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ON WEIGHTED TOEPLITZ OPERATORS

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ABSTRACT. A weighted Toeplitz operator on $H^2(\beta)$ is defined as $T_\phi f = P(\phi f)$ where P is the projection from $L^2(\beta)$ onto $H^2(\beta)$ and the symbol $\phi \in L^2(\beta)$ for a given sequence $\beta = \langle \beta_n \rangle_{n \in Z}$ of positive numbers. In this paper, a matrix characterization of a weighted multiplication operator on $L^2(\beta)$ is given and it is used to deduce the same for a weighted Toeplitz operator. The eigenvalues of some weighted Toeplitz operators are also determined.

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1. INTRODUCTION

Let $\beta = \{\beta_n\}_{n\in Z}$ be a sequence of positive numbers with $\beta_0 = 1$, $0 < \dfrac{\beta_n}{\beta_{n+1}} \leq 1$ for all $n \geq 0$ and $0 < \dfrac{\beta_n}{\beta_{n-1}} \leq 1$ for all $n \leq 0$. Consider the spaces [2], [4]:

$$L^2(\beta) = \left\{ f(z) = \sum_{n=-\infty}^{\infty} a_n z^n | a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

and

$$H^2(\beta) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n | a_n \in \mathbb{C}, \|f\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty \right\}$$

Then $(L^2(\beta), \|\cdot\|_\beta)$ is a Hilbert space [4] with an orthonormal basis given by $\left\{ e_k(z) = \dfrac{z^k}{\beta_k} \right\}_{k\in\mathbb{Z}}$ and with an inner product defined by

$$\left\langle \sum_{n=-\infty}^{\infty} a_n z^n, \sum_{n=-\infty}^{\infty} b_n z^n \right\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n \beta_n^2$$

Further, $H^2(\beta)$ is a subspace [4] of $L^2(\beta)$.

Now, let

$$L^\infty(\beta) = \left\{ \phi(z) = \sum_{n=-\infty}^{\infty} a_n z^n | \phi L^2(\beta) \subseteq L^2(\beta) \text{ and } \exists\, c \in \mathbb{R} \text{ such that } \|\phi f\|_\beta \leq c\|f\|_\beta \;\forall\, f \in L^2(\beta) \right\}$$

Then $L^\infty(\beta)$ is a Banach space with respect to the norm defined by

$$\|\phi\|_\infty = \inf\{c | \|\phi f\|_\beta \leq c\|f\|_\beta \;\forall\, f \in L^2(\beta)\}.$$

Let $P : L^2(\beta) \to H^2(\beta)$ be the orthogonal projection of $L^2(\beta)$ onto $H^2(\beta)$. Then the weighted Toeplitz operator on $H^2(\beta)$ with symbol $\phi \in L^\infty(\beta)$ [4] is defined as

$$T_\phi(f) = P(\phi f)\,.$$

The above mapping is well defined, for if $f \in H^2(\beta) \subset L^2(\beta)$, then by definition, $\phi f \in L^2(\beta)$ and hence $P(\phi f) \in H^2(\beta)$.

2. MATRIX CHARACTERIZATION OF A WEIGHTED TOEPLITZ OPERATOR

Clearly,

$$\begin{aligned} T_\phi e_j &= P(\phi e_j) \\ &= P\left(\sum_{n=-\infty}^{\infty} a_n z^n \frac{z^j}{\beta_j} \right) = P\left(\sum_{n=-\infty}^{\infty} a_{n-j} \frac{z^n}{\beta_j} \right) \\ &= \sum_{n=0}^{\infty} a_{n-j} \frac{z^n}{\beta_j} = \sum_{n=0}^{\infty} \left(\frac{\beta_n a_{n-j}}{\beta_j} \right) e_n\,. \end{aligned}$$

