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GOOD AND SPECIAL WEAKLY PICARD OPERATORS FOR THE STANCU OPERATORS WITH MODIFIED COEFFICIENTS

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ABSTRACT. In this paper some properties of good and special weakly Picard operators for the Stancu operators with modified coefficients are obtained. In the study of the sequence of iterates of these operators, we obtain the property of dual monotone iteration.

Key words and phrases: Contraction principle, Good and special weakly Picard operators, Monotone iteration, Stancu operators.

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1. INTRODUCTION

Through this paper we investigate some properties of the iterates of Stancu's operators with modified coefficients of uniform approximation from fixed point theory point of view. We obtain that these operators are good and special weakly Picard operators of type M. The notions of good and special weakly Picard operators are defined by I.A. Rus in [7] and the good and special convergence of type M of the sequence of successive approximation in metric space was introduced by L. D'Apuzzo in [2].

The Stancu operators with modified coefficients was introduced in [8], where was obtained properties about monotonicity, uniform convergence and error estimation in the approximation. The estimations are of Popoviciu, Lorentz and Voronovskaja type. Using the contraction principle, O. Agratini obtains in [1], the convergence of the iterates of Stancu's operators with modified coefficients giving the structure of the fixed point set.

Definition 1.1. ([5], [6], [7]) Let (X, d) be a metric space.
1) An operator $A : X \to X$ is weakly Picard operator (briefly WPO) if the sequence of successive approximations $(A^m(x_0))_{m\in\mathbb{N}}$ converges for all $x_0 \in X$ and the limit (which may depend on x_0) is a fixed point of A.
2) If the operator $A : X \to X$ is WPO and $F_A = \{x^*\}$, then by definition the operator A is Picard operator (briefly PO).

Theorem 1.1. *(Characterization theorem* - [5], [6], [7]*)An operator $A : X \to X$ is WPO if and only if there exits a partition of X, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, such that:*
(a) $X_\lambda \in I(A)$, $\forall\lambda \in \Lambda$;
(b) $A|_{X_\lambda} : X_\lambda \to X_\lambda$ is PO, $\forall\lambda \in \Lambda$.

Definition 1.2. Let (X, d) be a metric space and $A : X \to X$ a WPO.
1) $A : X \to X$ is good WPO, if the series $\sum_{m=1}^{\infty} d(A^{m-1}(x), A^m(x))$ converges, for all $x \in X$ (see [7]). In case that the sequence $(d(A^{m-1}(x), A^m(x)))_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, the operator A is good WPO of type M (see [2]).
2) $A : X \to X$ is special WPO, if the series $\sum_{m=1}^{\infty} d(A^m(x), A^\infty(x))$ converges, for all $x \in X$ (see [7]). When the sequence $(d(A^m(x), A^\infty(x)))_{m\in\mathbb{N}^*}$ is strictly decreasing for all $x \in X$, A is special WPO of type M (see [2]).

In what follows we investigate the good and special WPO property for the Stancu operators with modified coefficients. On the other hand, in this paper, we obtain some dual monotonicity properties of the sequence of iterates of Stancu's operators with modified coefficients.

2. PRELIMINARIES

It is well known that the Stancu operators [8] are defined by:

$$S_{n,\alpha}(f)(x) := \sum_{k=0}^{n} w_{n,k,\alpha}(x) f\left(\frac{k}{n}\right), \; f \in C[0,1], \; x \in [0,1] \tag{2.1}$$

where $w_{n,k,\alpha}(x) := \frac{\binom{n}{k} x^{[k,-\alpha]}(1-x)^{[n-k,\alpha]}}{1^{[n,-\alpha]}}$, $\forall\, k = \overline{1,n}$, represent the fundamental polynomials of Stancu of n degree. Here $y^{[m,-\alpha]}$ stands for the generalized factorial power with the step $-\alpha$, $y^{[0,-\alpha]} := 1$ and $y^{[m,-\alpha]} := y(y+\alpha)...(y+(m-1)\alpha)$, $m \in \mathbb{N}$.

