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FINITE AND INFINITE ORDER SOLUTIONS OF A CLASS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the growth of solutions of higher order linear differential equations where most of the coefficients have the same order and type with each other.

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1. INTRODUCTION

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory (see [6], [7]). In addition, for a non constant entire function $f : \mathbb{C} \to \mathbb{C}$, we will use the notations $\sigma(f)$, $\sigma_2(f)$ and $\tau(f)$ to denote respectively the order, hyper-order and type of f, defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r},$$

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

$$\tau(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\sigma}},$$

respectively, where $\sigma = \sigma(f)$ and $M(r,f) = \max_{|z|=r} |f(z)|$.

Given $\varepsilon \geq 0$ and $\theta_1, \theta_2 \in [0, 2\pi)$ such that $\varepsilon < \frac{\theta_2 - \theta_1}{2}$, we will use throughout this paper the following notations:

$$S(\varepsilon) = \{z \in \mathbb{C} : \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon\},$$

$$I(\varepsilon) = [\theta_1 + \varepsilon, \theta_2 - \varepsilon].$$

For $n \geq 2$, we consider the linear differential equation

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + \dots + A_0(z) f = 0 \tag{1.1}$$

where $A_0, A_1, \dots, A_{n-1}$ are entire functions with $A_0 \not\equiv 0$. It is well known that all solutions of (1.1) are entire functions. A classical result due to Wittich shows that all solutions of (1.1) are of finite order of growth if and only if all coefficients are polynomials. For a complete analysis of possible orders in the polynomial case, see [5]. If some (or all) of the coefficients are transcendental, it is natural to ask when and how many solutions of finite order may appear? Altough some partial answers were given to these questions (see, for example, [2], [3]), the problem remains open in its full generality.

In [3] Gundersen proved the following results.

Theorem 1.1. [3] *Let $A_0 \not\equiv 0$ and A_1 be entire functions such that for real constants α, β, θ_1, θ_2 with $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_0(z)| \geq \exp\left\{(1 + o(1))\,\alpha\,|z|^{\beta}\right\},$$

and

$$|A_1(z)| \leq \exp\left\{o(1)\,|z|^{\beta}\right\},$$

as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $f \not\equiv 0$ of the differential equation

$$f'' + A_1(z) f' + A_0(z) f = 0 \tag{1.2}$$

has infinite order.

Theorem 1.2. [3] *Let $A_0(z) \not\equiv 0$ and $A_1(z)$ be entire functions such that for real constants α, β, θ_1, θ_2 with $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_1(z)| \geq \exp\left\{(1 + o(1))\,\alpha\,|z|^{\beta}\right\}$$

and

$$|A_0(z)| \leq \exp\left\{o(1)\,|z|^{\beta}\right\}$$

as $z \to \infty$ with $\theta_1 \leq \arg z \leq \theta_2$. Given $\varepsilon > 0$ small enough. If f is a nontrivial solution of (1.2) of finite order, then the following conditions hold:
(i) There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Indeed,

$$|f(z) - b| \leq \exp\left\{-(1+o(1))\,\alpha\,|z|^{\beta}\right\}$$

for $z \in S(\varepsilon)$ with $|z|$ sufficiently large.
(ii) For each integer $k \geq 1$

$$\left|f^{(k)}(z)\right| \leq \exp\left\{-(1+o(1))\,\alpha\,|z|^{\beta}\right\}$$

for $z \in S(\varepsilon)$ with $|z|$ sufficiently large.

Theorem 1.1 and Theorem 1.2 have been improved and generalized to the higher order case by Belaidi, B., Hamouda, S. [1] and Laine, I., Yang, R. [8].

Recently in [9] , Tu, J. and Yi, C-F. investigated the case when most coefficients in (1.1) have the same order with each other and obtained the following result:

Theorem 1.3. [9] *Let $A_j(z)$ $(j = 0, ..., n-1)$ be entire functions satisfying $\sigma(A_0) = \sigma$, $\tau(A_0) = \tau$, $0 < \sigma < \infty$, $0 < \tau < \infty$, and let $\sigma(A_j) \leq \sigma$, $\tau(A_j) < \tau$ if $\sigma(A_j) = \sigma$ $(j = 1, ..., n-1)$, then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_2(f) = \sigma(A_0)$.*

In this paper, we will investigate the case when most coefficients in (1.1) have the same order and type with each other.