If we denote the matrix of T_ϕ by $\langle \lambda_{ij} \rangle_{i,j=0}^{\infty}$, then

$$\begin{aligned}\lambda_{ij} &= \langle T_\phi e_j, e_i \rangle \\ &= \left\langle \sum_{n=0}^{\infty} \left(\frac{\beta_n a_{n-j}}{\beta_j} \right) e_n, e_i \right\rangle \\ &= \frac{\beta_i}{\beta_j} a_{i-j} \quad \text{for all } \ i, j = 0, 1, 2, \ldots .\end{aligned}$$

One can observe that if we extend the matrix of T_ϕ to a bilaterally infinite matrix, then the matrix of M_ϕ, the weighted multiplication operator is obtained. In other words, if $\langle \lambda_{ij} \rangle_{i,j=-\infty}^{\infty}$ denotes the matrix of M_ϕ on $L^2(\beta)$ given by $M_\phi f = \phi f$ for all $f \in L^2(\beta)$, then

$$\lambda_{ij} = \frac{\beta_i}{\beta_j} a_{i-j}, \quad i, j = 0, \pm 1, \pm 2, \ldots \tag{2.1}$$

The above matrix is of the form

$$\left[\begin{array}{c|cc|ccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\hline
\cdots & a_0 & \frac{\beta_{-2}}{\beta_{-1}} a_{-1} & \frac{\beta_{-2}}{\beta_0} a_{-2} & \cdots & \cdots \\
\cdots & \frac{\beta_{-1}}{\beta_{-2}} a_1 & a_0 & \frac{\beta_{-1}}{\beta_0} a_{-1} & \frac{\beta_{-1}}{\beta_1} a_{-2} & \cdots \\
\cline{4-6}
\cdots & \frac{\beta_0}{\beta_{-2}} a_2 & \frac{\beta_0}{\beta_{-1}} a_1 & a_0 & \frac{\beta_0}{\beta_1} a_{-1} & \frac{\beta_0}{\beta_2} a_{-2} \\
\cdots & \frac{\beta_1}{\beta_{-2}} a_3 & \frac{\beta_1}{\beta_{-1}} a_2 & \frac{\beta_1}{\beta_0} a_1 & a_0 & \frac{\beta_1}{\beta_2} a_{-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}\right]$$

Further, the matrix of T_ϕ can be easily identified as the lower right part as shown above.

It is interesting to note that the inducing function ϕ can be recaptured from the matrix of T_ϕ. The non positive Fourier coefficients of ϕ can be obtained from the matrix of T_ϕ by multiplying the entries in the 0-th row by $1, \frac{\beta_1}{\beta_0}, \frac{\beta_2}{\beta_0}, \ldots$, respectively whereas the non-negative Fourier coefficients can be obtained by multiplying the entries in the 0-th column by $1, \frac{\beta_0}{\beta_1}, \frac{\beta_0}{\beta_2}, \ldots$, respectively.

Definition 2.1. Let $w = \langle w_n \rangle_{n \in \mathbb{Z}}$ be a sequence of positive numbers and $0 < \omega_n < \infty$ for each n. The weighted Laurent matrix corresponding to w is a bilaterally infinite matrix $\langle \lambda_{ij} \rangle$ such that

$$\lambda_{i+1,j+1} = \frac{w_i}{w_j} \lambda_{i,j}, \quad i, j = 0, \pm 1, \pm 2, \ldots . \tag{2.2}$$

Theorem 2.1. *A necessary and sufficient condition that an operator on $L^2(\beta)$ be a weighted multiplication operator is that its matrix with respect to the orthonormal basis $\left\{e_k(z) = \dfrac{z^k}{\beta_k}\right\}_{k\in\mathbb{Z}}$ be a weighted Laurent matrix corresponding to the weight sequence $w = \left\{w_k = \dfrac{\beta_{k+1}}{\beta_k}\right\}_{k\in\mathbb{Z}}$.*