The fundamental polynomials of Stancu have the following properties:

1) $\sum_{k=0}^{n} w_{n,k,\alpha}(x) = 1,\ \forall\, x \in [0,1];$

2) $w_{n,k,\alpha}(0) = 0,\ \forall\, k \in \overline{1,n}$ and $w_{n,0,\alpha}(0) = 1$

3) $w_{n,k,\alpha}(1) = 0,\ \forall\, k \in \overline{1,n-1}$ and $w_{n,0,\alpha}(1) = 1$

From the paper [8], we have the following properties of Stancu's operators with modified coefficients:

1) $S_{n,\alpha}$ is an linear and positive operator;

2) $S_{n,\alpha}(e_0)(x) = 1,\ S_{n,\alpha}(e_1)(x) = x,\ \forall\, x \in [0,1];$

3) $S_{n,\alpha}(f)(0) = f(0)$ and $S_{n,\alpha}(f)(1) = f(1), \forall\, f \in C[0,1]\,.$

Theorem 2.1. *(* [1]*) The Stancu operators with modified coefficients are WPO and for any* $n \in \mathbb{N}^*$, $S^{\infty}_{n,\alpha}(f) = \varphi^*_f,\ \forall\, f \in C[0,1]$ *where the function* $\varphi^*_f \in C[0,1]$ *is given by:*

$$\varphi^*_f(x) = f(0) + [f(1) - f(0)]\, x,\ \forall\, x \in [0,1]$$

The convergence exists on the space $(C[0,1]\,, \|\cdot\|_C)$.

In the application of Characterization theorem of WPO, it was used the following partition of $C[0,1]$:

$$C[0,1] := \bigcup_{\delta,\beta\in\mathbb{R}} X_{\delta,\beta}$$

where $X_{\delta,\beta} := \{f \in C[0,1] : f(0) = \delta,\ f(1) = \beta\}\,,\ \delta, \beta \in \mathbb{R}.$

Proposition 2.2. *(*[1]*) The Stancu operators with modified coefficients satisfied the following contraction property relative to above partition:*

$$\|S_{n,\alpha}(f) - S_{n,\alpha}(g)\|_C \leqslant \left(1 - \tfrac{1}{2^{n-1}.1^{[n,-\alpha]}}\right) \|f - g\|_C\,,\ \forall\, f, g \in X_{\delta,\beta}, \forall\, \delta, \beta \in \mathbb{R}$$

3. MAIN RESULTS

In this section, we shall investigate some properties of the iterates Stancu operators with modified coefficients in the sense of good and special convergence.

Theorem 3.1. *The Stancu operators with modified coefficients* $S_{n,\alpha}$ *are special WPO and good WPO of type M on* $C[0,1]$.

Proof. From Theorem 2.1, $S_{n,\alpha}$ is WPO.

Let $f \in C[0,1]$. Then $f \in X_{f(0),f(1)}$ and according to (2) we infer that $S_{n,\alpha}$ is contraction on $X_{f(0),f(1)}$. So, the operator $S_{n,\alpha}$ is special WPO of type M on $X_{f(0),f(1)}$. Finally, we get that $S_{n,\alpha}$ is special WPO of type M on $C[0,1]$.

