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SOME INEQUALITIES CONCERNING DERIVATIVE AND MAXIMUM MODULUS OF POLYNOMIALS

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ABSTRACT. In this paper, we prove some compact generalizations of some well-known Bernstein type inequalities concerning the maximum modulus of a polynomial and its derivative in terms of maximum modulus of a polynomial on the unit circle. Besides, an inequality for self-inversive polynomials has also been obtained, which in particular gives some known inequalities for this class of polynomials. All the inequalities obtained are sharp.

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1. INTRODUCTION

If $f(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n and $f'(z)$ its derivative, then it is well known that

$$\max_{|z|=1} |f'(z)| \leq n \max_{|z|=1} |f(z)|, \tag{1.1}$$

and

$$\max_{|z|=R>1} |f(z)| \leq R^n \max_{|z|=1} |f(z)|. \tag{1.2}$$

Inequality (1.1) is an immediate consequence of a famous result due to Bernstein on the derivative of a trigonometric polynomial (for reference see [4, 8, 13, 15, 16]), whereas inequality (1.2) is a simple deduction from the maximum modulus principle (see [14, p. 158 problem 269], or [17, p. 346]). Both the above inequalities are sharp and equality in each holds only when $f(z)$ is a constant multiple of z^n.

Concerning refinements of inequalities (1.1) and (1.2) for polynomials $f(z)$ satisfying $f(0) \neq 0$, see Frappier, Rahman and Ruscheweyh [6, p. 70].

It was observed by Bernstein [4] that (1.1) can be deduced from (1.2), by making use of Gauss-Lucas Theorem which says that the critical points of a polynomial lie in the closed convex hull of its zeros. The proof of the fact that (1.2) can also be deduced from (1.1) is given in Govil, Qazi and Rahman [9, p. 453].

For the class of polynomials $f(z) \neq 0$ in $|z| < 1$, inequalities (1.1) and (1.2) have respectively been replaced by

$$\max_{|z|=1} |f'(z)| \leq \frac{n}{2} \max_{|z|=1} |f(z)|, \tag{1.3}$$

and

$$\max_{|z|=R>1} |f(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |f(z)|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later proved by Lax [11], whereas inequality (1.4) was proved by Ankeny and Rivlin [1], for which they made use of (1.3). Both these inequalities are also sharp and equality in each holds for the polynomials $f(z) = \gamma z^n + \delta$, where $|\gamma| = |\delta|$.

Formulated a little differently, inequalities (1.1) and (1.2) say that, if f is a polynomial of degree at most n such that $|f(z)| \leq M(= |Mz^n|)$ for $|z| = 1$ then

$$|f'(z)| \leq |\frac{d}{dz}(Mz^n)|, \tag{1.5}$$

and

$$|f(Rz)| \leq |MR^n z^n| \text{ for } |z| = 1 \text{ and } R > 1. \tag{1.6}$$

Inequality (1.5) can be seen as a special case of the following result (see also [5] or [15]).

Theorem 1.1. Let $F(z) = \sum_{j=0}^{n} A_j z^j$ be a polynomial of degree n, having all its zeros in the closed unit disk. Further, if $f(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree at most n such that $|f(z)| \leq |F(z)|$ for $|z| = 1$, then

$$|f'(z)| \leq |F'(z)|, \ (1 \leq |z| < \infty). \tag{1.7}$$

Equality holds in (1.7) at some point outside the closed unit disk if and only if $f(z) = e^{i\gamma} F(z)$ for some $\gamma \in \boldsymbol{R}$.

Inequality (1.6) follows immediately from the following result which is a special case of Bernstein-Walsh lemma (see [15, Corollary 12.1.3]).