Proof. For necessity, let M_ϕ be a weighted multiplication operator on $L^2(\beta)$. Then,

$$\begin{aligned}
\lambda_{i,j} &= \langle M_\phi e_j, e_i\rangle \\
&= \langle \phi e_j, e_i\rangle \\
&= \left\langle \sum a_n z^n \frac{z^j}{\beta_j}, e_i\right\rangle \\
&= \left\langle \sum a_n \frac{\beta_{n+j}}{\beta_j} e_{n+j}, e_i\right\rangle \\
&= a_{i-j}\frac{\beta_i}{\beta_j} \qquad i,j = 0, \pm 1, \pm 2, \dots
\end{aligned}$$

Further,

$$\begin{aligned}
\lambda_{i+1,j+1} &= a_{i-j}\frac{\beta_{i+1}}{\beta_{j+1}} \\
&= \frac{w_i}{w_j}\lambda_{i,j} \quad \text{where } w_n = \frac{\beta_{n+1}}{\beta_n},\ n \in \mathbb{Z}
\end{aligned}$$

Hence from (2.2), the matrix of M_ϕ is a weighted Laurent matrix corresponding to the weight sequence $w = \langle w_n\rangle_{n\in\mathbb{Z}}$.

For sufficiency, let A be an operator on $L^2(\beta)$ with its matrix as the weighted Laurent matrix corresponding to the weighted sequence $w = \langle w_k\rangle_{k\in\mathbb{Z}}$ given by $w_k = \dfrac{\beta_{k+1}}{\beta_k}$. Now, since [2] an operator on $L^2(\beta)$ that commutes with the weighted shift operator M_z is a multiplication operator M_ϕ for some $\phi \in L^\infty(\beta)$, it is enough to prove that A commutes with M_z. The proof is immediate:

Given that

$$\langle Ae_{j+1}, e_{i+1}\rangle = \frac{w_i}{w_j}\langle Ae_j, e_i\rangle, \quad i,j = 0, \pm 1, \pm 2, \dots .$$

Now,

$$\begin{aligned}
\langle AM_z e_j, e_i\rangle &= \langle Aw_j e_{j+1}, e_i\rangle \\
&= w_j\langle Ae_{j+1}, e_i\rangle \\
&= w_j\frac{w_{i-1}}{w_j}\langle Ae_j, e_{i-1}\rangle \\
&= \langle Ae_j, w_{i-1}e_{i-1}\rangle \\
&= \langle Ae_j, M_z^* e_i\rangle \\
&= \langle M_z Ae_j, e_i\rangle, \quad i,j = 0, \pm 1, \pm 2, \dots .
\end{aligned}$$

Thus $AM_z = M_z A$. ∎

Since a weighted Toeplitz operator is defined to be the orthogonal projection of a weighted multiplication operator on $H^2(\beta)$, hence we are motivated to give the following definition.

Definition 2.2. Let $w = (w_0, w_1, w_2, \ldots)$ be a sequence of positive numbers and $0 < w_n < \infty$ for all non negative integers n. The weighted Toeplitz matrix corresponding to the weight sequence w is a unilaterally infinite matrix $\langle \lambda_{ij} \rangle$ such that $\lambda_{i+1,j+1} = \dfrac{w_i}{w_j}\lambda_{i,j}$, $i, j = 0, 1, 2, \ldots$.

Theorem 2.2. *A necessary and sufficient condition that an operator on $H^2(\beta)$ be a weighted Toeplitz operator T_ϕ is that its matrix $\langle \lambda_{ij} \rangle$ with respect to the orthonormal basis $\left\{ e_k(z) = \dfrac{z^k}{\beta_k} \right\}_{k \in \mathbb{Z}^+ \cup \{0\}}$ is a weighted Toeplitz matrix corresponding to the weight sequence $w = \langle w_n \rangle$ given by $w_n = \dfrac{\beta_{n+1}}{\beta_n}$, $n \in \mathbb{Z}^+ \cup \{0\}$.*

