The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

**Volume 8**, Issue 1, Article 14, pp. 1-39, 2011

# APPROXIMATION OF AN AQCQ-FUNCTIONAL EQUATION AND ITS APPLICATIONS

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*Received 2 October, 2010; accepted 4 May, 2011; published 28 November, 2011.*

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ABSTRACT. This paper is a survey on the generalized Hyers-Ulam stability of an AQCQ-functional equation in several spaces.

Its content is divided into the following sections:

1. Introduction and preliminaries.
2. Generalized Hyers-Ulam stability of an AQCQ-functional equation in Banach spaces: direct method.
3. Generalized Hyers-Ulam stability of an AQCQ-functional equation in Banach spaces: fixed point method.
4. Generalized Hyers-Ulam stability of an AQCQ-functional equation in random Banach spaces: direct method.
5. Generalized Hyers-Ulam stability of an AQCQ-functional equation in random Banach spaces: fixed point method.
6. Generalized Hyers-Ulam stability of an AQCQ-functional equation in non-Archimedean Banach spaces: direct method.
7. Generalized Hyers-Ulam stability of an AQCQ-functional equation in non-Archimedean Banach spaces: fixed point method.

*Key words and phrases:* Non-Archimedean Banach space, Fixed point, Generalized Hyers-Ulam stability, Additive-quadratic-cubic-quartic functional equation, Random Banach space.

2010 *Mathematics Subject Classification.* Primary 46S10, 39B52, 47H10, 54E70, 47S10, 26E30, 12J25, 46S50.

ISSN (electronic): 1449-5910



The second author was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2010-0009232).

## 1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [58] concerning the stability of group homomorphisms. Hyers [28] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [49] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [49] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruta [24] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th. M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [57] for mappings $f : X \to Y$, where $X$ is a normed space and $Y$ is a Banach space. Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain $X$ is replaced by an Abelian group. Czerwik [13] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 4, 6, 12, 14], [18]–[22], [29, 32, 33, 39, 41], [50]–[54]).

In [31], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [35], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.2}$$

It is easy to show that the function $f(x)=x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

Let $X$ be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on $X$ if $d$ satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
(3) $d(x,z) \le d(x,y)+d(y,z)$ for all $x,y,z \in X$.

We recall a fundamental result in fixed point theory.

**Theorem 1.1.** [7, 15] *Let $(X,d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

*for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that*

(1) $d(J^n x, J^{n+1}x)<\infty, \qquad \forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;*
(3) *$y^*$ is the unique fixed point of $J$ in the set $Y=\{y \in X \mid d(J^{n_0}x,y)<\infty\}$;*

(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th. M. Rassias [30] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [42, 43, 48]).

The aim of this paper is to investigate the generalized Hyers-Ulam stability of the additive-quadratic-cubic-quartic functional equation

$$\begin{aligned} f(x+2y) + f(x-2y) &= 4f(x+y) + 4f(x-y) - 6f(x) \\ &+ f(2y) + f(-2y) - 4f(y) - 4f(-y) \end{aligned} \tag{1.3}$$

in Banach spaces, in random Banach spaces and in non-Archimedean Banach spaces by using the direct method random and by the fixed point method.

## 2. Generalized Hyers-Ulam stability of an AQCQ-functional equation in Banach spaces: direct method

One can easily show that an odd mapping $f : X \to Y$ satisfies (1.3) if and only if the odd mapping mapping $f : X \to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x).$$

It was shown in Lemma 2.2 of [17] that $g(x) := f(2x) - 2f(x)$ and $h(x) := f(2x) - 8f(x)$ are cubic and additive, respectively, and that $f(x) = \frac{1}{6}g(x) - \frac{1}{6}h(x)$.

One can easily show that an even mapping $f : X \to Y$ satisfies (1.3) if and only if the even mapping $f : X \to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y) + f(x-2y) = 4f(x+y) + 4f(x-y) - 6f(x) + 2f(2y) - 8f(y).$$

It was shown in Lemma 2.1 of [16] that $g(x) := f(2x) - 4f(x)$ and $h(x) := f(2x) - 16f(x)$ are quartic and quadratic, respectively, and that $f(x) = \frac{1}{12}g(x) - \frac{1}{12}h(x)$.

In this section, assume that $X$ is a normed space and $Y$ is a Banach space.

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x,y) : &= f(x+2y) + f(x-2y) - 4f(x+y) - 4f(x-y) + 6f(x) \\ &- f(2y) - f(-2y) + 4f(y) + 4f(-y) \end{aligned}$$

for all $x, y \in X$.

Note that the main results of this section are contained in [47].

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in Banach spaces: an odd case.

**Theorem 2.1.** *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that*

$$\Phi(x,y) := \sum_{n=0}^{\infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{2.1}$$

*for all* $x, y \in X$. *Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{2.2}$$

*for all* $x, y \in X$. *Then there exists a unique cubic mapping* $C : X \to Y$ *such that*

$$\|f(2x) - 2f(x) - C(x)\| \le 4\Phi\left(\frac{x}{2}, \frac{x}{2}\right) + \Phi\left(x, \frac{x}{2}\right) \tag{2.3}$$

*for all* $x \in X$.

*Proof.* Letting $x = y$ in (2.2), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \leq \varphi(y, y) \tag{2.4}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (2.2), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \leq \varphi(2y, y) \tag{2.5}$$

for all $y \in X$.

By (2.4) and (2.5),

$$\begin{aligned}\|f(4y) - 10f(2y) + 16f(y)\| &\leq \|4(f(3y) - 4f(2y) + 5f(y))\| \\ &+ \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \\ &\leq 4\varphi(y, y) + \varphi(2y, y)\end{aligned}$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\|8^l g(\frac{x}{2^l}) - 8^m g(\frac{x}{2^m})\| \leq \sum_{j=l}^{m-1} 4 \cdot 8^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right) + \sum_{j=l}^{m-1} 8^j \varphi\left(\frac{x}{2^j}, \frac{x}{2^{j+1}}\right) \tag{2.6}$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (2.1) and (2.6) that the sequence $\{8^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is complete, the sequence $\{8^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := \lim_{k\to\infty} 8^k g\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

By (2.1) and (2.2),

$$\begin{aligned}\|DC(x, y)\| &= \lim_{k\to\infty} 8^k \left\|Dg\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\| \\ &\leq \lim_{k\to\infty} 8^k \left(\varphi\left(\frac{2x}{2^k}, \frac{2y}{2^k}\right) + 2\varphi\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right) = 0\end{aligned}$$

for all $x, y \in X$. So $DC(x, y) = 0$. Since $g : X \to Y$ is odd, $C : X \to Y$ is odd. So the mapping $C : X \to Y$ is cubic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.6), we get (2.3). So there exists a cubic mapping $C : X \to Y$ satisfying (2.3).

Now, let $C' : X \to Y$ be another cubic mapping satisfying (2.3). Then we have

$$\begin{aligned}\|C(x) - C'(x)\| &= 8^q \left\|C\left(\frac{x}{2^q}\right) - C'\left(\frac{x}{2^q}\right)\right\| \\ &\leq 8^q \left\|C\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right)\right\| + 8^q \left\|C'\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right)\right\| \\ &\leq 2 \cdot 4 \cdot 8^q \Phi\left(\frac{x}{2^{q+1}}, \frac{x}{2^{q+1}}\right) + 2 \cdot 8^q \Phi\left(\frac{x}{2^q}, \frac{x}{2^{q+1}}\right),\end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $C(x) = C'(x)$ for all $x \in X$. This proves the uniqueness of $C$. ∎

**Corollary 2.2.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{2.7}$$

*for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{2^p + 9}{2^p - 8}\theta\|x\|^p$$

*for all $x \in X$.*

Similarly, we can obtain the following. We will omit the proof.

**Theorem 2.3.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{n=0}^{\infty} \frac{1}{8^n}\varphi\left(2^n x, 2^n y\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{2}\Phi\left(x, x\right) + \frac{1}{8}\Phi\left(2x, x\right)$$

*for all $x \in X$.*

**Corollary 2.4.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 3$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{9 + 2^p}{8 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 2.5.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{n=0}^{\infty} 2^n\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq 4\Phi\left(\frac{x}{2}, \frac{x}{2}\right) + \Phi\left(x, \frac{x}{2}\right)$$

*for all $x \in X$.*

**Corollary 2.6.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{2^p + 9}{2^p - 2}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 2.7.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Phi(x, y) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi\left(2^n x, 2^n y\right) < \infty \tag{2.8}$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.2). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq 2\Phi\left(x, x\right) + \frac{1}{2}\Phi\left(2x, x\right)$$

*for all $x \in X$.*

**Corollary 2.8.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $f : X \to Y$ be an odd mapping satisfying (2.7). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{9 + 2^p}{2 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

Now we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in Banach spaces: an even case.