Theorem 1.2. Let $F(z)$ be a polynomial of degree n, having all its zeros in the closed unit disk $|z| \leq 1$. Furthermore, let $f(z)$ be a polynomial of degree at most n such that $|f(z)| \leq |F(z)|$ for $|z| = 1$. Then

$$|f(z)| < |F(z)| \; for \; |z| > 1,$$

unless $f(z) = e^{i\gamma}F(z)$ for some $\gamma \in \boldsymbol{R}$.

In this paper we will prove a more general result, which provides a compact generalization of inequalities (1.1) and (1.2), and includes Theorem 1.1 and Theorem 1.2 as special cases. We further show that inequalities (1.3) and (1.4) can also be deduced from our result. In fact, we will be proving

Theorem 1.3. Let $f(z)$ and $F(z)$ be two polynomials such that the degree of $f(z)$ does not exceed to that of $F(z)$. If $F(z)$ has all its zeros in $|z| \leq 1$ and $|f(z)| \leq |F(z)|$ for $|z| = 1$, then for any real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$|f(Rz) - \beta f(rz)| \leq |F(Rz) - \beta F(rz)| \; for \; |z| \geq 1, \tag{1.8}$$

and the strict inequality holds in (1.8) if $|z| > 1$. The result is sharp, and the equality holds for the polynomial $f(z) = e^{i\gamma}F(z)$ where $\gamma \in \boldsymbol{R}$, and $F(z)$ is any polynomial having all its zeros in $|z| \leq 1$.

The following result immediately follows from Theorem 1.3, if we take $r = 1$.

Corollary 1.4. Let $f(z)$ and $F(z)$ be two polynomials with degree of $f(z)$ not exceeding that of $F(z)$. If $F(z)$ has all its zeros in $|z| \leq 1$ and $|f(z)| \leq |F(z)| \; for \; |z| = 1$, then for any real or complex number β with $|\beta| \leq 1$ *and* $R \geq 1$,

$$|f(Rz) - \beta f(z)| \leq |F(Rz) - \beta F(z)| \, for \; |z| \geq 1. \tag{1.9}$$

The result is sharp, and the equality here holds for the polynomial $f(z) = e^{i\gamma}F(z)$, where $\gamma \in \boldsymbol{R}$, and $F(z)$ is any polynomial having all its zeros in $|z| \leq 1$.

Remark 1.1. Taking $\beta = 1$, then dividing the two sides of (1.9) by $(R - 1)$ and making $R \to 1$ we get Theorem 1.1, whereas Theorem 1.2 immediately follows from Corollary 1.4, if we take $\beta = 0$.

If in Theorem 1.3, we take $F(z) = Mz^n$, where $M = \max_{|z|=1} |f(z)|$, then we get the following:

Corollary 1.5. If $f(z)$ is a polynomial of degree n, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$|f(Rz) - \beta f(rz)| \leq |R^n - \beta r^n||z|^n \max_{|z|=1} |f(z)|, \; for \; |z| \geq 1. \tag{1.10}$$

The result is again sharp, and equality holds for $f(z) = \gamma z^n, \; \gamma \neq 0$.

If in (1.10), we take $\beta = 1$ and $r = 1$, then divide the two sides of it by $(R - 1)$ and make $R \to 1$, we get inequality (1.1), whereas inequality (1.2) is a special case of Corollary 1.5, if we just take $\beta = 0$.

If $P(z)$ is a polynomial of degree n that does not vanish in $|z| < 1$ then the polynomial $Q(z) = z^n\overline{P(1/\overline{z})}$, has all its zeros lying in $|z| \leq 1$. Hence, if we replace $f(z)$ by $P(z)$ and $F(z)$ by $Q(z)$ in Theorem 1.3, we get the following:

Corollary 1.6. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$ then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, *we have*

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| \, for \; |z| \geq 1,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Again, the result is sharp, and equality holds for any polynomial that has all its zeros on $|z| = 1$.

For $r = 1$, this result reduces to a result earlier proved by Aziz and Rather [3, Lemma 12].