Since in [4] was proved that any special WPO is good WPO, then we have that $S_{n,\alpha}$ is good WPO of type M on $C[0,1]$. ∎

Remark 3.1. Using the inequality (2) we obtain the estimation:

$$\begin{aligned} \left|S^1_{n,\alpha}(f)(x) - f^*(x)\right| &= \left|S^1_{n,\alpha}(f)(x) - S^1_{n,\alpha}(f^*(x))(x)\right| \leqslant \\ &\leqslant \left(1 - \tfrac{1}{2^{n-1}.1^{[n,-\alpha]}}\right) |f(x) - f^*(x)| = \\ &= \left(1 - \tfrac{1}{2^{n-1}.1^{[n,-\alpha]}}\right) |f(x) - \{f(0) + [f(1) - f(0)]\, x\}| \leqslant \\ &\leqslant 2a \cdot \left(1 - \tfrac{1}{2^{n-1}.1^{[n,-\alpha]}}\right) \end{aligned}$$

where $a = diam\,(f([0,1])) = \max\{|f(x) - f(y)| : x, y \in [0,1]\}$

By induction, for $m \in \mathbb{N}^*$ follows:

$$\left|S^m_{n,\alpha}(f)(x) - f^*(x)\right| \leqslant 2a \cdot \left(1 - \tfrac{1}{2^{n-1}.1^{[n,-\alpha]}}\right)^m$$

for all $x \in [0,1]$.

Consequently,

$\sum\limits_{m=1}^{\infty} \left| S_{n,\alpha}^{m}(f)(x) - f^{*}(x) \right| \leqslant 2a \cdot \left(2^{n-1} \cdot 1^{[n,-\alpha]} - 1 \right)$, for all $x \in [0,1]$.

Remark 3.2. The special and good WPO property of Stancu operators with modified coefficients can be obtained directly like this:

$\left| S_{n,\alpha}^{1}(f)(x) - S_{n,\alpha}^{0}(f)(x) \right| = \left| \sum\limits_{k=0}^{n} w_{n,k,\alpha}(x) f\left(\frac{k}{n}\right) - f(x) \right| =$

$= \left| \sum\limits_{k=0}^{n} w_{n,k,\alpha}(x) \left[f\left(\frac{k}{n}\right) - f(x) \right] \right| \leqslant a \cdot \sum\limits_{k=0}^{n} w_{n,k,\alpha}(x) = a,$

where $a = diam\,(f([0,1]))$.

By induction follows:

$\left| S_{n,\alpha}^{m}(f)(x) - S_{n,\alpha}^{m-1}(f)(x) \right| = \left| S_{n,\alpha}^{1}\left(S_{n,\alpha}^{m-1}(f)(x) \right) - S_{n,\alpha}^{1}\left(S_{n,\alpha}^{m-2}(f)(x) \right) \right| \leqslant$

$\leqslant \left(1 - \frac{1}{2^{n-1} \cdot 1^{[n,-\alpha]}} \right)^{m-1} \cdot a.$

Then, $\sum\limits_{m=1}^{\infty} \left| S_{n,\alpha}^{m}(f)(x) - S_{n,\alpha}^{m-1}(f)(x) \right| \leqslant a \cdot 2^{n-1},\ \forall\, x \in [0,1],\ f \in C[0,1].$

Remark 3.3. Let (X,d) be a metric space and $A : X \to X$ a WPO. We can say that A is special (good) WPO with uniform convergence on a closed subset $Y \subset X$, if the series: $\sum\limits_{m=1}^{\infty} d\left(A^{m}(x), A^{\infty}(x) \right)$ (respectively, $\sum\limits_{m=1}^{\infty} d\left(A^{m}(x), A^{m-1}(x) \right)$) uniformly converges on Y as series of functions by $x \in Y \subset X$.

Let $CV_{+}[0,1] = \{ f \in C[0,1] |\, f -$ convex and $f(x) \geqslant 0,\ \forall\, x \in [0,1] \}$. We can see that for any $f \in CV_{+}[0,1] \cap X_{\delta,\beta}$, $\delta, \beta \in \mathbb{R}$, we have $diam\,(f([0,1])) \leqslant \max(\delta, \beta)$ and using the estimations:

$d\left(S_{n,\alpha}^{m}(f), f^{*} \right) \leqslant 2 \left(1 - \frac{1}{2^{n-1} \cdot 1^{[n,-\alpha]}} \right)^{m} \cdot diam\,(f([0,1])) \leqslant$