**Theorem 2.9.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \tag{2.9}$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq 4\Psi\left(\frac{x}{2}, \frac{x}{2}\right) + \Psi\left(x, \frac{x}{2}\right)$$

*for all $x \in X$.*

*Proof.* Letting $x = y$ in (2.2), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq \varphi(y, y) \tag{2.10}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (2.2), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq \varphi(2y, y) \tag{2.11}$$

for all $y \in X$.

By (2.10) and (2.11),

$$\begin{aligned}
&\|f(4x) - 20f(2x) + 64f(x)\| \\
&\quad \leq \|4(f(3x) - 6f(2x) + 15f(x))\| \\
&\qquad + \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\| \\
&\quad \leq 4\varphi(x, x) + \varphi(2x, x)
\end{aligned}$$

for all $x \in X$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. ∎

**Corollary 2.10.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{2^p + 9}{2^p - 16}\theta\|x\|^p$$

*for all $x \in X$.*

Similarly, we can obtain the following. We will omit the proof.

**Theorem 2.11.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} \frac{1}{16^n} \varphi\left(2^n x, 2^n y\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \le \frac{1}{4}\Psi\left(x, x\right) + \frac{1}{16}\Psi\left(2x, x\right)$$

*for all $x \in X$.*

**Corollary 2.12.** *Let $\theta \ge 0$ and let $p$ be a real number with $0 < p < 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \le \frac{9 + 2^p}{16 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 2.13.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le 4\Psi\left(\frac{x}{2}, \frac{x}{2}\right) + \Psi\left(x, \frac{x}{2}\right)$$

*for all $x \in X$.*

**Corollary 2.14.** *Let $\theta \ge 0$ and let $p$ be a real number with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \frac{2^p + 9}{2^p - 4}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 2.15.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that*

$$\Psi(x, y) := \sum_{n=0}^{\infty} \frac{1}{4^n} \varphi\left(2^n x, 2^n y\right) < \infty$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.2). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \Psi\left(x, x\right) + \frac{1}{4}\Psi\left(2x, x\right)$$

*for all $x \in X$.*

**Corollary 2.16.** *Let $\theta \ge 0$ and let $p$ be a real number with $0 < p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \frac{9 + 2^p}{4 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then $f_o$ is odd and $f_e$ is even. $f_o$ and $f_e$ satisfy the functional equation (1.3). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$. Then $f_o(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x)$. Thus

$$f(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x) + \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x).$$

Hence we obtain the following results.

**Theorem 2.17.** *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying (2.9). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\le \frac{2}{3}\Phi_1\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{6}\Phi_1\left(x, \frac{x}{2}\right) + \frac{1}{3}\Psi_2\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{12}\Psi_2\left(x, \frac{x}{2}\right) \\
&+\frac{2}{3}\Phi_3\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{6}\Phi_3\left(x, \frac{x}{2}\right) + \frac{1}{3}\Psi_4\left(\frac{x}{2}, \frac{x}{2}\right) + \frac{1}{12}\Psi_4\left(x, \frac{x}{2}\right)
\end{aligned}$$

*for all $x \in X$. Here $\Phi_1 := \Phi, \Psi_2 := \Psi, \Phi_3 := \Phi$ and $\Psi_4 := \Psi$ are given in the statements of Theorems 2.5, 2.13, 2.1 and 2.9, respectively.*

**Corollary 2.18.** *Let $\theta \ge 0$ and let $p$ be a real number with $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\le \left( \frac{2^p+9}{6(2^p-2)} + \frac{2^p+9}{12(2^p-4)} + \frac{2^p+9}{6(2^p-8)} + \frac{2^p+9}{12(2^p-16)} \right) \theta\|x\|^p
\end{aligned}$$

*for all $x \in X$.*

**Theorem 2.19.** *Let $\varphi : X^2 \to [0,\infty)$ be a function satisfying (2.8). Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.2). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\le \frac{1}{3}\Phi_1(x,x) + \frac{1}{12}\Phi_1(2x,x) + \frac{1}{12}\Psi_2(x,x) + \frac{1}{48}\Psi_2(2x,x) \\
&+\frac{1}{12}\Phi_3(x,x) + \frac{1}{48}\Phi_3(2x,x) + \frac{1}{48}\Psi_4(x,x) + \frac{1}{192}\Psi_4(2x,x)
\end{aligned}$$

*for all $x \in X$. Here $\Phi_1 := \Phi, \Psi_2 := \Psi, \Phi_3 := \Phi$ and $\Psi_4 := \Psi$ are given in the statements of Theorems 2.7, 2.15, 2.3 and 2.11, respectively.*

**Corollary 2.20.** *Let $\theta \ge 0$ and let $p$ be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exist an additive mapping $A : X \to Y$,* *a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\|$$
$$\leq \left(\frac{2^p+9}{6(2-2^p)} + \frac{2^p+9}{12(4-2^p)} + \frac{2^p+9}{6(8-2^p)} + \frac{2^p+9}{12(16-2^p)}\right)\theta\|x\|^p$$

*for all $x \in X$.*

## 3. GENERALIZED HYERS-ULAM STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN BANACH SPACES: FIXED POINT METHOD

In this section, assume that $X$ is a normed space and $Y$ is a Banach space.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in Banach spaces: an odd case.

Note that the fundamental ideas in the proofs of the main results are contained in [7, 8, 9], and that the main results of this section are contained in [34].

**Theorem 3.1.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{8}\varphi(2x,2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\| \leq \varphi(x,y) \tag{3.1}$$

*for all $x, y \in X$. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{L}{8-8L}(4\varphi(x,x) + \varphi(2x,x)) \tag{3.2}$$

*for all $x \in X$.*

*Proof.* Letting $x = y$ in (3.1), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \leq \varphi(y,y) \tag{3.3}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (3.1), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \leq \varphi(2y,y) \tag{3.4}$$

for all $y \in X$.

By (3.3) and (3.4),

$$\begin{aligned} \|f(4y) - 10f(2y) + 16f(y)\| &\leq \|4(f(3y) - 4f(2y) + 5f(y))\| \\ &+ \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \\ &\leq 4\varphi(y,y) + \varphi(2y,y) \end{aligned}$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\|g(x) - 8g\left(\frac{x}{2}\right)\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{3.5}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu(4\varphi(x,x) + \varphi(2x,x)),\ \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see the proof of Lemma 2.1 of [37]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le 4\varphi(x, x) + \varphi(2x, x)$$

for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\| \le L(4\varphi(x, x) + \varphi(2x, x))$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$\|g(x) - 8g\left(\frac{x}{2}\right)\| \le \frac{L}{8}\left(4\varphi(x, x) + \varphi(2x, x)\right)$$

for all $x \in X$. So $d(g, Jg) \le \frac{L}{8}$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{3.6}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that $C$ is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|g(x) - C(x)\| \le \mu(4\varphi(x, x) + \varphi(2x, x))$$

for all $x \in X$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \le \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \le \frac{L}{8 - 8L}.$$

This implies that the inequality (3.2) holds.

By (3.1),

$$\left\|8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \le 8^n \left(\varphi\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right) + 2\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right)$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\left\|8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \le L^n(\varphi(2x, 2y) + 2\varphi(x, y))$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\|DC(x,y)\| = 0$$

for all $x, y \in X$. Thus the mapping $C : X \to Y$ is cubic, as desired. ∎

**Corollary 3.2.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.7}$$

*for all $x, y \in X$. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{2^p + 9}{2^p - 8}\theta\|x\|^p$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 3.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. ∎

Similarly, we can obtain the following. We will omit the proof.

**Theorem 3.3.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.1). Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{8 - 8L}(4\varphi(x,x) + \varphi(2x,x))$$

*for all $x \in X$.*

**Corollary 3.4.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (3.7). Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{9 + 2^p}{8 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 3.5.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{2}\varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.1). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L}{2 - 2L}(4\varphi(x,x) + \varphi(2x,x))$$

*for all $x \in X$.*

**Corollary 3.6.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (3.7). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{2^p + 9}{2^p - 2}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 3.7.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (3.1). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{2 - 2L}(4\varphi(x, x) + \varphi(2x, x))$$

*for all $x \in X$.*

**Corollary 3.8.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (3.7). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{9 + 2^p}{2 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in Banach spaces: an even case.