2. MAIN RESULTS

Theorem 2.1. *Let $A_0(z) \not\equiv 0, A_1(z), ..., A_{n-1}(z)$ be entire functions such that for real constants θ_1, θ_2 with $0 \leq \theta_1 < \theta_2 < 2\pi$ and for any $K > 0$*

$$\frac{\sum_{j=1}^{n-1} |A_j(z)| + 1}{|A_0(z)|} \leq \frac{1}{|z|^K} \tag{2.1}$$

for all $|z|$ sufficiently large with $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $f \not\equiv 0$ of (1.1) is of infinite order.

From Theorem 2.1 we get the following corollary

Corollary 2.2. *Let $P_1(z), ..., P_{n-1}(z)$ be polynomials and let $A(z)$ be a transcendental entire function with order $\sigma(A) = 0$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f^{(n)} + P_{n-1}(z)\,e^z f^{(n-1)} + ... + P_1(z)\,e^z f' + A(z)\,e^z f = 0,$$

is of infinite order.

Now we are going to present a counterpart of Theorem 2.1 by taking $A_s(z)$ instead of $A_0(z)$ as follows:

Theorem 2.3. *Let $A_0(z) \not\equiv 0, A_1(z), ..., A_{n-1}(z)$ be entire functions such that for real constants θ_1, θ_2 with $0 \leq \theta_1 < \theta_2 < 2\pi$ and for any $K > 0$*

$$\frac{\sum_{j=0 (j\neq s)}^{n-1} |A_j(z)| + 1}{|A_s(z)|} \leq \frac{1}{|z|^K} \tag{2.2}$$

for all $|z|$ sufficiently large with $\theta_1 \leq \arg z \leq \theta_2$, where $s \in \{1, ..., n-1\}$. Given $\varepsilon > 0$ small enough, if f is a transcendental solution of (1.1) of finite order $\sigma < \infty$, then the following

conditions hold:
(i) There exists $j \in \{0, ..., s-1\}$ *and a complex constant* $b_j \neq 0$ *such that* $f^{(j)}(z) \to b_j$ *as* $z \to \infty$ *in the sector* $S(\varepsilon)$. *More precisely, for any* $K > 0$ *we have*

$$\left| f^{(j)}(z) - b_j \right| \leq \frac{1}{|z|^K}$$

for all $z \in S(\varepsilon)$ *with* $|z|$ *sufficiently large.*
(ii) For each integer $m \geq j+1$, $f^{(m)}(z) \to 0$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *More precisely, for any* $K > 0$ *we have*

$$\left| f^{(m)}(z) \right| \leq \frac{1}{|z|^K}$$

for all $z \in S(\varepsilon)$ *with* $|z|$ *sufficiently large.*

Corollary 2.4. *Let* $A_0(z) \not\equiv 0, A_1(z), ..., A_{n-1}(z)$ *be entire functions such that, for some* $s \in \{1, ..., n-1\}$, $A_s(z)$ *is a transcendental entire function with* $\sigma(A_s) = 0$ *and for all* $j \neq s$ $A_j(z)$ *is a polynomial. If* f *is a transcendental solution of finite order of the differential equation*

$$f^{(n)} + A_{n-1}(z) e^z f^{(n-1)} + ... + A_s(z) e^z f^{(s)} + ... + A_0(z) e^z f = 0, \tag{2.3}$$

then the conditions (i) and (ii) of Theorem 2.3 hold.