$\leqslant 2 \left(1 - \frac{1}{2^{n-1} \cdot 1^{[n,-\alpha]}} \right)^{m} \cdot \max(\delta, \beta)$ and

$d\left(S_{n,\alpha}^{m}(f), S_{n,\alpha}^{m-1}(f) \right) \leqslant \left(1 - \frac{1}{2^{n-1} \cdot 1^{[n,-\alpha]}} \right)^{m-1} \cdot diam\,(f([0,1])) \leqslant$

$\leqslant \left(1 - \frac{1}{2^{n-1} \cdot 1^{[n,-\alpha]}} \right)^{m-1} \cdot \max(\delta, \beta)$

By the Weierstrass criterion of uniform convergence follows that the series:

$$\sum_{m=1}^{\infty} d\left(S_{n,\alpha}^{m}(f), S_{n,\alpha}^{\infty}(f) \right) \text{ and } \sum_{m=1}^{\infty} d\left(S_{n,\alpha}^{m}(f), S_{n,\alpha}^{m-1}(f) \right)$$

uniformly converges on each $CV_{+}[0,1] \cap X_{\delta,\beta}$, $\delta, \beta \in \mathbb{R}$.

So, as above, we infer that $S_{n,\alpha}$ are special and good WPO with uniform convergence on each $CV_{+}[0,1] \cap X_{\delta,\beta}$, $\delta, \beta \in \mathbb{R}$.

In what follows, we recall some notions which are introduced in the paper [3].

Definition 3.1. ([3]) Let $(X, d, \leqslant)$ be an ordered metric space and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ sequences in X such that $x_n \to x$ and $y_n \to y$ for $n \to \infty$. We say that the pair $\left((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \right)$ is dual monotone convergent if and only if $x_n \leqslant x_{n+1}$, $y_{n+1} \geq y_n$ (or $x_n \geqslant x_{n+1}$, $y_{n+1} \leq y_n$), $\forall n \in \mathbb{N}$ and $x_0 = y,\ y_0 = x$.

Definition 3.2. ([3])Let $(X, d, \leqslant)$ be an ordered metric space and $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ sequences of operators from X to X. The pair $\left((A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \right)$ is dual monotone convergence if and only if for any $x \in X$, the pair $\left((A_n(x))_{n\in\mathbb{N}}, (B_n(x))_{n\in\mathbb{N}} \right)$ is dual monotone convergent. For closed $Y \subset X$ and $A_n : Y \to X,\ B_n : Y \to X$, $n \in \mathbb{N}$, the dual monotone convergence of the pair $\left((A_n)_{n\in\mathbb{N}}, (B_n)_{n\in\mathbb{N}} \right)$ on Y is defined in the same way.

Definition 3.3. ([3]) Let $(X, d, \leqslant)$ be an ordered metric space and $Y \subset X$ closed set. We say that the sequence of operators $(A_n)_{n\in\mathbb{N}}$, $A_n : Y \to X$, $n \in \mathbb{N}$, have dual monotone iteration on Y if and only if there exist $N_0 \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $n \geqslant N_0$, the pair $\left((A_k)_{k\in\mathbb{N}^*}, (A_n^m)_{m\in\mathbb{N}}\right)$ is dual monotone convergent on Y.

Theorem 3.2. *The Stancu operators with modified coefficients have monotone iteration on Y, where: $Y = \{f \in C[0,1] | f - \text{convex}\}$ and the sequence $(S_{n,\alpha})_{n\in\mathbb{N}^*}$ have dual monotone iteration on Y.*

Proof. It is known that $C[0,1]$ is ordered complete metric space.
Using the Jensen's inequality we infer that Y is closed in $C[0,1]$.
Again, from the Jensen's inequality, for f convex follows that:

$$f\left(\sum_{k=0}^{n} w_{n,k,\alpha}(x) \cdot \frac{k}{n}\right) = f\left(\sum_{k=0}^{n} \frac{\binom{n}{k} x^{[k,-\alpha]}(1-x)^{[n-k,\alpha]}}{1^{[n,-\alpha]}} \cdot \frac{k}{n}\right) \leqslant$$