**Theorem 3.9.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{16}\varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{L}{16 - 16L}(4\varphi(x, x) + \varphi(2x, x))$$

*for all $x \in X$.*

*Proof.* Letting $x = y$ in (3.1), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq \varphi(y, y) \tag{3.8}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (3.1), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq \varphi(2y, y) \tag{3.9}$$

for all $y \in X$.

By (3.8) and (3.9),

$$\begin{aligned}
&\|f(4x) - 20f(2x) + 64f(x)\| \\
&\quad \leq \|4(f(3x) - 6f(2x) + 15f(x))\| \\
&\qquad + \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\| \\
&\quad \leq 4\varphi(x, x) + \varphi(2x, x)
\end{aligned}$$

for all $x \in X$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq 4\varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right) \tag{3.10}$$

for all $x \in X$.

Let $(S, d)$ be the generalized metric space defined in the proof of Theorem 3.1.

It follows from (3.10) that

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq \frac{L}{16}(4\varphi(x, x) + \varphi(2x, x))$$

for all $x \in X$. So $d(g, Jg) \leq \frac{L}{16}$.

The rest of the proof is similar to the proof of Theorem 3.1. ∎

**Corollary 3.10.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then there is unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{2^p + 9}{2^p - 16}\theta\|x\|^p$$

*for all $x \in X$.*

Similarly, we can obtain the following. We will omit the proof.

**Theorem 3.11.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 16L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{1}{16 - 16L}(4\varphi(x, x) + \varphi(2x, x))$$

*for all $x \in X$.*

**Corollary 3.12.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{9 + 2^p}{16 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 3.13.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{4}\varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{L}{4 - 4L}(4\varphi(x, x) + \varphi(2x, x))$$

*for all $x \in X$.*

**Corollary 3.14.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{2^p + 9}{2^p - 4}\theta\|x\|^p$$

*for all $x \in X$.*

**Theorem 3.15.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.1). Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{1}{4 - 4L}(4\varphi(x, x) + \varphi(2x, x))$$

*for all $x \in X$.*

**Corollary 3.16.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (3.7). Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{9 + 2^p}{4 - 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

Hence we obtain the following results.

**Theorem 3.17.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{16}\varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
&\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\| \\
&\leq \left(\frac{L}{12 - 12L} + \frac{L}{48 - 48L} + \frac{L}{48 - 48L} + \frac{L}{192 - 192L}\right) \\
&\quad \times (4\varphi(x, x) + \varphi(2x, x))
\end{aligned}$$

*for all $x \in X$.*

**Corollary 3.18.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
&\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\| \\
&\leq \left(\frac{2^p + 9}{6(2^p - 2)} + \frac{2^p + 9}{12(2^p - 4)} + \frac{2^p + 9}{6(2^p - 8)} + \frac{2^p + 9}{12(2^p - 16)}\right)\theta\|x\|^p
\end{aligned}$$

*for all $x \in X$.*

**Theorem 3.19.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\| \\ \leq \left(\frac{1}{12-12L} + \frac{1}{48-48L} + \frac{1}{48-48L} + \frac{1}{192-192L}\right) \\ \times(4\varphi(x, x) + \varphi(2x, x))\end{aligned}$$

*for all $x \in X$.*

**Corollary 3.20.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (3.7). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}\left\|f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x)\right\| \\ \leq \left(\frac{2^p+9}{6(2-2^p)} + \frac{2^p+9}{12(4-2^p)} + \frac{2^p+9}{6(8-2^p)} + \frac{2^p+9}{12(16-2^p)}\right)\theta\|x\|^p\end{aligned}$$

*for all $x \in X$.*

## 4. Generalized Hyers-Ulam stability of an AQCQ-functional equation in random Banach spaces: direct method

Fuzzy set theory is a powerful tool set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g., population dynamics [5], chaos control [23], computer programming [25], nonlinear operators [40], etc. Recently, the fuzzy topology has proved to be a very useful tool to deal with such situations where the use of classical theories breaks down.

In the sequel, we adopt the usual terminology, notations and conventions of the theory of random normed spaces, as in [10, 36, 37, 55, 56]. Throughout this paper, $\Delta^+$ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1]$ such that $F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. $D^+$ is a subset of $\Delta^+$ consisting of all functions $F \in \Delta^+$ for which $l^-F(+\infty) = 1$, where $l^-f(x)$ denotes the left limit of the function $f$ at the point $x$, that is, $l^-f(x) = \lim_{t\to x^-} f(t)$. The space $\Delta^+$ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t$ in $\mathbb{R}$. The maximal element for $\Delta^+$ in this order is the distribution function $\varepsilon_0$ given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

**Definition 4.1.** ([55]) A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous triangular norm (briefly, a continuous $t$-norm) if $T$ satisfies the following conditions:
$(a)$ $T$ is commutative and associative;
$(b)$ $T$ is continuous;

$(c)$ $T(a, 1) = a$ for all $a \in [0, 1]$;
$(d)$ $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Typical examples of continuous $t$-norms are $T_P(a, b) = ab$, $T_M(a, b) = \min(a, b)$ and $T_L(a, b) = \max(a + b - 1, 0)$ (the Lukasiewicz $t$-norm). Recall (see [26, 27]) that if $T$ is a $t$-norm and $\{x_n\}$ is a given sequence of numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recurrently by $T_{i=1}^1 x_i = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i-1}$. It is known ([27]) that for the Lukasiewicz $t$-norm the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^\infty x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty.$$

**Definition 4.2.** ([56]) A *random normed space* (briefly, RN-space) is a triple $(X, \mu, T)$, where $X$ is a vector space, $T$ is a continuous $t$-norm and $\mu$ is a mapping from $X$ into $D^+$ such that the following conditions hold:
$(RN_1)$ $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
$(RN_2)$ $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $x \in X$, $\alpha \neq 0$;
$(RN_3)$ $\mu_{x+y}(t + s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Every normed space $(X, \|.\|)$ defines a random normed space $(X, \mu, T_M)$, where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and $T_M$ is the minimum $t$-norm. This space is called the induced random normed space.

**Definition 4.3.** Let $(X, \mu, T)$ be an RN-space.
(1) A sequence $\{x_n\}$ in $X$ is said to be *convergent* to $x$ in $X$ if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
(2) A sequence $\{x_n\}$ in $X$ is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N$ such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.
(3) An RN-space $(X, \mu, T)$ is said to be *complete* if and only if every Cauchy sequence in $X$ is convergent to a point in $X$.

**Theorem 4.4.** ([55]) *If $(X, \mu, T)$ is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Throughout this section, assume that $X$ is a real vector space and that $(Y, \mu, T)$ is a complete RN-space.

Note that the main results of this section are contained in [46].

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in complete RN-spaces: an odd case.

**Theorem 4.5.** *Let $f : X \to Y$ be an odd mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) such that*

$$\mu_{Df(x,y)}(t) \geq \rho_{x,y}(t) \tag{4.1}$$

*for all $x, y \in X$ and all $t > 0$. If*

$$\begin{aligned} &\lim_{n\to\infty} T_{k=1}^\infty \left(T\left(\rho_{2^{k+n-1}x, 2^{k+n-1}x}\left(2^{k+n-3}t\right), \rho_{2^{k+n}x, 2^{k+n-1}x}\left(2^{k+n-1}t\right)\right)\right) \\ &\qquad = 1 \end{aligned} \tag{4.2}$$

*and*

$$\lim_{n\to\infty} \rho_{2^n x, 2^n y}(2^n t) = 1 \tag{4.3}$$

*for all $x, y \in X$ and all $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}x,2^{k-1}x}\left(2^{k-3}t\right), \rho_{2^k x,2^{k-1}x}\left(2^{k-1}t\right)\right)\right), \tag{4.4}$$