Note that if $P(z)$ is any polynomial of degree n and $M = \max_{|z|=1} |P(z)|$, then for every α with $|\alpha| > 1$, the polynomial $P(z) + \alpha M$ does not vanish in $|z| < 1$, and so applying Corollary 1.6 to the polynomial $P(z) + \alpha M$, we get that for $|z| \geq 1$,

$$|P(Rz) - \beta P(rz) + \alpha(1-\beta)M| \leq |Q(Rz) - \beta Q(rz) + \bar{\alpha}(R^n - \beta r^n) z^n M|. \tag{1.11}$$

Choosing the argument of α in the right hand side of (1.11) suitably, we get

$$|P(Rz) - \beta P(rz)| - |\alpha||1-\beta|M \leq |\alpha||R^n - \beta r^n||z|^n M - |Q(Rz) - \beta Q(rz)|.$$

The fact that the right hand side in the above inequality is nonnegative follows from Corollary 1.5, if we apply it to the polynomial $Q(z)$, which is clearly of degree n.

Now, if in the above, we make $|\alpha| \to 1$, we get the following result:

Corollary 1.7. Let $P(z)$ be a polynomial of degree n and $Q(z) = z^n\overline{P(1/\overline{z})}$. Then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have for $|z| \geq 1$,

$$|P(Rz) - \beta P(rz)| + |Q(Rz) - \beta Q(rz)| \leq \Big(|R^n - \beta r^n||z|^n + |1-\beta|\Big)M,$$

where $M = \max_{|z|=1} |P(z)|$. The result is best possible and equality holds for $P(z) = e^{i\gamma}\left(\frac{z^n+1}{2}\right)$, where $\gamma \in \boldsymbol{R}$.

Corollary 1.7 includes many known polynomial inequalities as special cases. For example, taking $\beta = 1$ and $r = 1$, then dividing both the sides by $(R-1)$ and making $R \to 1$, we get that for any polynomial $P(z)$ of degree n, and for $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|,$$

where as usual $Q(z) = z^n\overline{P(1/\overline{z})}$.

The above inequality can be found in Govil [7, Lemma 3] (also see Govil and Rahman [10, Lemma 10], where a result in more general form has been proved), and easily yields inequality (1.3) due to Lax [11]. Further, if in Corollary 1.7, we just take $\beta = 0$, we get that for any polynomial $P(z)$ of degree n and for $|z| = R \geq 1$,

$$|P(z)| + |Q(z)| \leq (R^n + 1) \max_{|z|=1} |P(z)|.$$

The above inequality which can be found in Rahman and Schmeisser [15], Theorem 12.4.1] easily yields inequality (1.4), due to Ankeny and Rivlin [1].

Next, on combining Corollary 1.6 and Corollary 1.7, we obtain the following result which also includes inequalities (1.3) and (1.4) as special cases.

Corollary 1.8. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$|P(Rz) - \beta P(rz)| \leq \left\{\frac{|R^n - \beta r^n||z|^n + |1-\beta|}{2}\right\} \max_{|z|=1} |P(z)| \, \textit{for} \ |z| \geq 1.$$

The result is sharp and equality holds for $P(z) = e^{i\gamma}\left(\frac{z^n+1}{2}\right)$, where $\gamma \in \boldsymbol{R}$.

If we replace $F(z)$ by $P(z)$ and $f(z)$ by mz^n where $m = \min_{|z|=1} |P(z)|$, in Theorem 1.3, we get the following:

Corollary 1.9. If $P(z)$ is a polynomial of degree n which has all its zeros in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$|P(Rz) - \beta P(rz)| \geq m|z^n||R^n - \beta r^n| \; for \; |z| \geq 1.$$

Again, the result is sharp with equality holding for $P(z) = e^{i\gamma}z^n,$ for any $\gamma \in \boldsymbol{R}$.

