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TURAN TYPE INEQUALITIES FOR SOME SPECIAL FUNCTIONS

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ABSTRACT. In this paper new results concerning the q-polygamma and q-zeta functions are presented. Other generealizations of some known results are also obtained.

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1. INTRODUCTION

Let c be a complex number, the $q-$shifted factorial are defined by

$$(c;q)_0 = 1, \quad (c;q)_n = \prod_{k=o}^{n-1}(1-cq^k), \quad n = 1, 2, ... \tag{1.1}$$

$$(c;q)_\infty = \lim_{n\to\infty}(c;q)_n = \prod_{k=0}^{\infty}(1-cq^k). \tag{1.2}$$

For x complex we denote

$$[x]_q = \frac{1-q^x}{1-q}. \tag{1.3}$$

The $q-$Jackson integrals from 0 to c are defined by [5, 6]

$$\int_0^c f(x)d_qx = (1-q)c\sum_{n=0}^{\infty} f(cq^n)q^n, \tag{1.4}$$

and

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{1.5}$$

provided the sum converges absolutely.

The $q-$analogue of the Gamma function is defined by Jakson [6] as follows

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \neq 0, -1, -2, ... \quad , \tag{1.6}$$

and it is satisfying the following

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \quad \Gamma_q(1) = 1, \tag{1.7}$$

and tends to $\Gamma(x)$ as $q \to 1$.
The $q-$integral representation of the Gamma function is (see [2, 4]) as follows

$$\Gamma_q(x) = K_q(x)\int_0^\infty t^{x-1}e_q^{-t}d_qt, \tag{1.8}$$

where

$$e_q^t = \frac{1}{((1-q)t;q)_\infty},$$

and

$$K_q(t) = \frac{(1-q)^{-x}}{1+(1-q)^{-1}} \times \frac{(-(1-q);q)_\infty(-(1-q)^{-1};q)_\infty}{(-(1-q)q^t;q)_\infty(-(1-q)^{-1}q^{1-t};q)_\infty}.$$

The $q-$analogue of the psi function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is defined as the logarithmic derivative of the $q-$gamma function, that is $\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}$.

From (1.6), we obtain for $x > 0$

$$\begin{aligned}
\psi_q(x) &= -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}} \\
&= -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^{n}}, \\
&= -\ln(1-q) + \frac{\ln q}{1-q} \int_0^q \frac{t^{x-1}}{1-t} d_q t. \qquad (1.9)
\end{aligned}$$

For $x > 0$, we put

$$\alpha(x) = \frac{\ln(x)}{\ln(q)} - E\left(\frac{\ln(x)}{\ln(q)}\right)$$

and

$$\{x\}_q = \frac{[x]_q}{q^{x+\alpha([x]_q)}},$$

where $E\left(\frac{\ln(x)}{\ln(q)}\right)$ is the integer part of $\frac{\ln(x)}{\ln(q)}$.

The $q-$Zeta function is defined (see[3]) as

$$\xi_q(s) = \sum_{n=1}^{\infty} \frac{1}{\{n\}_q^s} = \sum_{n=1}^{\infty} \frac{q^{(n+\alpha([n]_q))s}}{[n]_q^s}. \qquad (1.10)$$

It has been proved in [3], that the $q-$analogue of the classical Riemann Zeta function is

$$\xi_q(s) = \frac{1}{\widetilde{\Gamma}(s)} \int_0^{\infty} t^{s-1} Z_q(t) d_q t, \quad s \in C, \quad R(s) > 1,$$

where for all $t > 0$,

$$Z_q(t) = \sum_{n=1}^{\infty} e_q^{-\{n\}_q t} \quad and \quad \widetilde{\Gamma}_q(t) = \frac{\Gamma_q(t)}{K_q(t)}.$$

In his paper, K. Brahim [1], gave the following results:

Theorem 1.1. *For $n = 1, 2, ...$, let $\psi_{q,n} = \psi_q^{(n)}$, the $n-$th derivative of the function ψ_q. Then*

$$\psi_{q,m}(x)\psi_{q,n}(x) \geq \psi^2_{q,\frac{m+n}{2}}(x), \qquad (1.11)$$

where $\frac{m+n}{2}$ is an integer.