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}x,2^{k-1}x}\left(8^{k-1}t\right), \rho_{2^k x,2^{k-1}x}\left(4\cdot 8^{k-1}t\right)\right)\right) \tag{4.5}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Putting $x = y$ in (4.1), we get

$$\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq \rho_{y,y}(t) \tag{4.6}$$

for all $y \in X$ and all $t > 0$. Replacing $x$ by $2y$ in (4.1), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \rho_{2y,y}(t) \tag{4.7}$$

for all $y \in X$ and all $t > 0$. It follows from (4.6) and (4.7) that

$$\begin{aligned}
&\mu_{f(4x)-10f(2x)+16f(x)}(t) \\
&= \mu_{(4f(3x)-16f(2x)+20f(x))+(f(4x)-4f(3x)+6f(2x)-4f(x))}(t) \\
&\geq T\left(\mu_{4f(3x)-16f(2x)+20f(x)}\left(\frac{t}{2}\right), \mu_{f(4x)-4f(3x)+6f(2x)-4f(x)}\left(\frac{t}{2}\right)\right) \\
&\geq T\left(\rho_{x,x}\left(\frac{t}{8}\right), \rho_{2x,x}\left(\frac{t}{2}\right)\right)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 8f(x)$. Then we conclude that

$$\mu_{g(2x)-2g(x)}(t) \geq T\left(\rho_{x,x}\left(\frac{t}{8}\right), \rho_{2x,x}\left(\frac{t}{2}\right)\right)$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{g(2x)}{2}-g(x)}(t) \geq T\left(\rho_{x,x}\left(\frac{t}{4}\right), \rho_{2x,x}(t)\right)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{\frac{g(2^{k+1}x)}{2^{k+1}}-\frac{g(2^k x)}{2^k}}(t) \geq T\left(\rho_{2^k x,2^k x}\left(2^{k-2}t\right), \rho_{2^{k+1}x,2^k x}\left(2^k t\right)\right)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. From $1 > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$, it follows that

$$\begin{aligned}
\mu_{\frac{g(2^n x)}{2^n}-g(x)}(t) &\geq T_{k=1}^{n}\left(\mu_{\frac{g(2^k x)}{2^k}-\frac{g(2^{k-1}x)}{2^{k-1}}}\left(\frac{t}{2^k}\right)\right) \\
&\geq T_{k=1}^{n}\left(T\left(\rho_{2^{k-1}x,2^{k-1}x}\left(2^{k-3}t\right), \rho_{2^k x,2^{k-1}x}\left(2^{k-1}t\right)\right)\right)
\end{aligned} \tag{4.8}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{g(2^n x)}{2^n}\}$, replacing $x$ with $2^m x$ in (4.8), we obtain that

$$\begin{aligned}
&\mu_{\frac{g(2^{n+m}x)}{2^{n+m}}-\frac{g(2^m x)}{2^m}}(t) \\
&\geq T_{k=1}^{n}\left(T\left(\rho_{2^{k+m-1}x,2^{k+m-1}x}\left(2^{k+m-3}t\right), \rho_{2^{k+m}x,2^{k+m-1}x}\left(2^{k+m-1}t\right)\right)\right).
\end{aligned} \tag{4.9}$$

Since the right hand side of the inequality (4.9) tends to 1 as $m$ and $n$ tend to infinity, the sequence $\{\frac{g(2^n x)}{2^n}\}$ is a Cauchy sequence. Thus we may define $A(x) = \lim_{n\to\infty} \frac{g(2^n x)}{2^n}$ for all $x \in X$.

Now we show that $A$ is an additive mapping. Replacing $x$ and $y$ with $2^n x$ and $2^n y$ in (4.1), respectively, we get

$$\mu_{\frac{Df(2^n x, 2^n y)}{2^n}}(t) \geq \rho_{2^n x, 2^n y}(2^n t).$$

Taking the limit as $n \to \infty$, we find that $A : X \to Y$ satisfies (1.3) for all $x, y \in X$. Since $f : X \to Y$ is odd, $A : X \to Y$ is odd. By Lemma 2.2 of [17], the mapping $A : X \to Y$ is additive. Letting the limit as $n \to \infty$ in (4.8), we get (4.4).

Next, we prove the uniqueness of the additive mapping $A : X \to Y$ subject to (4.4). Let us assume that there exists another additive mapping $L : X \to Y$ which satisfies (4.4). Since $A(2^n x) = 2^n A(x)$, $L(2^n x) = 2^n L(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (4.4), it follows that

$$\begin{aligned}
&\mu_{A(x)-L(x)}(2t) \\
&= \mu_{A(2^n x)-L(2^n x)}(2^{n+1}t) \\
&\geq T(\mu_{A(2^n x)-g(2^n x)}(2^n t), \mu_{g(2^n x)-L(2^n x)}(2^n t)) \\
&\geq T(T_{k=1}^{\infty}(T(\rho_{2^{n+k-1}x, 2^{n+k-1}x}(2^{n+k-3}t), \rho_{2^{n+k}x, 2^{n+k-1}x}(2^{n+k-1}t))), \\
&\quad T_{k=1}^{\infty}(T(\rho_{2^{n+k-1}x, 2^{n+k-1}x}(2^{n+k-3}t), \rho_{2^{n+k}x, 2^{n+k-1}x}(2^{n+k-1}t))))
\end{aligned} \tag{4.10}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (4.10), we conclude that $A = L$.

Let $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 2f(x)$. Then we conclude that

$$\mu_{h(2x)-8h(x)}(t) \geq T\left(\rho_{x,x}\left(\frac{t}{8}\right), \rho_{2x,x}\left(\frac{t}{2}\right)\right)$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{h(2x)}{8}-h(x)}(t) \geq T\left(\rho_{x,x}\left(t\right), \rho_{2x,x}\left(4t\right)\right)$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{\frac{h(2^{k+1}x)}{8^{k+1}}-\frac{h(2^k x)}{8^k}}(t) \geq T\left(\rho_{2^k x, 2^k x}\left(8^k t\right), \rho_{2^{k+1}x, 2^k x}\left(4 \cdot 8^k t\right)\right)$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. From $1 > \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$, it follows that

$$\begin{aligned}
\mu_{\frac{h(2^n x)}{8^n}-h(x)}(t) &\geq T_{k=1}^{n}(\mu_{\frac{h(2^k x)}{8^k}-\frac{h(2^{k-1}x)}{8^{k-1}}}(\frac{t}{8^k})) \\
&\geq T_{k=1}^{n}\left(T\left(\rho_{2^{k-1}x, 2^{k-1}x}\left(8^{k-1}t\right), \rho_{2^k x, 2^{k-1}x}\left(4 \cdot 8^{k-1}t\right)\right)\right)
\end{aligned} \tag{4.11}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{\frac{h(2^n x)}{8^n}\}$, replacing $x$ with $2^m x$ in (4.11), we obtain that

$$\begin{aligned}
\mu_{\frac{h(2^{n+m}x)}{8^{n+m}}-\frac{h(2^m x)}{8^m}}(t) \geq T_{k=1}^{n}\Big(T\Big(&\rho_{2^{k+m-1}x, 2^{k+m-1}x}\left(8^{k+m-1}t\right), \\
&\rho_{2^{k+m}x, 2^{k+m-1}x}\left(4 \cdot 8^{k+m-1}t\right)\Big)\Big).
\end{aligned} \tag{4.12}$$

Since the right hand side of the inequality (4.12) tends to $1$ as $m$ and $n$ tend to infinity, the sequence $\{\frac{h(2^n x)}{8^n}\}$ is a Cauchy sequence. Thus we may define $C(x) = \lim_{n\to\infty} \frac{h(2^n x)}{8^n}$ for all $x \in X$.

Now we show that $C$ is a cubic mapping. Replacing $x$ and $y$ with $2^n x$ and $2^n y$ in (4.1), respectively, we get

$$\mu_{\frac{Df(2^n x, 2^n y)}{8^n}}(t) \geq \rho_{2^n x, 2^n y}(8^n t).$$

Taking the limit as $n \to \infty$, we find that $C : X \to Y$ satisfies (1.3) for all $x, y \in X$. Since $f : X \to Y$ is odd, $C : X \to Y$ is odd. By Lemma 2.2 of [17], the mapping $C : X \to Y$ is cubic. Letting the limit as $n \to \infty$ in (4.11), we get (4.5).

The proof of the uniqueness of $C : X \to Y$ is similar to the proof of the uniqueness of $A : X \to Y$. ∎

Similarly, one can obtain the following result.