In particular, the above inequality gives

$$\min_{|z|=1} |P(Rz) - \beta P(rz)| \geq |R^n - \beta r^n| \min_{|z|=1} |P(z)|. \tag{1.12}$$

If in the above inequality (1.12) we take $\beta = r = 1$, then divide both sides by $(R-1)$ and make $R \to 1$ we get as a special case of Corollary 1.9, the result of Aziz and Dawood [2, Theorem 1].

Again, let $m = \min_{|z|=1} |P(z)|$ and that the polynomial $P(z)$ has no zeros in $|z| \leq 1$. Then clearly, for any α with $|\alpha| < 1$, the polynomial $P(z) + m\alpha z^n$ has no zeros in $|z| \leq 1$, and that the polynomial $Q(z) + \bar{\alpha}m$, where $Q(z) = z^n\overline{P(1/\bar{z})}$, has all its zeros in $|z| \leq 1$. Therefore, if in Theorem 1.3 we take $f(z) = P(z) + m\alpha z^n$ and $F(z) = Q(z) + \bar{\alpha}m$, we get from inequality (1.8), that for $|z| \geq 1$ and $R \geq r \geq 1$,

$$|P(Rz) - \beta P(rz) + \alpha m(R^n - \beta r^n)z^n| \leq |Q(Rz) - \beta Q(rz) + \bar{\alpha}m(1-\beta)|. \tag{1.13}$$

If we now choose the argument of α in the left hand side of inequality (1.13) such that

$$|P(Rz) - \beta P(rz) + \alpha m(R^n - \beta r^n)z^n| = |P(Rz) - \beta P(rz)| + m|\alpha||R^n - \beta r^n||z|^n,$$

we get for $|z| \geq 1$ and $R \geq r \geq 1$,

$$|P(Rz) - \beta P(rz)| + m|\alpha||R^n - \beta r^n||z|^n \leq |Q(Rz) - \beta Q(rz)| + m|\alpha||(1-\beta)|.$$

If we now let $|\alpha| \to 1$, we obtain that for every β with $|\beta| \leq 1$ and $R \geq r \geq 1$,

$$|P(Rz) - \beta P(rz)| \leq |Q(Rz) - \beta Q(rz)| - m\Big\{|R^n - \beta r^n||z|^n - |1-\beta|\Big\} \text{ for } z| \geq 1. \tag{1.14}$$

The above result in conjunction with Corollary 1.7 gives the following result, which is a generalization of a result of Aziz and Dawood [2, Theorem 2].

Corollary 1.10. If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1$ and $R \geq r \geq 1$, we have

$$|P(Rz) - \beta P(rz)| \leq \left\{\frac{|R^n - \beta r^n||z|^n + |1-\beta|}{2}\right\} \max_{|z|=1} |P(z)|$$
$$-\left\{\frac{|R^n - \beta r^n||z|^n - |1-\beta|}{2}\right\} \min_{|z|=1} |P(z)| \; for \; |z| \geq 1.$$

The above inequality is sharp, and equality holds for $P(z) = z^n + \gamma,\ |\gamma| = 1$.

Note that a polynomial of degree n is said to be self-inversive if $P(z) = uQ(z)$, $|u| = 1$, where as usual $Q(z) = z^n\overline{P(1/\bar{z})}$. Therefore, from Corollary 1.7 we easily get the following result which, as special cases, includes some known polynomial inequalities for self-inversive polynomials.

Corollary 1.11. If $P(z)$ is a self-inversive polynomial of degree n, then for $R \geq r \geq 1$,

$$|P(Rz) - \beta P(rz)| \leq \left\{\frac{|R^n - \beta r^n||z|^n + |1-\beta|}{2}\right\} \max_{|z|=1} |P(z)|, \; for \; |z| \geq 1.$$

This result is also sharp and equality holds for $P(z) = z^n + 1$, or, in fact, for any polynomial that has all its zeros on $|z| = 1$.

