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A NEW METHOD FOR COMPARING CLOSED INTERVALS

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ABSTRACT. The usual ordering "$\leq$" on $\mathbb{R}$ is a total ordering, that is, for any two real numbers in $\mathbb{R}$, we can determine their order without difficulty. However, for any two closed intervals in $\mathbb{R}$, there is not a natural ordering among the set of all closed intervals in $\mathbb{R}$. Several methods have been developed to compare two intervals. In this paper, we define the μ-ordering which is a new method for ordering closed intervals.

Key words and phrases: partial ordering, total ordering, μ-ordering, interval analysis.

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1. INTRODUCTION

In practice, there are many optimization problems formulated using imprecise parameters. Frequently, such parameters may be considered as intervals [6, 7, 8, 9]. In formulation of realistic problems, set of intervals may appear as coefficients in inequality or equality constraints of an optimization problem. Consequently, there should be arise one key question related to the comparison of any two intervals.

Theoretically, intervals can only be partially ordered and hence cannot be compared. However, when intervals are used in practical applications or when a choice has to be made among alternatives, the comparison of intervals becomes necessary. There are numbers of definitions of the ordering relation over intervals [2, 3, 4, 5, 14, 15].

The foremost work was done by Moore [11] who studied the arithmetic of interval numbers. There were two transitive order relations defined over intervals; one as an extension of “$\leq$" on the real line and another as an extension of “$\subseteq$", the concept of set inclusion. However, these order relations cannot explained ranking between two partially overlapping intervals. Ishibuchi and Tanaka [8] suggested three order relations “$\leq_1, \leq_2$ and $\leq_3$" which depends on the endpoints of intervals, or the midpoint and the radius of intervals. However, there exist a set of pair of intervals for which both order relations do not hold. Moreover these order relations did not discuss anything about how large is an interval when it is compared to another interval. In [12], Sengupta and others introduced the acceptability index which depends on the midpoint and radius of intervals.

It was noted in reviews of the most known approaches that, although some of these methods have shown more consistency and better performance in difficult cases, no single method of interval comparison may be put forward as the best. The existing approaches to interval comparison may be clustered into three groups: methods of only qualitative intervals ordering [2, 3, 10], methods permitting quantitative ordering by means of some indices obtained from the base definitions of fuzzy sets theory [2, 4, 5] and methods based on representation of fuzzy number as α-level sets [14, 15].

In this paper we present further development and construct the method which provides the comparison in the form of closed interval. In Section 2, we present a brief survey of the existing works on comparing and ranking any two intervals on the real line. In Section 3, we give a new approach which is called μ-ordering to compare two intervals. Moreover, we prove an important theorem which gives the relation between two intervals. Finally, Section 4 covers conclusions and further research directions.

2. COMPARISON RELATIONS: EXISTING IDEAS

In this section, the order relations which present the decision maker’s preference between intervals are defined.

Let $A = [\underline{a}, \bar{a}]$ be a real closed interval, then the midpoint and the radius of A, respectively, are:

$$m_A = \frac{\underline{a} + \bar{a}}{2} \qquad , \qquad r_A = \bar{a} - m_A.$$

Interval A is alternatively represented as $A = \langle m_A, r_A \rangle$.

There are many different methods for interval comparison proposed in the literature. In [11], there are two transitive order relations defined over intervals: the first one as an extension of $\leq$ on the real line as $A \leq B$ iff $\bar{a} \leq \underline{b}$, and the other as an extension of the concept of set inclusion, i.e., $A \subseteq B$ iff $\underline{a} \geq \underline{b}$ and $\bar{a} \leq \bar{b}$. These order relations cannot explain ranking between two overlapping intervals.

Ishibuchi and Tanaka [8] approached the problem of ranking two intervals more prominently. They defined three definitions to rank intervals. The first definition for order relation is determined by left and right limits of an interval.

Definition 2.1. We define the order relation $\leq_1$ between two closed intervals A and B as

$$A \leq_1 B \qquad \text{iff} \qquad \underline{a} \leq \underline{b} \quad \text{and} \quad \bar{a} \leq \bar{b}.$$

It is clear from definition 2.1 that if $A \leq_1 B$, then $m_A \leq m_B$, and if $\underline{a} = \bar{a}$ and $\underline{b} = \bar{b}$, then $\leq_1$ is the ordinary inequality relation $\leq$ on the set of real numbers.