**Theorem 4.6.** *Let $f : X \to Y$ be an odd mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) satisfying (4.1). If*

$$\lim_{n\to\infty} T_{k=1}^{\infty}\left(T\left(\rho_{\frac{x}{2^{k+n}},\frac{x}{2^{k+n}}}\left(\frac{t}{4\cdot 8^{k+n}}\right),\rho_{\frac{x}{2^{k+n-1}},\frac{x}{2^{k+n}}}\left(\frac{t}{8^{k+n}}\right)\right)\right)=1$$

*and*

$$\lim_{n\to\infty}\rho_{\frac{x}{2^n},\frac{y}{2^n}}\left(\frac{t}{8^n}\right)=1$$

*for all $x, y \in X$ and all $t > 0$, then there exist a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that*

$$\begin{aligned}
\mu_{f(2x)-8f(x)-A(x)}(t) &\geq T_{k=1}^{\infty}\left(T\left(\rho_{\frac{x}{2^k},\frac{x}{2^k}}\left(\frac{t}{2^{k+2}}\right),\rho_{\frac{x}{2^{k-1}},\frac{x}{2^k}}\left(\frac{t}{2^k}\right)\right)\right),\\
\mu_{f(2x)-2f(x)-C(x)}(t) &\geq T_{k=1}^{\infty}\left(T\left(\rho_{\frac{x}{2^k},\frac{x}{2^k}}\left(\frac{t}{4\cdot 8^k}\right),\rho_{\frac{x}{2^{k-1}},\frac{x}{2^k}}\left(\frac{t}{8^k}\right)\right)\right)
\end{aligned}$$

*for all $x \in X$ and all $t > 0$.*

Now we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete RN-spaces: an even case.

**Theorem 4.7.** *Let $f : X \to Y$ be an even mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) satisfying $f(0) = 0$ and (4.1). If*

$$\begin{aligned}
\lim_{n\to\infty} T_{k=1}^{\infty}&\left(T\left(\rho_{2^{k+n-1}x,2^{k+n-1}x}\left(2\cdot 4^{k+n-2}t\right),\rho_{2^{k+n}x,2^{k+n-1}x}\left(2\cdot 4^{k+n-1}t\right)\right)\right)\\
&=1
\end{aligned}$$

*and*

$$\lim_{n\to\infty}\rho_{2^n x,2^n y}(4^n t)=1$$

*for all $x, y \in X$ and all $t > 0$, then there exist a unique quadratic mapping $P : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
&\mu_{f(2x)-16f(x)-P(x)}(t)\\
&\quad\geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}x,2^{k-1}x}\left(2\cdot 4^{k-2}t\right),\rho_{2^k x,2^{k-1}x}\left(2\cdot 4^{k-1}t\right)\right)\right),\\
&\mu_{f(2x)-4f(x)-Q(x)}(t)\\
&\quad\geq T_{k=1}^{\infty}\left(T\left(\rho_{2^{k-1}x,2^{k-1}x}\left(2\cdot 16^{k-1}t\right),\rho_{2^k x,2^{k-1}x}\left(8\cdot 16^{k-1}t\right)\right)\right)
\end{aligned}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Putting $x = y$ in (4.1), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq \rho_{y,y}(t) \tag{4.13}$$

for all $y \in X$ and all $t > 0$. Replacing $x$ by $2y$ in (4.1), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq \rho_{2y,y}(t) \tag{4.14}$$

for all $y \in X$ and all $t > 0$. It follows from (4.13) and (4.14) that

$$\begin{aligned}
&\mu_{f(4x)-20f(2x)+64f(x)}(t) \\
&= \mu_{(4f(3x)-24f(2x)+60f(x))+(f(4x)-4f(3x)+4f(2x)+4f(x))}(t) \\
&\geq T\left(\mu_{4f(3x)-24f(2x)+60f(x)}\left(\frac{t}{2}\right), \mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}\left(\frac{t}{2}\right)\right) \\
&\geq T\left(\rho_{x,x}\left(\frac{t}{8}\right), \rho_{2x,x}\left(\frac{t}{2}\right)\right)
\end{aligned}$$

for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$. Then we conclude that

$$\mu_{g(2x)-4g(x)}(t) \geq T\left(\rho_{x,x}\left(\frac{t}{8}\right), \rho_{2x,x}\left(\frac{t}{2}\right)\right)$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{\frac{g(2x)}{4}-g(x)}(t) \geq T\left(\rho_{x,x}\left(\frac{t}{2}\right), \rho_{2x,x}(2t)\right)$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 4.5. ∎

Similarly, one can obtain the following result.

**Theorem 4.8.** *Let $f : X \to Y$ be an even mapping for which there is a $\rho : X^2 \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) satisfying $f(0) = 0$ and (4.1). If*

$$\lim_{n\to\infty} T_{k=1}^{\infty}\left(T\left(\rho_{\frac{x}{2^{k+n}},\frac{x}{2^{k+n}}}\left(\frac{t}{4\cdot 16^{k+n}}\right), \rho_{\frac{x}{2^{k+n-1}},\frac{x}{2^{k+n}}}\left(\frac{t}{16^{k+n}}\right)\right)\right) = 1$$

*and*

$$\lim_{n\to\infty} \rho_{\frac{x}{2^n},\frac{y}{2^n}}\left(\frac{t}{16^n}\right) = 1$$

*for all $x, y \in X$ and all $t > 0$, then there exist a unique quadratic mapping $P : X \to Y$ and a unique quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}
\mu_{f(2x)-16f(x)-P(x)}(t) &\geq T_{k=1}^{\infty}\left(T\left(\rho_{\frac{x}{2^k},\frac{x}{2^k}}\left(\frac{t}{4^{k+1}}\right), \rho_{\frac{x}{2^{k-1}},\frac{x}{2^k}}\left(\frac{t}{4^k}\right)\right)\right), \\
\mu_{f(2x)-4f(x)-Q(x)}(t) &\geq T_{k=1}^{\infty}\left(T\left(\rho_{\frac{x}{2^k},\frac{x}{2^k}}\left(\frac{t}{4\cdot 16^k}\right), \rho_{\frac{x}{2^{k-1}},\frac{x}{16^k}}\left(\frac{t}{2^k}\right)\right)\right)
\end{aligned}$$

*for all $x \in X$ and all $t > 0$.*

## 5. GENERALIZED HYERS-ULAM STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN RANDOM BANACH SPACES: FIXED POINT METHOD

Throughout this section, assume that $X$ is a real vector space and that $(Y, \mu, T := \min)$ is a complete RN-space.

Note that the main results of this section are contained in [2].

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in complete RN-spaces: an odd case.

**Theorem 5.1.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{8} \varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \varphi(x, y)} \tag{5.1}$$

*for all $x, y \in X$ and all $t > 0$. Then*

$$C(x) := \lim_{n\to\infty} 8^n \left( f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$$

*exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + 5L(\varphi(x, x) + \varphi(2x, x))} \tag{5.2}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $x = y$ in (5.1), we get

$$\mu_{f(3y)-4f(2y)+5f(y)}(t) \geq \frac{t}{t + \varphi(y, y)} \tag{5.3}$$

for all $y \in X$ and all $t > 0$.

Replacing $x$ by $2y$ in (5.1), we get

$$\mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t) \geq \frac{t}{t + \varphi(2y, y)} \tag{5.4}$$

for all $y \in X$ and all $t > 0$.

By (5.3) and (5.4),

$$\begin{aligned} &\mu_{f(4y)-10f(2y)+16f(y)}(4t + t) \\ &\quad \geq \min\left\{\mu_{4(f(3y)-4f(2y)+5f(y))}(4t), \mu_{f(4y)-4f(3y)+6f(2y)-4f(y)}(t)\right\} \\ &\quad \geq \frac{t}{t + \varphi(y, y) + \varphi(2y, y)} \end{aligned} \tag{5.5}$$

for all $y \in X$ and all $t > 0$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$ in (5.5), we get

$$\mu_{g(x)-8g\left(\frac{x}{2}\right)}(5t) \geq \frac{t}{t + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \tag{5.6}$$

for all $x \in X$ and all $t > 0$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$\begin{aligned} d(g, h) = \inf\{\nu \in \mathbb{R}_+ : \mu_{g(x)-h(x)}(\nu t) \\ \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}, \ \forall x \in X, \forall t > 0\}, \end{aligned}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see the proof of Lemma 2.1 of [37]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\mu_{g(x)-h(x)}(\varepsilon t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned}
&\mu_{Jg(x)-Jh(x)}(L\varepsilon t) = \mu_{8g\left(\frac{x}{2}\right)-8h\left(\frac{x}{2}\right)}(L\varepsilon t) = \mu_{g\left(\frac{x}{2}\right)-h\left(\frac{x}{2}\right)}\left(\frac{L}{8}\varepsilon t\right) \\
&\geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right) + \varphi\left(x, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \frac{L}{8}(\varphi(x, x) + \varphi(2x, x))} \\
&= \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}
\end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (5.6) that

$$\mu_{g(x)-8g\left(\frac{x}{2}\right)}\left(\frac{5L}{8}t\right) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(g, Jg) \leq \frac{5L}{8}$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{5.7}$$

for all $x \in X$. Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that $C$ is a unique mapping satisfying (5.7) such that there exists a $\nu \in (0, \infty)$ satisfying

$$\mu_{g(x)-C(x)}(\nu t) \geq \frac{t}{t + \varphi(x, x) + \varphi(2x, x)}$$

for all $x \in X$ and all $t > 0$;

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L}d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{5L}{8 - 8L}.$$

This implies that the inequality (5.2) holds.