Corollary 2.5. *Let* $A_0(z) \not\equiv 0, A_1(z), ..., A_{n-1}(z)$ *be entire functions such that for real constants* α, β, μ, θ_1, θ_2 *with* $0 \leq \beta < \alpha$, $\mu > 0$ *and* $0 \leq \theta_1 < \theta_2 < 2\pi$, *we have, for some* $s \in \{1, ..., n-1\}$,

$$|A_s(z)| \geq \exp_p \{\alpha |z|^\mu\} \tag{2.4}$$

and

$$|A_j(z)| \leq \exp_p \{\beta |z|^\mu\} \text{ for all } j \in \{0, 1, ..., n-1\} - \{s\} \tag{2.5}$$

for all $|z|$ *sufficiently large with* $\theta_1 \leq \arg z \leq \theta_2$, *where* $p \geq 1$ *is an integer,* $\left(\exp_1(z) = \exp(z) \text{ and } \exp_{k+1}(z) = \exp\{\exp_k(z)\} \text{ for } k \geq 1 \right)$. *If* f *is a transcendental solution of (1.1) of finite order, then the conditions (i) and (ii) of Theorem 2.3 hold.*

In order to prove these results, we need the following lemmas.

3. Preliminary lemmas

Lemma 3.1. [4] *Let* f *be a transcendental entire function of finite order* σ, *let* $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \geq 0$ $(i = 1, ..., m)$, *and let* $\epsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0, 2\pi)$ *that has linear measure zero, such that if* $\psi_0 \in [0, 2\pi) - E$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \geq R_0$, *and for all* $(k, j) \in \Gamma$, *we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma - 1 + \epsilon)}. \tag{3.1}$$

Remark 3.1. It is easy to show that Lemma 3.1 is valid in the case when f is a polynomial by taking $\sigma = 0$.

Lemma 3.2. *(*[3], [8]*) Let $f(z)$ be an entire function and suppose that $\left|f^{(k)}(z)\right|$ is unbounded on some ray $\arg z=\theta$. Then there exists an infinite sequence of points $z_j=r_je^{i\theta}$ $(j=1,2,\dots)$, where $r_j\to+\infty$, such that $f^{(k)}(z_j)\to\infty$ and*

$$\left|\frac{f^{(q)}(z_j)}{f^{(k)}(z_j)}\right|\leq\frac{1}{(k-q)!}(1+o(1))\left|z_j\right|^{k-q}$$

for all $q\in\{0,\dots,k-1\}$.

Lemma 3.3. *If f is an entire function such that for $K>1$ we have*

$$\left|f\left(re^{i\theta}\right)\right|\leq\frac{1}{r^K}$$

for all r sufficiently large, then $\int\limits_r^{\infty}\left|f\left(te^{i\theta}\right)\right|dt$ converges and we have

$$\int\limits_r^{\infty}\left|f\left(te^{i\theta}\right)\right|dt\leq\frac{1}{(K-1)\,r^{K-1}}$$

for all r sufficiently large.

Proof. It is easy to show that $\int_r^{\infty}\left|f\left(te^{i\theta}\right)\right|dt$ converges. For r large enough, we have

$$\int\limits_r^{\infty}\left|f\left(te^{i\theta}\right)\right|dt\leq\int\limits_r^{\infty}\frac{1}{t^K}dt=\frac{1}{(K-1)\,r^{K-1}}.$$

∎

4. PROOF OF THEOREMS

Proof of theorem 2.1. Suppose that $f\not\equiv0$ is a solution of (1.1) of finite order $\sigma(f)=\sigma<\infty$. From Lemma 3.1 and Remark 3.1, there exists a set $E\subset[0,2\pi)$ that has linear measure zero, such that if $\psi_0\in I(0)-E$, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right|\leq|z|^{k\sigma},\ \text{for all } k\in\{1,\dots,n\},\tag{4.1}$$

for all z such that $|z|$ sufficiently large and $\arg z=\psi_0$. From (1.1) we can write

$$\begin{aligned}1\ \leq\ &\frac{1}{|A_0(z)|}\left|\frac{f^{(n)}(z)}{f(z)}\right|+\frac{|A_{n-1}(z)|}{|A_0(z)|}\left|\frac{f^{(n-1)}(z)}{f(z)}\right|+\dots\\&+\frac{|A_1(z)|}{|A_0(z)|}\left|\frac{f'(z)}{f(z)}\right|.\end{aligned}\tag{4.2}$$