It is also clear that the order $\leq_1$ is a partial order which is transitive, reflexive and antisymmetric. However, this order relation can be applied to special intervals, and can not be applied to all intervals. For example, we can not compare the intervals $[1, 2]$ and $[0, 3]$.

There is another definition of ordering intervals which depends on the midpoint and the radius of intervals.

Definition 2.2. We define the order relation $\leq_2$ between the closed intervals A and B as

$$A \leq_2 B \qquad \text{iff} \qquad m_A \leq m_B \quad \text{and} \quad r_A \leq r_B.$$

The order relation $\leq_2$ is also a partial order. It is clear from Definition 2.2 that if $A \leq_2 B$, then $\bar{a} \leq \bar{b}$. Moreover, if $m_A = m_B = 0$, then $\leq_2$ is the ordinary inequality relation $\leq$ on the set of real numbers.

The order relations $\leq_1$ and ≤ 2 defined above never conflict with each other in the sense that there is no such pair A and B that $A \neq B$, $A \leq_1 B$, and $B \leq_2 A$, i.e., if $A \leq_1 B$ and $B \leq_2 A$ then $A = B$.

The third definition depends on the midpoint and the endpoint.

Definition 2.3. We define the order relation $\leq_3$ between two closed intervals A and B as

$$A \leq_3 B \qquad \text{iff} \qquad \bar{a} \leq \bar{b} \quad \text{and} \quad m_A \leq m_B.$$

The following Proposition gives the relation between the above definitions.

Proposition 2.1. [8] *The following relationship holds*

$$A \leq_3 B \qquad \textit{iff} \qquad A \leq_1 B \quad \textit{or} \quad A \leq_2 B.$$

In [3], Chanas and Kuchta introduced the t_0, t_1-cut.

Definition 2.4. Let $A = [\underline{a}, \bar{a}]$ be a closed interval, t_0 and t_1 are real numbers such that $0 \leq t_0 < t_1 \leq 1$. The t_0, t_1-cut of the interval A is defined by:

$$A/_{[t_0,t_1]} := [\underline{a} + 2t_0 r_A, \bar{a} + 2t_1 r_A]$$

Then the cut order relation is defined by

$$A \leq_i /_{[t_0,t_1]} B \Leftrightarrow A/_{[t_0,t_1]} \leq_i B/_{[t_0,t_1]}$$

where $i = 1, 2, 3$.

In [12], Sengupta and Pal showed that there exists a set of pairs of intervals for which both of $\leq_i$, for $i = 1, 2, 3$ do not hold.

In [13], Sengupta and others used the acceptability index to compare two intervals.

3. μ-ORDERING

In this section we define a new approach to compare two closed intervals.

Definition 3.1 (Intervals Measure Function)**.** Let $\mathscr{I}$ be the set of all closed and bounded intervals on the real line $\mathbb{R}$. We define the intervals measure function $\mu : \mathscr{I} \times \mathscr{I} \longrightarrow \mathbb{R}$ as follows:

$$\mu(A,B) = \begin{cases} m_B - m_A + 2\,\text{sgn}(m_B - m_A), & \text{if } r_B + r_A = 0 \\ \frac{m_B - m_A}{r_B + r_A} + \text{sgn}(m_B - m_A), & \text{if } m_A \neq m_B \text{ and } r_B + r_A \neq 0 \\ \frac{r_B - r_A}{\max\{r_B, r_A\}}, & \text{if } m_A = m_B \text{ and } r_B + r_A \neq 0 \end{cases}$$

Then the order relation is defined as follows:

Definition 3.2. If $A, B \in \mathscr{I}$, then the order relation $\leq_\mu$ over intervals is defined by:

$$A \leq_\mu B \text{ if and only if } \mu(A,B) \geq 0.$$

From Definitions 3.1 and 3.2, we can conclude the following Proposition:

Proposition 3.1. (1) *If A and B are real numbers, then $\leq_\mu$ is the ordinary inequality relation "$\leq$" on the set of real numbers.*
(2) *$\mu(A,B) = 0$ iff $A = B$.*
(3) *If $0 < \mu(A,B) \leq 1$ then $A \subset B$, (proper subset).*
(4) *If $1 < \mu(A,B) \leq 2$ then $A \bigcap B \neq \phi$;*
Moreover, if $1 < \mu(A,B) \leq 2 - \frac{2\min\{r_A, r_B\}}{r_B + r_A}$, then .