By (5.1),

$$\mu_{8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}(8^n t) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\mu_{8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}(t) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$\mu_{DC(x,y)}(t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. ∎

**Corollary 5.2.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\mu_{Df(x,y)}(t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{5.8}$$

*for all $x, y \in X$ and all $t > 0$. Then $C(x) := \lim_{n\to\infty} 8^n \left(f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(2^p - 8)t}{(2^p - 8)t + 5(3 + 2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* The proof follows from Theorem 5.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. ∎

Similarly, we can obtain the following. We will omit the proof.

**Theorem 5.3.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \leq 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (5.1). Then*

$$C(x) := \lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)$$

*exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8 - 8L)t}{(8 - 8L)t + 5\varphi(x,x) + 5\varphi(2x,x)}$$

*for all $x \in X$ and all $t > 0$.*

**Corollary 5.4.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 3$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (5.8). Then $C(x) := \lim_{n\to\infty} \frac{1}{8^n}\left(f\left(2^{n+1}x\right) - 2f(2^n x)\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$\mu_{f(2x)-2f(x)-C(x)}(t) \geq \frac{(8 - 2^p)t}{(8 - 2^p)t + 5(3 + 2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

**Theorem 5.5.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{2}\varphi(2x,2y)$$

*for all $x,y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (5.1). Then*

$$A(x) := N\text{-}\lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \ge \frac{(2-2L)t}{(2-2L)t + 5L(\varphi(x,x)+\varphi(2x,x))}$$

*for all $x \in X$ and all $t > 0$.*

**Corollary 5.6.** *Let $\theta \ge 0$ and let $p$ be a real number with $p > 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (5.8). Then $A(x) := \lim_{n\to\infty} 2^n \left(f\left(\frac{x}{2^{n-1}}\right) - 8f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \ge \frac{(2^p-2)t}{(2^p-2)t + 5(3+2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

**Theorem 5.7.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le 2L\varphi\left(\frac{x}{2},\frac{y}{2}\right)$$

*for all $x,y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (5.1). Then*

$$A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)$$

*exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \ge \frac{(2-2L)t}{(2-2L)t + 5\varphi(x,x) + 5\varphi(2x,x)}$$

*for all $x \in X$ and all $t > 0$.*

**Corollary 5.8.** *Let $\theta \ge 0$ and let $p$ be a real number with $0 < p < 1$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying (5.8). Then $A(x) := \lim_{n\to\infty} \frac{1}{2^n}\left(f\left(2^{n+1}x\right) - 8f(2^n x)\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \to Y$ such that*

$$\mu_{f(2x)-8f(x)-A(x)}(t) \ge \frac{(2-2^p)t}{(2-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in complete random normed spaces: an even case.

**Theorem 5.9.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{16}\varphi(2x,2y)$$

*for all $x,y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.1). Then*

$$Q(x) := \lim_{n\to\infty} 16^n \left(f\left(\frac{x}{2^{n-1}}\right) - 4f\left(\frac{x}{2^n}\right)\right)$$

*exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(16-16L)t}{(16-16L)t + 5L(\varphi(x,x)+\varphi(2x,x))}$$

*for all $x \in X$ and all $t > 0$.*

*Proof.* Letting $x = y$ in (5.1), we get

$$\mu_{f(3y)-6f(2y)+15f(y)}(t) \geq \frac{t}{t+\varphi(y,y)} \tag{5.9}$$

for all $y \in X$ and all $t > 0$.

Replacing $x$ by $2y$ in (5.1), we get

$$\mu_{f(4y)-4f(3y)+4f(2y)+4f(y)}(t) \geq \frac{t}{t+\varphi(2y,y)} \tag{5.10}$$

for all $y \in X$ and all $t > 0$.

By (5.9) and (5.10),

$$\begin{aligned} &\mu_{f(4x)-20f(2x)+64f(x)}(4t+t) \\ &\quad \geq \min\left\{\mu_{4(f(3x)-6f(2x)+15f(x))}(4t), \mu_{f(4x)-4f(3x)+4f(2x)+4f(x))}(t)\right\} \\ &\quad \geq \frac{t}{t+\varphi(x,x)+\varphi(2x,x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\mu_{g(x)-16g\left(\frac{x}{2}\right)}(5t) \geq \frac{t}{t+\varphi\left(\frac{x}{2},\frac{x}{2}\right)+\varphi\left(x,\frac{x}{2}\right)}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 5.1. ∎

**Corollary 5.10.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 4$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.8). Then $Q(x) := \lim_{n\to\infty} \frac{1}{16^n}\left(f\left(2^{n+1}x\right) - 4f(2^n x)\right)$ exists for each $x \in X$ and defines a quartic mapping $Q : X \to Y$ such that*

$$\mu_{f(2x)-4f(x)-Q(x)}(t) \geq \frac{(16-2^p)t}{(16-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

Similarly, we can obtain the following. We will omit the proof.

**Theorem 5.11.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \leq \frac{L}{4}\varphi(2x,2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.1). Then*

$$T(x) := \lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16f\left(\frac{x}{2^n}\right)\right)$$

*exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(4-4L)t}{(4-4L)t + 5L(\varphi(x,x)+\varphi(2x,x))}$$

*for all $x \in X$ and all $t > 0$.*

**Corollary 5.12.** *Let $\theta \geq 0$ and let $p$ be a real number with $p > 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.8). Then $T(x) := \lim_{n\to\infty} 4^n \left(f\left(\frac{x}{2^{n-1}}\right) - 16 f\left(\frac{x}{2^n}\right)\right)$ exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(2^p-4)t}{(2^p-4)t + 5(3+2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

**Theorem 5.13.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists a constant $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.1). Then*

$$T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right)$$

*exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(4-4L)t}{(4-4L)t + 5\varphi(x,x) + 5\varphi(2x,x)}$$

*for all $x \in X$ and all $t > 0$.*

**Corollary 5.14.** *Let $\theta \geq 0$ and let $p$ be a real number with $0 < p < 2$. Let $X$ be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (5.8). Then $T(x) := \lim_{n\to\infty} \frac{1}{4^n}\left(f\left(2^{n+1}x\right) - 16f(2^n x)\right)$ exists for each $x \in X$ and defines a quadratic mapping $T : X \to Y$ such that*

$$\mu_{f(2x)-16f(x)-T(x)}(t) \geq \frac{(4-2^p)t}{(4-2^p)t + 5(3+2^p)\theta\|x\|^p}$$

*for all $x \in X$ and all $t > 0$.*

## 6. GENERALIZED HYERS-ULAM STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN BANACH SPACES: DIRECT METHOD

A *valuation* is a function $|\cdot|$ from a field $K$ into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s|, \qquad \forall r, s \in K.$$

A field $K$ is called a *valued field* if $K$ carries a valuation. Throughout this paper, we assume that the base field is a valued field, hence call it simply a field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \qquad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

**Definition 6.1.** ([38]) Let $X$ be a vector space over a field $K$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|rx\| = |r|\|x\| \qquad (r \in K, x \in X)$;

(iii) the strong triangle inequality

$$\|x + y\| \le \max\{\|x\|, \|y\|\}, \qquad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

**Definition 6.2.** (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer $N$ such that

$$\|x_n - x_m\| \le \varepsilon$$

for all $n, m \ge N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $X$. Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer $N$ and an $x \in X$ such that

$$\|x_n - x\| \le \varepsilon$$

for all $n \ge N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(iii) If every Cauchy sequence in $X$ converges, then the non-Archimedean normed space $X$ is called a *non-Archimedean Banach space*.