Theorem 1.2. *For all $s > 1$ we have*

$$[s+1]_q \frac{\xi_q(s)}{\xi_q(s+1)} \geq q[s]_q \frac{\xi_q(s+1)}{\xi_q(s+2)}. \qquad (1.12)$$

2. IMPROVEMENT AND GENERALIZATION

We start by giving a short easy proof for the following Lemma, without go to the $q-$definition for functions f and g.

Lemma 2.1. [1]. *Let $a \in R_+ \cup \{\infty\}$ and let f and g be two nonnegative functions. Then*

$$\left(\int_0^b g(x) f^{\frac{m+n}{2}}(x) d_q x\right)^2 \le \left(\int_0^b g(x) f^m(x) d_q x\right)\left(\int_0^b g(x) f^n(x) d_q x\right). \tag{2.1}$$

Proof. We have, by extending the limits of integration between a and b, $0 \le a < b \le \infty$,

$$\int_a^b \left(\sqrt{g(x) f^m(x)} \sqrt{\int_a^b g(x) f^n(x) d_q x} - \sqrt{g(x) f^n(x)} \sqrt{\int_a^b g(x) f^m(x) d_q x}\right)^2 d_q x \ge 0.$$

Opening the above inequality gives

$$\begin{aligned} & 2\int_a^b g(x) f^m(x) d_q x \int_a^b g(x) f^n(x) d_q x \\ \ge\ & 2\int_a^b g(x) f^{\frac{m+n}{2}}(x) d_q x \sqrt{\int_a^b g(x) f^n(x) d_q x} \times \sqrt{\int_a^b g(x) f^m(x) d_q x}. \end{aligned}$$

Canceling and squaring, the result follows.
The above inequality can also be generalized as follows:
By using the AG-inequality

$$cd \le \frac{c^p}{p} + \frac{d^q}{q}, \quad p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad c, d \ge 0. \tag{2.2}$$

In fact, we have, by putting

$$c = g^{\frac{1}{p}}(x) f^{\frac{m}{p}}(x) / \left(\int_a^b g(x) f^m(x) d_q x\right)^{1/p},$$

$$d = g^{\frac{1}{q}}(x) f^{\frac{n}{q}}(x) / \left(\int_a^b g(x) f^n(x) d_q x\right)^{1/q},$$

we obtain, after multiplying (2.2) by $d_q x$ and then integrating from a to b,

$$\int_a^b g(x) f^{\frac{m}{p}+\frac{n}{q}}(x) d_q x \le \left(\int_a^b g(x) f^m(x) d_q x\right)^{1/p} \left(\int_a^b g(x) f^n(x) d_q x\right)^{1/q}. \tag{2.3}$$

Inequality (2.1) follows from inequality (2.3) by putting $p = q = 2$.

For more extension, we give the following. ∎

Theorem 2.2. *For $n = 1, 2, \ldots$, let $\psi_{q,n} = \psi_q^{(n)}$, the $n-$th derivative of the function ψ_q. Then*

$$\psi_{q, \frac{m}{s}+\frac{n}{t}}\left(\frac{x}{s} + \frac{y}{t}\right) \le \psi_{q,m}^{1/s}(x) \psi_{q,n}^{1/t}(y), \tag{2.4}$$

where $\frac{m+n}{2}$ is an integer, $s > 1$, $\frac{1}{s} + \frac{1}{t} = 1$.