$$\begin{cases} A \subset B & \text{if } r_B \geq r_A \\ B \subset A, & \text{if } r_B < r_A \end{cases}$$

(5) *$\mu(A,B) > 2$ iff $A \bigcap B = \phi$.*

Proof. (1) Follows immediately from the definition of μ.
(2) Let $\mu(A,B) = 0$, if A and B are real numbers, then

$$B - A + 2\,\text{sgn}(B - A) = 0$$

which implies that $A = B$. If A and B are intervals then

$$\frac{m_B - m_A}{r_B + r_A} + \text{sgn}(m_B - m_A) \neq 0$$

because $m_A \neq m_B$; therefore, $m_A = m_B$ and

$$\frac{r_B - r_A}{\max\{r_B, r_A\}} = 0,$$

which implies that $r_A = r_B$, thus $A = B$. On the other hand, if $A = B$, then it is clear from the definition of μ that $\mu(A,B) = 0$.
(3) If $0 < \mu(A,B) \leq 1$, then A and B can not be real numbers because

$$B - A + 2\,\text{sgn}(B - A) = 0 \quad \text{if} \quad A = B,$$

and

$$|B - A + 2\,\text{sgn}(B - A)| > 2 \quad \text{if} \quad A \neq B.$$

Therefore, A and B are intervals. Now if $m_A \neq m_B$, then

$$\left|\frac{m_B - m_A}{r_B + r_A} + \text{sgn}(m_B - m_A)\right| > 1;$$

consequently, $m_A = m_B$ and

$$0 < \frac{r_B - r_A}{\max\{r_B, r_A\}} \leq 1,$$

which implies that $r_A \neq r_B$, and $r_B > r_A \geq 0$, thus $A \subset B$.

(4) If $1 < \mu(A, B) \leq 2$, then A and B can not be real numbers. Now if $m_A \neq m_B$, then

$$1 < \mu(A, B) = \frac{m_B - m_A}{r_B + r_A} + \text{sgn}(m_B - m_A) \leq 2;$$

this implies that $m_B > m_A$ and therefore,

$$0 < \frac{m_B - m_A}{r_B + r_A} \leq 1,$$

which means $A \bigcap B \neq \phi$. If $m_A = m_B$ then the following condition

$$1 < \mu(A, B) = \frac{r_B - r_A}{\max\{r_B, r_A\}} \leq 2,$$

can not be satisfied.
Now if $r_B \geq r_A$ and $1 < \mu(A, B) \leq 2 - \frac{2r_A}{r_B + r_A}$, then

$$1 < \mu(A, B) = \frac{m_B - m_A}{r_B + r_A} + \text{sgn}(m_B - m_A) \leq 2 - \frac{2r_A}{r_B + r_A};$$

which implies that

$$0 < \mu(A, B) = \frac{m_B - m_A}{r_B + r_A} \leq 1 - \frac{2r_A}{r_B + r_A};$$

therefore, $m_B - m_A + r_A \leq r_B$, and hence $A \subset B$. If $r_B < r_A$, then we can easily prove that $B \subset A$

(5) If $\mu(A, B) > 2$, then A and B are either different real numbers, or they are intervals. If they are intervals then the following condition is always satisfied

$$\frac{r_B - r_A}{\max\{r_B, r_A\}} \leq 1;$$

and therefore $m_A \neq m_B$, and we have

$$\mu(A, B) = \frac{m_B - m_A}{r_B + r_A} + \text{sgn}(m_B - m_A) > 2;$$

this implies that $m_B > m_A$ and

$$\frac{m_B - m_A}{r_B + r_A} > 1;$$

therefore, $A \bigcap B = \phi$. On the other hand, if $A \bigcap B = \phi$ then it can be easily shown that $\mu(A, B) > 2$.

∎

From the previous theorem one can notice that as μ goes to zero A almost equal B. Moreover, as μ goes to ∞, the distance between intervals A and B goes to ∞.

4. CONCLUSION

The problem of interval comparison is perennial interest because of its direct relevance in modelling and optimization of real world processes. This problem is considered on the methodological level. To get an effective comparison test, a new method is elaborated. The method allows all possible cases of interval location and intersection and of ordering of interval and real number to be taken into account. Additionally, this method allows the widths of the intervals to be taken into account in the ordering procedure. We have shown that the elaborated method for interval comparison is a useful practical tool.

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