Proof. For necessity, let T_ϕ be a weighted Toeplitz operator on $H^2(\beta)$. Then

$$\begin{aligned}
\lambda_{i+1,j+1} &= \langle T_\phi e_{j+1}, e_{i+1} \rangle \\
&= \langle P M_\phi e_{j+1}, e_{i+1} \rangle \\
&= \langle M_\phi e_{j+1}, P^* e_{i+1} \rangle \\
&= \langle M_\phi e_{j+1}, e_{i+1} \rangle \\
&= \frac{w_i}{w_j}\lambda_{i,j}, \quad i, j = 0, 1, 2, \ldots .
\end{aligned}$$

Thus the matrix of T_ϕ is a weighted Toeplitz matrix. For sufficiency, let A be an operator on $H^2(\beta)$ such that $\langle Ae_{j+1}, e_{i+1} \rangle = \dfrac{w_i}{w_j}\langle Ae_j, e_i \rangle$ where $w_k = \dfrac{\beta_{k+1}}{\beta_k}$ and $i, j, k = 0, 1, 2, \ldots$.

We now prove that A is a weighted Toeplitz operator on $H^2(\beta)$.

Let $N : L^2(\beta) \to L^2(\beta)$ be an operator given by $Ne_j = \dfrac{1}{w_j} e_{j+1}$.

Also, let M_z be denoted by M.

For each non negative integer n, consider the operator on $L^2(\beta)$ given by

$$A_n = N^{*n} A P M^n .$$

Case (i): If $i, j \geq 0$ then,

$$\begin{aligned}
\langle A_n e_j, e_i \rangle &= \langle N^{*n} A P M^n e_j, e_i \rangle \\
&= \langle N^{*n-1} A P M^{n-1} M e_j, N e_i \rangle \\
&= \frac{w_j}{w_i} \langle A_{n-1} e_{j+1}, e_{i+1} \rangle \\
&= \prod_{k=0}^{n-1} \left(\frac{w_{j+k}}{w_{i+k}} \right) \langle A_0 e_{j+n}, e_{i+n} \rangle \\
&= \prod_{k=0}^{n-1} \left(\frac{w_{j+k}}{w_{i+k}} \right) \langle A e_{j+n}, e_{i+n} \rangle \qquad (2.3)
\end{aligned}$$

On the other hand,

$$\langle A e_{j+n}, e_{i+n} \rangle = \prod_{k=0}^{n-1} \left(\frac{w_{i+k}}{w_{j+k}} \right) \langle A e_j, e_i \rangle \qquad (2.4)$$

From (2.3) and (2.4), we get that for $i, j \geq 0$

$$\langle A_n e_j, e_i \rangle = \langle A e_j, e_i \rangle$$

Case (ii): If i or j or both are negative, then for sufficiently large values of n, $j+n$ and $i+n$ are positive; so that the sequence $\{\langle A_n e_j, e_i\rangle\}$ is convergent.

Thus if p and q are trigonometric polynomials (finite linear combinations of the e_i's, $i = 0, \pm1, \pm2, \ldots$), then the sequence $\{\langle A_n p, q\rangle\}$ is convergent.

Also, $\|A_n\| = \|N^{*n}APM^n\| \leq \|N^{*n}\|\|A\|\|P\|\|M^n\| = \|A\|$.

Next we show that $\lim_{n\to\infty}\langle A_n f, g\rangle$ exists $\forall\, f, g \in L^2(\beta)$.

Let $f, g \in L^2(\beta)$. By Weierstrass Approximation theorem, every continuous function can be approximated by a polynomial. Hence $\forall\, \epsilon > 0$, $\exists$ polynomials p and q such that [1],

$$\|g - q\| < \frac{\epsilon}{4(\|A\|+1)(\|f\|+1)}$$

and

$$\|f - p\| < \frac{\epsilon}{8(\|A\|+1)(\|q\|+1)}$$

Now for $n \geq m$, consider

$$\begin{aligned}
|\langle A_n f, g\rangle - \langle A_m f, g\rangle| &= |\langle A_n f, g\rangle - \langle A_n f, q\rangle + \langle A_n f, q\rangle - \langle A_m f, q\rangle + \langle A_m f, q\rangle - \langle A_m f, g\rangle| \\
&= \|A_n\|\|f\|\|g-q\| + |\langle A_n f, q\rangle - \langle A_m f, q\rangle| + \|A_m\|\|f\|\|g-q\| \\
&\leq 2\|A\|\|f\|\|g-q\| + |\langle A_n f, q\rangle - \langle A_m f, q\rangle| \qquad (2.5)
\end{aligned}$$