Proof. Let m and n be two integers of the same parity. From (1.9), it follows that

$$\psi_{q,n}(x) = \frac{\ln q}{1-q}\int_0^q \frac{(\ln(1/u))^n u^{x-1}}{1-u} d_q u.$$

By above, we have

$$\begin{aligned}
\psi_{q,\frac{m}{s}+\frac{n}{t}}\left(\frac{x}{s}+\frac{y}{t}\right) &= \frac{\ln q}{1-q}\int_0^q \frac{(\ln(1/u))^{\frac{m}{s}+\frac{n}{t}} u^{\frac{x}{s}+\frac{y}{t}-1}}{1-u} d_q u \\
&= \frac{\ln q}{1-q}\int_0^q \frac{(\ln(1/u))^{\frac{m}{s}} u^{\frac{x-1}{s}}}{(1-u)^{\frac{1}{s}}}\frac{(\ln(1/u))^{\frac{n}{t}} u^{\frac{y-1}{t}}}{(1-u)^{\frac{1}{t}}} d_q u \\
&\le \frac{\ln q}{1-q}\left(\int_0^q \frac{(\ln(1/u))^m u^{x-1}}{1-u} d_q u\right)^{1/s}\left(\int_0^q \frac{(\ln(1/u))^n u^{y-1}}{1-u} d_q u\right)^{1/t} \\
&= \psi_{q,m}^{1/s}(x)\psi_{q,n}^{1/t}(y).
\end{aligned}$$

∎

Remark 2.1. On putting $y = x$ in Theorem 2.2, we obtain a generalization for Theorem 1.1.

Another type via Minkowski's inequality is the following.

Theorem 2.3. *For $n = 1, 2, ...$, let $\psi_{q,n} = \psi_q^{(n)}$, the $n-$th derivative of the function ψ_q. Then*

$$(\psi_{q,m}(x) + \psi_{q,n}(y))^{1/p} \le \psi_{q,m}^{1/p}(x) + \psi_{q,n}^{1/p}(y), \tag{2.5}$$

where $\frac{m+n}{2}$ is an integer, $p \ge 1$.

Proof. Since

$$(a+b)^p \ge a^p + b^p, \quad a, b \ge 0, \quad p \ge 1,$$

then, we have, via Minkowski's inequality

$$\begin{aligned}
&(\psi_{q,m}(x) + \psi_{q,n}(y))^{1/p} \\
&= \left(\frac{\ln q}{1-q}\right)^{1/p}\left(\int_0^q \left(\left(\frac{t^{\frac{x-1}{p}}(\ln(1/t))^{m/p}}{(1-t)^{1/p}}\right)^p + \left(\frac{t^{\frac{y-1}{p}}(\ln(1/t))^{n/p}}{(1-t)^{1/p}}\right)^p\right) d_q t\right)^{1/p} \\
&\le \left(\frac{\ln q}{1-q}\right)^{1/p}\left(\int_0^q \left(\frac{t^{\frac{x-1}{p}}(\ln(1/t))^{m/p}}{(1-t)^{1/p}} + \frac{t^{\frac{y-1}{p}}(\ln(1/t))^{n/p}}{(1-t)^{1/p}}\right)^p d_q t\right)^{1/p} \\
&\le \left(\frac{\ln q}{1-q}\right)^{1/p}\left(\left(\int_0^q \left(\frac{t^{\frac{x-1}{p}}(\ln(1/t))^{m/p}}{(1-t)^{1/p}}\right)^p d_q t\right)^{1/p} + \left(\left(\int_0^q \left(\frac{t^{\frac{y-1}{p}}(\ln(1/t))^{n/p}}{(1-t)^{1/p}}\right)^p d_q t\right)^{1/p}\right)\right) \\
&= \left(\frac{\ln q}{1-q}\right)^{1/p}\left(\left(\int_0^q \frac{t^{x-1}(\ln(1/t))^m}{1-t} d_q t\right)^{1/p} + \left(\int_0^q \frac{t^{y-1}(\ln(1/t))^n}{1-t} d_q t\right)^{1/p}\right) \\
&= \psi_{q,m}^{1/p}(x) + \psi_{q,n}^{1/p}(y).
\end{aligned}$$

