2.15), as asserted.

2) Similarly, we consider the function $h_2:[a,b]\to\mathbb{R}$, defined by
$$h_2(x) = \int_a^x f(t)g\,(t)\,\mathrm{d}t \ - f\left(\frac{a+x}{2}\right)\int_a^x g\,(t)\,\mathrm{d}t$$
and we compute its first derivative
$$h_2'(x) = \left(f(x) - f\left(\frac{a+x}{2}\right)\right)g\,(x) \ - \frac{1}{2}f'\left(\frac{a+x}{2}\right)\int_a^x g\,(t)\,\mathrm{d}t.$$
There exist $k_1\in\left[\frac{a+x}{2},x\right]$ and $k_2\in[a,x]$ such that
$$h_2'(x) = \left(f'\,(k_1)\,g\,(x) \ - f'\left(\frac{a+x}{2}\right)g\,(k_2)\right)\frac{x-a}{2}.$$
Therefore, due to the monotonicity we have $f'\,(k_1)\ge f'\left(\frac{a+x}{2}\right)$ and $g\,(x)\ge g\,(k_2)$, which leads that h_2 is increasing on its domain and $h_2\,(b)\ge h_2\,(x)\ge h_2\,(a)=0$.

Thus the proof is completed. ∎

The following result incorporates the classic statement of the Hermite-Hadamard inequality.

Corollary 2.5. *Suppose $f : [a,b] \to \mathbb{R}$ is differentiable and convex. Then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t \geq \frac{x-a}{b-a}\left(\frac{f(a)+f(x)}{2} - \frac{1}{x-a}\int_a^x f(t)\mathrm{d}t\right) \geq 0 \tag{2.17}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right) \geq \frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x f(t)\mathrm{d}t - f\left(\frac{a+x}{2}\right)\right) \geq 0 \tag{2.18}$$

for all $x \in (a,b)$.

Proof. We can follow the steps in the proof of Theorem 2.4 with with $g(x) = 1$, which satisfies both monotonicity conditions. ∎

Notice that in the previous corollary the condition $f' \geq 0$ which appears in the statement of Theorem 2.4 was no longer necessary and has been cancelled.

Remark 2.5. Under the same assumptions as in Proposition 1.2, when we apply (2.18) to the convex functions $f(x) - m\frac{x^2}{2}$ and $M\frac{x^2}{2} - f(x)$, we recover and improve the inequalities (1.3) as follows:

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right) - m\frac{(b-a)^2}{24} \\ \geq\ &\frac{x-a}{b-a}\left(\frac{1}{x-a}\int_a^x f(t)\mathrm{d}t - f\left(\frac{a+x}{2}\right) - m\frac{(x-a)^2}{24}\right) \geq 0 \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} &\frac{M}{24}(b-a)^2 - \left(\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\frac{a+b}{2}\right)\right) \\ \geq\ &\frac{x-a}{b-a}\left[\frac{M}{24}(x-a)^2 - \left(\frac{1}{x-a}\int_a^x f(t)\mathrm{d}t - f\left(\frac{a+x}{2}\right)\right)\right] \geq 0. \end{aligned} \tag{2.20}$$

Similarly if we use (2.17) we get improvements of (1.4) which at this moment can easily be written by the interested reader. Obviously same steps could be followed from Theorem 2.4, improving that way Theorem 2.3.

We end this section with the weighted statement of a known result concerning convex functions.

In the light of Proposition 1.1, the following statement appears as a trivial generalization of a result due to Vasić and Lacković [11], and Lupaş [10] (cf. J. E. Pečarić et al. [13, pp. 143]) and we omit its proof.

Proposition 2.6. *Let p and q be two positive numbers and $a_1 \leq a \leq b \leq b_1$. Let $g : [a,b] \to \mathbb{R}_+$ be integrable and symmetric about $A = \frac{pa+qb}{p+q}$. Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right)\int_{A-y}^{A+y} g(t)\,\mathrm{d}t \leq \int_{A-y}^{A+y} f(x)\,g(x)\,\mathrm{d}x \leq \frac{pf(a)+qf(b)}{p+q}\int_{A-y}^{A+y} g(t)\,\mathrm{d}t \tag{2.21}$$

hold for $y > 0$ and all continuous convex functions $f : [a_1,b_1] \to \mathbb{R}$ if and only if

$$y \leq \frac{b-a}{p+q}\min\{p,q\}.$$

3. Application to special means

From the inequality (2.17) applied to the convex function t^p, with $p \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$ we have

$$(b-a)\left\{[A_p(a,b)]^p - [L_p(a,b)]^p\right\} \geq (x-a)\left\{[A_p(a,x)]^p - [L_p(a,x)]^p\right\}, \tag{3.1}$$

where $x \in [a,b]$. Here $A_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p}$ is the power mean and $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p}$ is the p-logarithmic mean. Also the limit case $p \to -1$ (or we may equivalently say the case of the convex function $1/t$) gives us

$$(b-a)\left\{\frac{1}{H(a,b)} - \frac{1}{L(a,b)}\right\} \geq (x-a)\left\{\frac{1}{H(a,x)} - \frac{1}{L(a,x)}\right\},$$

where $H(a,b) = \frac{2ab}{a+b}$ is the harmonic mean and $L(a,b) = \frac{b-a}{\log b - \log a}$ is the logarithmic mean.

It is also useful to consider the inequality (2.17) applied for the convex function $-\log t$, when we get

$$\left[\frac{G(a,b)}{I(a,b)}\right]^{b-a} \leq \left[\frac{G(a,x)}{I(a,x)}\right]^{x-a},$$

for $a \neq b$, $x \in [a,b]$, where $G(a,b) = \sqrt{ab}$ is the geometric mean and $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$ is the identric mean.

In the remainder, we focus on two immediate particular cases of Theorem 2.1 that help us to give improvements of the well known arithmetic-geometric mean inequality (also known as Young's inequality).

1) We apply the theorem to the function $f : [a,b] \to \mathbb{R}$ $(a > 0)$ defined by $f(x) = -\log x$, which leads to

$$e^{\frac{\lambda(1-\lambda)(a-b)^2}{2b^2}} \leq \frac{\lambda a + (1-\lambda)b}{a^\lambda b^{1-\lambda}} \leq e^{\frac{\lambda(1-\lambda)(a-b)^2}{2a^2}}. \tag{3.2}$$

Since $e^{\frac{\lambda(1-\lambda)(a-b)^2}{2b^2}} \geq 1$, we obtain a refinement of Young's inequality where $\lambda \in [0,1]$.

We also obtained a reverse inequality for Young's inequality.

2) Next, we apply the theorem to the function $f : [\log a, \log b] \to \mathbb{R}$, defined by $f(x) = \exp x$ and we arrive at

$$\frac{\lambda(1-\lambda)a}{2}\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq \frac{\lambda(1-\lambda)b}{2}\log^2\left(\frac{a}{b}\right), \tag{3.3}$$

where $a, b > 0$ and $\lambda \in [0,1]$.

The inequality (3.3) gives an improvement of Young's inequality.

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