For $\beta = 0$ the above inequality gives that inequality (1.4) also holds for self-inversive polynomials, while in the above inequality if we take $\beta = 1$, $r = 1$, divide by $(R-1)$, and then make $R \to 1$, we get that the inequality (1.3) is as well true for self-inversive polynomials.

2. Lemmas

For the proof of Theorem 1.3, we will need the following lemma.

Lemma 2.1. If $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having all its zeros in $|z| \leq k$ where $k \geq 0$, then for every $R \geq r$ *and* $rR \geq k^2$,

$$|f(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |f(rz)| \mathit{for}\ |z| = 1. \tag{2.1}$$

Proof. The proof of this lemma is similar to the proof of Lemma 2 in Govil, Qazi and Rahman [9, p. 458], however for the sake of completeness we present its brief outlines.
Since all the zeros of $f(z)$ lie in $|z| \leq k$, we can clearly write

$$f(z) = a_n \prod_{j=1}^{n} (z - k_j e^{i\theta_j}),\ \text{where}\ k_j \leq k,\ j = 1, 2, \cdots, n.$$

Now for $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}\left|\frac{Re^{i\theta} - k_j e^{i\theta_j}}{re^{i\theta} - k_j e^{i\theta_j}}\right|^2 &= \frac{R^2 + k_j^2 - 2Rk_j \cos(\theta - \theta_j)}{r^2 + k_j^2 - 2rk_j \cos(\theta - \theta_j)} \\ &\geq \left(\frac{R+k_j}{r+k_j}\right)^2,\ \text{for}\ R > r\ \text{and}\ rR \geq k^2 \\ &\geq \left(\frac{R+k}{r+k}\right)^2,\ \text{for}\ j = 1, 2, \cdots, n.\end{aligned}$$

The above inequality gives

$$\left|\frac{Re^{i\theta} - k_j e^{i\theta_j}}{re^{i\theta} - k_j e^{i\theta_j}}\right| \geq \frac{R+k}{r+k},\ j = 1, 2, \cdots, n,$$

which implies

$$\begin{aligned}\left|\frac{f(Re^{i\theta}}{f(re^{i\theta}}\right| &= \prod_{j=1}^{n} \frac{|Re^{i\theta} - k_j e^{i\theta_j}|}{|re^{i\theta} - k_j e^{i\theta_j}|} \\ &\geq \prod_{j=1}^{n} \left(\frac{R+k}{r+k}\right) = \left(\frac{R+k}{r+k}\right)^n,\ \text{for}\ 0 \leq \theta \leq 2\pi.\end{aligned}$$

The above is clearly equivalent to

$$|f(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |f(rz)|,\ \text{for}\ |z| = 1,\ R > r\ \text{and}\ rR \geq k^2,$$

and the proof of the Lemma is thus complete. ∎

3. Proof of Theorem 1.3

As the Theorem holds trivially for $R = r$, therefore for the proof of Theorem 1.3 we can clearly assume that $R > r$. Also, since by hypothesis $|f(z)| \leq |F(z)|$ for $|z| = 1$, hence, for every real or complex number δ with $|\delta| > 1$, we have $|f(z)| < |\delta F(z)|$ for $|z| = 1$. Further, because all the zeros of $F(z)$ lie in $|z| \leq 1$, it follows by Rouché's theorem [12, p. 2], that all the zeros of $g(z) = f(z) - \delta F(z)$, with $|\delta| > 1$, also lie in $|z| \leq 1$. Hence if we apply the above Lemma with $k = 1$ to the polynomial $g(z)$, which has all its zeros in $|z| \leq 1$, we get that for $R > r \geq 1$ and for each θ with $0 \leq \theta \leq 2\pi$,

$$|g(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^n |g(re^{i\theta})|. \tag{3.1}$$