∎

Theorem 2.4. *For $x, y > 1$ we have*

$$\xi_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right) \le \frac{(\widetilde{\Gamma}_q(x))^{1/s}(\widetilde{\Gamma}_q(y))^{1/t}}{\widetilde{\Gamma}_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right)}\xi_q^{1/s}(x)\xi_q^{1/t}(y+2) \tag{2.6}$$

Proof. For $x, y > 1$, we have

$$\begin{aligned}
\xi_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right) &= \frac{1}{\widetilde{\Gamma}_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right)}\int_0^\infty u^{\frac{x-1}{s}+\frac{y+1}{t}}Z_q(u)d_qu \\
&= \frac{1}{\widetilde{\Gamma}_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right)}\int_0^\infty u^{\frac{x-1}{s}}Z_q^{1/s}(u)u^{\frac{y+1}{t}}Z_q^{1/t}(u)d_qu \\
&\le \frac{1}{\widetilde{\Gamma}_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right)}\left(\int_0^\infty u^{x-1}Z_q(u)d_qu\right)^{1/s}\left(\int_0^\infty u^{y+1}Z_q(u)d_qu\right)^{1/t} \\
&= \frac{(\widetilde{\Gamma}_q(x))^{1/s}(\widetilde{\Gamma}_q(y))^{1/t}}{\widetilde{\Gamma}_q\left(\frac{x-1}{s}+\frac{y+1}{t}\right)}\xi_q^{1/s}(x)\xi_q^{1/t}(y+2).
\end{aligned}$$

∎

Remark 2.2. On putting $y = x$, in Theorem 2.4 we get a generalization for Theorem 1.2.

3. MONOTONICITY

Theorem 3.1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma_q^a(1+bx)}{\Gamma_q^b(1+ax)}, \quad \forall x > 0, \quad ab > 0, \tag{3.1}$$

in which $1 + ax > 0, 1 + bx > 0$, then f is nondecreasing for $a \ge b$ and nonincreasing for $a \le b$.

Proof. We have

$$g(x) := \ln f(x) = a \ln \Gamma_q(1+bx) - b\ln\Gamma_q(1+ax,)$$

which implies

$$\begin{aligned}
g'(x) &= \frac{f'(x)}{f(x)} = ab\frac{\Gamma_q'(1+bx)}{\Gamma_q(1+bx)} - ab\frac{\Gamma_q'(1+ax)}{\Gamma_q(1+ax)} \\
&= ab\psi_q(1+bx) - ba\psi_q(1+ax) \\
&= ab\ln q\sum_{n=1}^\infty \frac{q^{n(1+bx)}-q^{n(1+ax)}}{1-q^n} \ge 0, \quad for \quad a \ge b.
\end{aligned}$$

Therefore g is nondecreasing, and hence $f(x) = e^{g(x)}$ is nondecreasing.
∎

Corollary 3.2. *For all $x \in [0, 1], a > b, ab > 0,$*

$$1 \le \frac{\Gamma_q^a(1+bx)}{\Gamma_q^b(1+ax)} \le \frac{\Gamma_q^a(1+b)}{\Gamma_q^b(1+a)}. \tag{3.2}$$

If $a < b$, (3.2) reverses.

Proof. The proof follows by applying Theorem 3.1 twice noticing that $\Gamma_q(1) = 1$. ∎

Corollary 3.3. *For all* $x, y \in [0, \infty), x \leq y, a > b, ab > 0,$

$$\frac{\Gamma_q^a(1+bx)}{\Gamma_q^b(1+ax)} \leq \frac{\Gamma_q^a(1+by)}{\Gamma_q^b(1+ay)}. \tag{3.3}$$

If $x \geq y, a < b, (3.2)$ *reverses.*

Proof. The proof follows by applying Theorem 3.1. ∎

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