Using (2.1), (4.1) and (4.2) and taking the limit as $z\to\infty$ with $\arg z=\psi_0\in I(0)-E$, we get a contradiction. So, every solution $f\not\equiv0$ of (1.1) is of infinite order. ∎

Proof of theorem 2.2. First we prove that $f^{(s)}(z)$ is bounded in $S(\varepsilon)$, for $\varepsilon>0$ small enough. Given $\epsilon\in(0,1)$, from Lemma 3.1 it follows that there exists a set $E\subset[0,2\pi)$ that has linear measure zero, such that for all $k\in\{s+1,\dots,n\}$

$$\left|\frac{f^{(k)}(z)}{f^{(s)}(z)}\right|\leq|z|^{(k-s)(\sigma-1+\epsilon)}\leq|z|^{n\sigma},\tag{4.3}$$

for any z such that $\arg z \in I(0) - E$ and $|z|$ sufficiently large. If we suppose that $f^{(s)}(z)$ is unbounded on some ray $\arg z = \phi \in S(0) - E$, then by Lemma 3.2 there exists an infinite sequence of points $z_j = r_j e^{i\phi}$ $(j = 1, 2, ...)$, with $r_j \to +\infty$, such that $f^{(s)}(z_j) \to \infty$ and

$$\left|\frac{f^{(q)}(z_j)}{f^{(s)}(z_j)}\right| \leq \frac{1}{(s-q)!}(1 + o(1))|z_j|^{s-q} \leq |z_j|^n \tag{4.4}$$

for every $q \in \{0, ..., s-1\}$ and j large enough. From (1.1), we can write

$$\begin{aligned} 1 \leq \; & \frac{1}{|A_s(z)|}\left|\frac{f^{(n)}(z)}{f^{(s)}(z)}\right| + \frac{|A_{n-1}(z)|}{|A_s(z)|}\left|\frac{f^{(n-1)}(z)}{f^{(s)}(z)}\right| + ... \\ & ... + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}(z)}{f^{(s)}(z)}\right| + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}(z)}{f^{(s)}(z)}\right| + ... \\ & ... + \frac{|A_0(z)|}{|A_s(z)|}\left|\frac{f(z)}{f^{(s)}(z)}\right|. \end{aligned} \tag{4.5}$$

Combining (2.2), (4.3), (4.4) and (4.5) and letting $j \to +\infty$ we obtain a contradiction. Therefore, $f^{(s)}(z)$ remains bounded on all rays $\arg z = \phi \in I(0) - E$. By Phragmen-Lindelöf theorem, we conclude that $f^{(s)}(z)$ is bounded, say $\left|f^{(s)}(z)\right| \leq M$, in a whole sector $S\left(\frac{\varepsilon}{2}\right)$, for some $\varepsilon > 0$ small enough.

For an integer $m \leq s$, by integrating $s - m$ times along the line segment $[0, z]$ in $S\left(\frac{\varepsilon}{2}\right)$, we have

$$\begin{aligned} f^{(m)}(z) = \; & f^{(m)}(0) + f^{(m+1)}(0)z + ... \\ & + \frac{1}{(s-m-1)!}f^{(s-1)}(0)z^{s-m-1} + \\ & \int_0^z ... \int_0^z f^{(s)}\left(te^{i\phi}\right)dt...dt, \end{aligned}$$

and therefore we get

$$\left|f^{(m)}(z)\right| \leq M'|z|^{s-m}, \tag{4.6}$$

for a certain constant $M' > 0$.