$\leqslant \sum_{k=0}^{n} w_{n,k,\alpha}(x) \cdot f\left(\frac{k}{n}\right) = S_{n,k}(f)(x)$, $\forall\, x \in [0,1]$ because $\sum_{k=0}^{n} w_{n,k,\alpha}(x) \geqslant 0$ and $\sum_{k=0}^{n} w_{n,k,\alpha}(x) = 1$, $\forall\, x \in [0,1]$.

Thus, because $S_{n,\alpha}(e_1) = e_1$, we have: $f(x) \leqslant S_{n,\alpha}(f)(x)$, $\forall\, x \in [0,1]$.

Consequently,

$$f(x) \leqslant S_{n,\alpha}^1(f)(x) \leqslant \ldots \leqslant S_{n,\alpha}^m(f)(x) \leqslant S_{n,\alpha}^{m+1}(f)(x) \leqslant \ldots \leqslant$$

$\leqslant S_{n,\alpha}^\infty(f)(x) = f(0) + [f(1) - f(0)]x = S_{1,\alpha}(f)(x)$, $\forall\, x \in [0,1]$, $\forall\, f \in Y$, which means monotone iteration for $\left(S_{n,\alpha}^m\right)_{m\in\mathbb{N}^*}$.

So, the sequence $\left(S_{n,\alpha}^m\right)_{m\in\mathbb{N}^*}$ is creasing on Y. The sequence $(S_{n,\alpha}(f))_{n\in\mathbb{N}^*}$ is decreasing for f convex (see [8]). It is known that $S_{n,\alpha}(f) \xrightarrow{unif} f$, $\forall\, f \in C[0,1]$, $k \to \infty$ (see [8]) and

$$S_{1,\alpha}(f)(x) = f^*(x) = f(0) + [f(1) - f(0)]x, \ \forall\, x \in [0,1]$$

Moreover, for $f \in Y$ we have:

$$f(x) \leqslant S_{k+1,\alpha}(f)(x) \leqslant S_{k,\alpha}(f)(x)$$

and for any $n \in \mathbb{N}^*$ fixed follows,

$$S_{n,\alpha}^0(f)(x) = f(x), \ S_{n,\alpha}^m(f)(x) \leqslant S_{n,\alpha}^{m+1}(f)(x)$$

with $\lim_{m\to\infty} S_{n,\alpha}^m(f)(x) = S_{n,\alpha}^\infty(f)(x) = S_{1,\alpha}(f)(x)$, $\forall\, x \in [0,1]$. From these, we infer that the pair $\left((S_{k,\alpha})_{k\in\mathbb{N}^*}, \left(S_{n,\alpha}^m\right)_{m\in\mathbb{N}}\right)$ is dual monotone convergent, for any $n \in \mathbb{N}^*$, on Y.

Then, the sequence $(S_{n,\alpha})_{n\in\mathbb{N}^*}$ have dual monotone iteration on Y. ∎

Corollary 3.3. *Considering the set*

$$Z = \{f \in C[0,1] | f - \text{concave}\}$$

the Stancu operators with modified coefficients have monotone iterations on Z and the sequence $(S_{n,\alpha})_{n\in\mathbb{N}^*}$ have dual monotone iteration on Z.

Proof. Follows analogous, having:

$$S_{n,\alpha}^\infty(f) \leqslant \ldots \leqslant S_{n,\alpha}^{m+1}(f) \leqslant S_{n,\alpha}^m(f) \leqslant \ldots \leqslant S_{n,\alpha}(f) \leqslant f$$

and $S_{k,\alpha}(f) \leqslant S_{k+1,\alpha}(f) \leqslant f$, for any $f \in Z$. ∎

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