Since $g(Re^{i\theta}) \neq 0$ for every $R > r \geq 1$, $0 \leq \theta < 2\pi$, we get that

$$|g(Re^{i\theta})| > \left(\frac{r+1}{R+1}\right)^n |g(Re^{i\theta})|,$$

and this when combined with (3.1) gives that for $R > r \geq 1$, and for each θ with $0 \leq \theta < 2\pi$,

$$|g(Re^{i\theta})| > |g(re^{i\theta})|,$$

which is equivalent to

$$|g(rz)| < |g(Rz)| \text{ for } |z| = 1 \text{ and } R > r \geq 1. \tag{3.2}$$

If β is any real or complex number with $|\beta| \leq 1$, then it follows from (3.2) that $|\beta g(rz)| < |g(Rz)|$ for $|z| = 1$ and $R > r \geq 1$. As all the zeros of $g(Rz)$ lie in $|z| \leq \frac{1}{R} < 1$, therefore again by using Rouché's theorem, we get that the polynomial

$$h(z) = g(Rz) - \beta g(rz) = f(Rz) - \beta f(rz) - \delta(F(Rz) - \beta F(rz)) \tag{3.3}$$

has all its zeros in $|z| < 1$, for every real or complex number δ with $|\delta| > 1$ and $R > r \geq 1$, and this clearly implies that for $|z| \geq 1$ and $R > r \geq 1$,

$$|f(Rz) - \beta f(rz)| \leq |F(Rz) - \beta F(rz)|. \tag{3.4}$$

To see that the inequality (3.4) holds, note that if the inequality (3.4) is not true, then there is a point $z = z_0$ with $|z_0| \geq 1$, such that

$$|f(Rz_0) - \beta f(rz_0)| > |F(Rz_0) - \beta F(rz_0)|. \tag{3.5}$$

Now, because by hypothesis all the zeros of $F(z)$ lie in $|z| \leq 1$, the polynomial $F(Rz)$ has all its zeros in $|z| \leq \frac{1}{R} < 1$ and therefore if we use Rouché's Theorem and arguments similar to the above we will get that all the zeros of $F(Rz) - \beta F(rz)$ lie in $|z| < 1$ for every $R > r \geq 1$, that is, $F(Rz_0) - \beta F(rz_0) \neq 0$ for every z_0 with $|z_0| \geq 1$. Therefore, if we take $\delta = f(Rz_0) - \beta f(rz_0)/F(Rz_0) - \beta F(rz_0)$, then δ is a well defined real or complex number, and in view of (3.5) we also have $|\delta| > 1$. Therefore, with this choice of δ, we get from (3.3), that $h(z_0) = 0$ for some z_0, satisfying $|z_0| \geq 1$, which is clearly a contradiction to the fact that all the zeros of $h(z)$ lie in $|z| < 1$. Thus for $R > r \geq 1$, we get

$$|f(Rz) - \beta f(rz)| \leq |F(Rz) - \beta F(rz)|, \text{ for } |z| \geq 1,$$

and this proves inequality (1.8).

Now in order to complete the proof of Theorem 1.3 we need only to show that for $|z| > 1$, the inequality (1.8) becomes a strict inequality, and this we do as follows.

Note that, $f(Rz) - \beta f(rz)$ and $F(Rz) - \beta F(rz)$ are two polynomials where the degree of $f(Rz) - \beta f(rz)$ does not exceed that of $F(Rz) - \beta F(rz)$, and the polynomial $F(Rz) - \beta F(rz)$ has all its zeros in $|z| \leq 1$. Further, by (1.8) we have

$$|f(Rz) - \beta f(rz)| \leq |F(Rz) - \beta F(rz)|, \text{ for } |z| = 1.$$

Therefore, if we apply Theorem 1.2 to the polynomials $f(Rz) - \beta f(rz)$ and $F(Rz) - \beta F(rz)$, which satisfy its hypotheses, we get that

$$|f(Rz) - \beta f(rz)| < |F(Rz) - \beta F(rz)| \text{ for } |z| > 1,$$

that is, (1.8) becomes a strict inequality for $|z| > 1$, and Theorem 1.3 is thus completely proved.

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