Using (1.1), we can write

$$\begin{aligned} \left|f^{(s)}(z)\right| \leq \; & \frac{|f(z)|}{|A_s(z)|}\left(\left|\frac{f^{(n)}(z)}{f(z)}\right| + |A_{n-1}(z)|\left|\frac{f^{(n-1)}(z)}{f(z)}\right| + ... \right. \\ & ... + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}(z)}{f(z)}\right| + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}(z)}{f(z)}\right| + ... \\ & \left. ... + |A_1(z)|\left|\frac{f'(z)}{f(z)}\right| + |A_0(z)|\right). \end{aligned} \tag{4.7}$$

Using Lemma 3.1, (4.6) (for $m = 0$), (4.7) and the hypothesis (2.2), we conclude that for any $K > 0$

$$\left|f^{(s)}(z)\right| \leq \frac{1}{|z|^K} \tag{4.8}$$

for any z with $|z|$ sufficiently large and $\arg z = \phi \in I\left(\frac{\varepsilon}{2}\right) - E$.

For $m > s$, consider $z = re^{i\theta} \in S(\varepsilon)$ such that the circle $\Gamma(z)$ centered at z and of radius $\rho = ((m-s)!)^{1/(m-s)}$ is contained in $S\left(\frac{\varepsilon}{2}\right)$, i.e. consider $r \geq \rho / \sin\frac{\varepsilon}{2}$. By the Cauchy formula applied to the function $f^{(s)}(z)$ we have

$$f^{(m)}(z) = \frac{(m-s)!}{2\pi} \int\limits_{\Gamma(z)} \frac{f^{(s)}(\mu)}{(z-\mu)^{m-s+1}} d\mu,$$

and using (4.8), we get

$$\left|f^{(m)}(z)\right| \leq \frac{1}{|z|^K} \tag{4.9}$$

for any $K > 0$ and all $z \in S(\varepsilon)$ with $|z|$ sufficiently large.
Until now, we have proved the second assertion for $m \geq s$.
We start to prove the first assertion for $j = s-1$. Set

$$a_s = \int\limits_0^{+\infty} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt.$$

By (4.8), it is easy to see that $\int\limits_0^{+\infty} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt$ converges. Moreover, a_s is independent of θ, because by using (4.8), the integral of $f^{(s)}(\mu)$ over the arc $Re^{i\theta}$, $\theta \in (\phi, \varphi) \subset I\left(\frac{\varepsilon}{2}\right)$, tends to zero as $R \to \infty$. Define now $b_{s-1} = f^{(s-1)}(0) + a_s$, and suppose that $b_{s-1} \neq 0$. Let $z = re^{i\theta}$ (r large enough) be an arbitrary point in $S(\varepsilon)$. Then by applying (4.8) and Lemma 3.3, we get

$$\begin{aligned} \left|f^{(s-1)}(z) - b_{s-1}\right| &= \left|\int\limits_{+\infty}^{|z|} f^{(s)}\left(te^{i\theta}\right) e^{i\theta} dt\right| \\ &\leq \int\limits_{|z|}^{+\infty} \left|f^{(s)}\left(te^{i\theta}\right)\right| dt \leq \frac{1}{|z|^K} \end{aligned} \tag{4.10}$$

for any $K > 0$ and all $z \in S(\varepsilon)$ with $|z|$ sufficiently large. Thus, we have completed the proof in the case $b_{s-1} \neq 0$.
If $b_{s-1} = 0$, we define $a_{s-1} = \int\limits_0^{+\infty} f^{(s-1)}\left(te^{i\theta}\right) e^{i\theta} dt$ and $b_{s-2} = f^{(s-2)}(0) + a_{s-1}$ and we apply Lemma 3.3 with (4.10) to obtain, for any $K > 0$,

$$\left|f^{(s-2)}(z) - b_{s-2}\right| \leq \frac{1}{|z|^K}$$

for all $z \in S(\varepsilon)$ with $|z|$ sufficiently large. By the same method, if $b_{s-1} = b_{s-2} = ... = b_{j+1} = 0$ and $b_j \neq 0$ $(j \in \{0, ..., s-1\})$, then for any $K > 0$

$$\left|f^{(j)}(z) - b_j\right| \leq \frac{1}{|z|^K}$$

and

$$\left|f^{(m)}(z)\right| \leq \frac{1}{|z|^K} \quad (\text{ for all } m \geq j+1) \tag{4.11}$$