Throughout this section, assume that $X$ is a normed space and that $Y$ is a non-Archimedean Banach space.

Note that the main results of this section are contained in [45].

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean Banach spaces: an odd case.

**Theorem 6.3.** *Let $\theta$ and $p$ be positive real numbers. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{6.1}$$

*for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{2^p + 1}{2^p}\theta\|x\|^p \tag{6.2}$$

*for all $x \in X$.*

*Proof.* Letting $x = y$ in (6.1), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \le 2\theta\|y\|^p \tag{6.3}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (6.1), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \le (2^p + 1)\theta\|y\|^p \tag{6.4}$$

for all $y \in X$.

By (6.3) and (6.4),

$$
\begin{aligned}
\|f(4y)-10f(2y)+16f(y)\| &\le \max\{\|4(f(3y)-4f(2y)+5f(y))\|,\\
&\qquad \|f(4y)-4f(3y)+6f(2y)-4f(y)\|\}\\
&\le \max\{\|f(3y)-4f(2y)+5f(y)\|,\\
&\qquad \|f(4y)-4f(3y)+6f(2y)-4f(y)\|\}\\
&\le \max\{2\theta\|y\|^p, (2^p+1)\theta\|y\|^p\} = (2^p+1)\theta\|y\|^p
\end{aligned} \tag{6.5}
$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \le \frac{2^p+1}{2^p}\theta\|x\|^p$$

for all $x \in X$. Hence

$$
\begin{aligned}
&\left\|8^l g\left(\frac{x}{2^l}\right) - 8^m g\left(\frac{x}{2^m}\right)\right\|\\
&\le \max\left\{\left\|8^l g\left(\frac{x}{2^l}\right) - 8^{l+1} g\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots,\right.\\
&\qquad \left.\left\|8^{m-1} g\left(\frac{x}{2^{m-1}}\right) - 8^m g\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\le \max\left\{\left\|g\left(\frac{x}{2^l}\right) - 8g\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|g\left(\frac{x}{2^{m-1}}\right) - 8g\left(\frac{x}{2^m}\right)\right\|\right\}\\
&\le \frac{2^p+1}{2^p}\max\left\{\frac{\theta\|x\|^p}{2^{pl}}, \cdots, \frac{\theta\|x\|^p}{2^{p(m-1)}}\right\} = \frac{2^p+1}{2^{p(l+1)}}\theta\|x\|^p
\end{aligned} \tag{6.6}
$$

for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (6.6) that the sequence $\{8^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since $Y$ is a non-Archimedean Banach space, the sequence $\{8^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := \lim_{k\to\infty} 8^k g\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

By (6.1),

$$
\begin{aligned}
\|DC(x,y)\| &= \lim_{k\to\infty}\left\|8^k Dg\left(\frac{x}{2^k}, \frac{y}{2^k}\right)\right\|\\
&\le \max\left\{\frac{2^p\theta}{2^{pk}}(\|x\|^p + \|y\|^p), \frac{\theta}{2^{pk}}(\|x\|^p + \|y\|^p)\right\}\\
&= \lim_{k\to\infty}\left(\frac{2^p\theta}{2^{pk}}(\|x\|^p + \|y\|^p)\right) = 0
\end{aligned}
$$

for all $x, y \in X$. So $DC(x,y) = 0$. Since $g : X \to Y$ is odd, $C : X \to Y$ is odd. So the mapping $C : X \to Y$ is cubic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (6.6), we get (6.2). So there exists a cubic mapping $C : X \to Y$ satisfying (6.2).

Now, let $C' : X \to Y$ be another cubic mapping satisfying (6.2). Then we have

$$
\begin{aligned}
\|C(x) - C'(x)\| &= \left\|8^q C\left(\frac{x}{2^q}\right) - 8^q C'\left(\frac{x}{2^q}\right)\right\|\\
&\le \max\left\{\left\|C\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right)\right\|, \left\|C'\left(\frac{x}{2^q}\right) - g\left(\frac{x}{2^q}\right)\right\|\right\}\\
&\le \frac{2^p+1}{2^{p(q+1)}}\theta\|x\|^p,
\end{aligned}
$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $C(x) = C'(x)$ for all $x \in X$. This proves the uniqueness of $C$. ∎

**Theorem 6.4.** *Let $\theta$ and $p$ be positive real numbers. Let $f : X \to Y$ be an odd mapping satisfying (6.1). Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{2^p + 1}{2^p}\theta\|x\|^p$$

*for all $x \in X$.*

*Proof.* Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 8f(x)$ in (6.5), we get

$$\left\|g(x) - 2g\left(\frac{x}{2}\right)\right\| \leq \frac{2^p + 1}{2^p}\theta\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 6.3. ∎

Now we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean Banach spaces: an even case.

**Theorem 6.5.** *Let $\theta$ and $p$ be positive real numbers. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (6.1). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{2^p + 1}{2^p}\theta\|x\|^p$$

*for all $x \in X$.*

*Proof.* Letting $x = y$ in (6.1), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq 2\theta\|y\|^p \tag{6.7}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (6.1), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq (2^p + 1)\theta\|y\|^p \tag{6.8}$$

for all $y \in X$.

By (6.7) and (6.8),

$$\begin{aligned}\|f(4x) - 20f(2x) + 64f(x)\| &\leq \max\{\|4(f(3x) - 6f(2x) + 15f(x))\|,\\ &\qquad \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\|\}\\ &\leq \max\{\|f(3x) - 6f(2x) + 15f(x)\|,\\ &\qquad \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\|\}\\ &\leq \max\{2\theta\|y\|^p, (2^p + 1)\theta\|y\|^p\} = (2^p + 1)\theta\|y\|^p\end{aligned}$$

for all $x \in X$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq \frac{2^p + 1}{2^p}\theta\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 6.3. ∎

**Theorem 6.6.** *Let $\theta$ and $p$ be positive real numbers. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (6.1). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{2^p + 1}{2^p}\theta\|x\|^p$$

*for all $x \in X$.*

*Proof.* Letting $g(x) := f(2x) - 16f(x)$ in (6.9), we get

$$\left\| g(x) - 16g\left(\frac{x}{2}\right) \right\| \le \frac{2^p+1}{2^p}\theta\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 6.3. ∎

**Theorem 6.7.** *Let $\theta$ and $p$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (6.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \le \frac{2^p+1}{|12| \cdot 2^p}\theta\|x\|^p$$

*for all $x \in X$.*

## 7. Generalized Hyers-Ulam stability of an AQCQ-functional equation in non-Archimedean Banach spaces: fixed point method

Throughout this section, assume that $X$ is a non-Archimedean normed vector space and that $Y$ is a non-Archimedean Banach space.

Note that the main results of this section are contained in [44].

We prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y) = 0$ in non-Archimedean Banach spaces: an odd case.

**Theorem 7.1.** *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le \frac{L}{|8|}\varphi(2x,2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\| \le \varphi(x,y) \tag{7.1}$$

*for all $x, y \in X$. Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \le \frac{L}{|8| - |8|L} max\{|4|\varphi(x,x), \varphi(2x,x)\} \tag{7.2}$$

*for all $x \in X$.*

*Proof.* Letting $x = y$ in (7.1), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \le \varphi(y,y) \tag{7.3}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (7.1), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \le \varphi(2y,y) \tag{7.4}$$

for all $y \in X$.

By (7.3) and (7.4),

$$\begin{aligned} \|f(4y) - 10f(2y) + 16f(y)\| &\le \max\{\|4(f(3y) - 4f(2y) + 5f(y))\|, \\ &\qquad \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\|\} \\ &\le \max\{|4| \cdot \|f(3y) - 4f(2y) + 5f(y)\|, \\ &\qquad \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\|\} \\ &\le \max\{|4|\varphi(y,y), \varphi(2y,y)\} \end{aligned} \tag{7.5}$$

for all $y \in X$.

Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \leq \max\left\{|4|\varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right)\right\} \tag{7.6}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on $S$:

$$\begin{aligned} d(g, h) = \inf\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \\ \leq \mu(\max\{|4|\varphi(x, x), \varphi(2x, x),\ \forall x \in X\})\}, \end{aligned}$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that $(S, d)$ is complete (see the proof of Lemma 2.1 of [37]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon \cdot \max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right)\right\| \\ &\leq |8|\varepsilon\frac{L}{|8|}\max\{|4|\varphi(x, x), \varphi(2x, x)\} \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (7.6) that

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \leq \frac{L}{|8|}\left(\max\{|4|\varphi(x, x), \varphi(2x, x)\}\right)$$

for all $x \in X$. So $d(g, Jg) \leq \frac{L}{|8|}$.