for all $z \in S(\varepsilon)$ with $|z|$ sufficiently large. Now it remains to show that the case $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$ is not possible. In this case we have, for any $K > 0$

$$\left|f^{(m)}(z)\right| \leq \frac{1}{|z|^K} \quad (\text{ for all } m \geq 0) \tag{4.12}$$

for all $z \in S(\varepsilon)$ with $|z|$ sufficiently large; i.e. for $m \geq 0$ and any $K > 0$, there exists $r_0(K,m) > 0$ such that if $|z| \geq r_0$ then $\left|f^{(m)}(z)\right| \leq \frac{1}{|z|^K}$. Now we take $z \in S(\varepsilon)$ such that $|z| \geq r_1(K) = \max\limits_{m=0,\ldots,s} r_0(K,m)$; we remark here that if z is fixed then (4.12) is valid for only some $K > 0$ and not for any $K > 0$. From (1.1) we can write

$$\begin{aligned} \frac{\left|f^{(s)}(z)\right|}{|f(z)|} \leq & \left(\frac{1}{|A_s(z)|}\frac{\left|f^{(n)}(z)\right|}{|f(z)|} + \frac{|A_{n-1}(z)|}{|A_s(z)|}\frac{\left|f^{(n-1)}(z)\right|}{|f(z)|} + \right. \\ & \ldots + \frac{|A_{s+1}(z)|}{|A_s(z)|}\left|\frac{f^{(s+1)}(z)}{f(z)}\right| + \frac{|A_{s-1}(z)|}{|A_s(z)|}\left|\frac{f^{(s-1)}(z)}{f(z)}\right| + \\ & \left. \ldots + \frac{|A_1(z)|}{|A_s(z)|}\frac{|f'(z)|}{|f(z)|} + \frac{|A_0(z)|}{|A_s(z)|}\right), \end{aligned} \tag{4.13}$$

and by using (2.2) and Lemma 3.1 in (4.13), we obtain

$$\frac{\left|f^{(s)}(z)\right|}{|f(z)|} \leq \frac{1}{|z|^K}; \tag{4.14}$$

and by using also (4.12) for $m = 0$ in (4.14), we get

$$\left|f^{(s)}(z)\right| \leq \frac{1}{|z|^{2K}} \tag{4.15}$$

for $|z| \geq r_1(K)$ and $\arg z \in I(\varepsilon) - E$, hence in $S\left(\varepsilon + \frac{\varepsilon}{2}\right)$ by Phragmén-Lindelöf principle. Repeating the reasoning of (4.10)–(4.11) with (4.15), we obtain

$$|f(z)| \leq \frac{1}{|z|^{2K}};$$

and by combining with (4.14), we get

$$\left|f^{(s)}(z)\right| \leq \frac{1}{|z|^{3K}}$$

in $S\left(\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2}\right)$. Inductively, by the same reasoning, after $(T-1)$ steps, we obtain

$$\left|f^{(s)}(z)\right| \leq \frac{1}{|z|^{TK}} \tag{4.16}$$

in $S\left(\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^{T-1}}\right) = S\left(\varepsilon\left(2 - \frac{1}{2^{T-1}}\right)\right)$ with $|z| \geq r_1(K)$. Thus we have proved, in this special case of $b_{s-1} = b_{s-2} = \ldots = b_0 = 0$, that (4.16) is valid in $S(2\varepsilon)$ for all $T \in \mathbb{N}$, provided $|z| \geq r_1$. Now if we fix a point $z \in S(2\varepsilon)$ with $|z| \geq r_1$, then by taking $T \to \infty$ in (4.16) we will see that $f^{(s)}$ vanishes at this point. Thus, we conclude that $f^{(s)}$ vanishes identically on all $z \in S(2\varepsilon)$ with $|z| \geq r_1$. Therefore, by the standard uniqueness theorem of entire functions, f has to be a polynomial, a contradiction. ∎

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