By Theorem 1.1, there exists a mapping $C : X \to Y$ satisfying the following:

(1) $C$ is a fixed point of $J$, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{7.7}$$

for all $x \in X$. The mapping $C$ is a unique fixed point of $J$ in the set

$$M = \{h \in S : d(g, h) < \infty\}.$$

This implies that $C$ is a unique mapping satisfying (7.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|g(x) - C(x)\| \leq \mu \cdot \max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

for all $x \in X$; Since $g : X \to Y$ is odd, $C : X \to Y$ is an odd mapping.

(2) $d(J^n g, C) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 8^n g\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(g, C) \leq \frac{1}{1-L} d(g, Jg)$, which implies the inequality

$$d(g, C) \leq \frac{L}{|8| - |8|L}.$$

This implies that the inequality (7.2) holds.

By (7.1),

$$\left\| 8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \leq |8|^n \max\left\{\varphi\left(\frac{2x}{2^n}, \frac{2y}{2^n}\right), |2|\varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\left\| 8^n Dg\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \leq |8|^n \frac{L^n}{|8|^n} \max\{\varphi(2x, 2y), |2|\varphi(x, y)\}$$

for all $x, y \in X$ and all $n \in \mathbb{N}$. So

$$\|DC(x, y)\| = 0$$

for all $x, y \in X$. Thus the mapping $C : X \to Y$ is cubic, as desired. ∎

**Corollary 7.2.** *Let $\theta$ and $p$ be positive real numbers with $p < 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{7.8}$$

*for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \mathit{max}\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |8|} \|x\|^p$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 7.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = \frac{|8|}{|2|^p}$ and we get the desired result. ∎

Similarly, we can obtain the following. We will omit the proof.

**Theorem 7.3.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq |8|L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (7.1). Then there is a unique cubic mapping $C : X \to Y$ such that*

$$\|f(2x) - 2f(x) - C(x)\| \leq \frac{1}{|8| - |8|L} \mathit{max}\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all $x \in X$.*

**Theorem 7.4.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{|2|}\varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (7.1). Then there is a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{L}{|2| - |2|L} \mathit{max}\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all* $x \in X$.

**Corollary 7.5.** *Let* $\theta$ *and* $p$ *be positive real numbers with* $p < 1$. *Let* $f : X \to Y$ *be an odd mapping satisfying (7.8). Then there exists a unique additive mapping* $C : X \to Y$ *such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |2|} \|x\|^p$$

*for all* $x \in X$.

**Theorem 7.6.** *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq |2| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all* $x, y \in X$. *Let* $f : X \to Y$ *be an odd mapping satisfying (7.1). Then there is a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \frac{1}{|2| - |2|L} max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all* $x \in X$.

Now we prove the generalized Hyers-Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean Banach spaces: an even case.

**Theorem 7.7.** *Let* $\varphi : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y) \leq \frac{L}{|16|} \varphi(2x, 2y)$$

*for all* $x, y \in X$. *Let* $f : X \to Y$ *be an even mapping satisfying (7.1) and* $f(0) = 0$. *Then there is a unique quartic mapping* $Q : X \to Y$ *such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{L}{|16| - |16|L} max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all* $x \in X$.

*Proof.* Letting $x = y$ in (7.1), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq \varphi(y, y) \tag{7.9}$$

for all $y \in X$.

Replacing $x$ by $2y$ in (7.1), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq \varphi(2y, y) \tag{7.10}$$

for all $y \in X$.

By (7.9) and (7.10),

$$\begin{aligned}\|f(4y) - 20f(2y) + 64f(y)\| &\leq \max\{\|4(f(3y) - 6f(2y) + 15f(y))\|, \\ &\qquad \|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\|\} \\ &\leq \max\{|4| \cdot \|f(3y) - 6f(2y) + 15f(y)\|, \\ &\qquad \|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\|\} \\ &\leq \max\{|4|\varphi(y, y), \varphi(2y, y)\}\end{aligned}$$

for all $y \in X$.

Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq \max\left\{|4|\varphi\left(\frac{x}{2}, \frac{x}{2}\right), \varphi\left(x, \frac{x}{2}\right)\right\}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 7.1. ∎

**Corollary 7.8.** *Let $\theta$ and $p$ be positive real numbers with $p < 4$. Let $f : X \to Y$ be an even mapping satisfying (7.8) and $f(0) = 0$. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |16|} \|x\|^p$$

*for all $x \in X$.*

*Proof.* The proof follows from Theorem 7.7 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = \frac{|16|}{|2|^p}$ and we get the desired result. ∎

Similarly, we can obtain the following. We will omit the proof.

**Theorem 7.9.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq |16| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (7.1) and $f(0) = 0$. Then there is a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \frac{1}{|16| - |16|L} max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all $x \in X$.*

**Theorem 7.10.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{|4|} \varphi(2x, 2y)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (7.1) and $f(0) = 0$. Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{L}{|4| - |4|L} max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all $x \in X$.*

**Corollary 7.11.** *Let $\theta$ and $p$ be positive real numbers with $p < 2$. Let $f : X \to Y$ be an even mapping satisfying (7.8) and $f(0) = 0$. Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq max\{2 \cdot |4|, |2|^p + 1\} \frac{\theta}{|2|^p - |4|} \|x\|^p$$

*for all $x \in X$.*

**Theorem 7.12.** *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq |4| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying (7.1) and $f(0) = 0$. Then there is a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \leq \frac{1}{|4| - |4|L} max\{|4|\varphi(x, x), \varphi(2x, x)\}$$

*for all* $x \in X$.

Hence we obtain the following results.

**Theorem 7.13.** *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leq \frac{L}{|2|}\varphi(2x,2y)$$

*for all* $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and (7.1). Then there exist an additive mapping* $A : X \to Y$, *a quadratic mapping* $T : X \to Y$, *a cubic mapping* $C : X \to Y$ *and a quartic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\leq \mathit{max}\left\{ \frac{L}{|6|\cdot|2|(1-L)}, \frac{L}{|12|\cdot|4|(1-L)}, \right. \\
&\qquad \left. \frac{L}{|6|\cdot|8|(1-L)}, \frac{L}{|12|\cdot|16|(1-L)} \right\} \\
&\qquad \times \frac{1}{|2|}\mathit{max}\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\} \\
&\leq \frac{L}{|12|\cdot|16|\cdot|2|(1-L)} \\
&\quad \times \mathit{max}\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\}
\end{aligned}$$

*for all* $x \in X$.

**Corollary 7.14.** *Let* $\theta$ *and* $p$ *be positive real numbers with* $p < 1$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and (7.8). Then there exist an additive mapping* $A : X \to Y$, *a quadratic mapping* $T : X \to Y$, *a cubic mapping* $C : X \to Y$ *and a quartic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\qquad \leq \mathit{max}\{2\cdot|4|, |2|^p + 1\} \cdot \frac{\theta}{|12|(|2|^p - |2|)}\|x\|^p
\end{aligned}$$

*for all* $x \in X$.

**Theorem 7.15.** *Let* $\varphi : X^2 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x,y) \leq |16|L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

*for all* $x, y \in X$. *Let* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$ *and (7.1). Then there exist an additive mapping* $A : X \to Y$, *a quadratic mapping* $T : X \to Y$, *a cubic mapping*

$C : X \to Y$ *and a quartic mapping* $Q : X \to Y$ *such that*

$$\begin{aligned}
&\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\
&\quad \leq \mathit{max}\left\{ \frac{1}{|6| \cdot |2|(1-L)}, \frac{1}{|12| \cdot |4|(1-L)}, \right. \\
&\qquad \left. \frac{1}{|6| \cdot |8|(1-L)}, \frac{1}{|12| \cdot |16|(1-L)} \right\} \\
&\qquad\qquad \times \frac{1}{|2|}\mathit{max}\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\} \\
&\quad \leq \frac{1}{|12| \cdot |16| \cdot |2|(1-L)} \\
&\qquad \times \mathit{max}\{|4|\varphi(x,x), \varphi(2x,x), |4|\varphi(-x,-x), \varphi(-2x,-x)\}
\end{aligned}$$

*for all* $x \in X$.

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