Consider

$$\begin{aligned}
|\langle A_n f, q\rangle - \langle A_m f, q\rangle| &\leq |\langle A_n f, q\rangle - \langle A_n p, q\rangle + \langle A_n p, q\rangle - \langle A_m p, q\rangle + \langle A_m p, q\rangle - \langle A_m f, q\rangle| \\
&\leq \|A_n\|\|f-p\|\|q\| + \|A_m\|\|f-p\|\|q\| + |\langle A_n p, q\rangle - \langle A_m p, q\rangle| \\
&\leq 2\|A\|\|f-p\|\|q\| + \frac{\epsilon}{4} \\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} \qquad (2.6)
\end{aligned}$$

Putting from (2.6) in (2.5),

$$|\langle A_n f, g\rangle - \langle A_m f, g\rangle| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $\langle A_n f, g\rangle$ is Cauchy in $\mathbb{C}$. Hence $\langle A_n f, g\rangle$ is convergent.

Now let us define

$$\Phi : L^2(\beta) \times L^2(\beta) \to \mathbb{C}$$

as

$$\Phi\langle f, g\rangle = \lim_{n\to\infty}\langle A_n f, g\rangle$$

with addition and scalar multiplication defined as follows:

(i) $\Phi\langle f_1 + f_2, g\rangle = \Phi\langle f_1, g\rangle + \Phi\langle f_2, g\rangle$
(ii) $\Phi\langle \alpha f, g\rangle = \alpha\Phi\langle f, g\rangle$
(iii) $\Phi\langle f, g_1 + g_2\rangle = \Phi\langle f, g_1\rangle + \Phi\langle f, g_2\rangle$
(iv) $\Phi\langle f, \alpha g\rangle = \bar{\alpha}\Phi\langle f, g\rangle$

Also, $|\Phi\langle f, g\rangle| \leq \|A\|\|f\|\|g\|$. Then Φ is a bounded sesquilinear function defined on $L^2(\beta) \times L^2(\beta)$. Hence there exists a unique bounded linear operator A_∞ on $L^2(\beta)$ such that

$$\Phi\langle f, g\rangle = \langle A_\infty f, g\rangle \quad \forall\, f, g \in L^2(\beta).$$

i.e.

$$\lim_{n\to\infty}\langle A_n f, g\rangle = \langle A_\infty f, g\rangle \quad \forall\, f, g \in L^2(\beta).$$

Thus, the sequence $\{A_n\}$ of operators is weakly convergent to an operator A_∞ on $L^2(\beta)$.

Further for all i and j,

$$\begin{aligned}\langle A_\infty e_j, e_i\rangle &= \lim_{n\to\infty} \langle N^{*n} APM^n e_j, e_i\rangle \\ &= \lim_{n\to\infty} \langle N^{*n+1} APM^{n+1} e_j, e_i\rangle \\ &= \lim_{n\to\infty} \langle N^{*n} APM^n M e_j, N e_i\rangle \\ &= \frac{w_j}{w_i} \lim_{n\to\infty} \langle N^* APM^n e_{j+1}, e_{i+1}\rangle \\ &= \frac{w_j}{w_i} \langle A_\infty e_{j+1}, e_{i+1}\rangle\end{aligned}$$

Hence $\langle A_\infty e_{j+1}, e_{i+1}\rangle = \dfrac{w_i}{w_j}\langle A_\infty e_j, e_i\rangle$.

Thus, A_∞ is a Laurent operator on $L^2(\beta)$.

For $f, g \in H^2(\beta)$,

$$\begin{aligned}\langle PA_\infty f, g\rangle &= \langle A_\infty f, Pg\rangle \\ &= \langle A_\infty f, g\rangle \\ &= \lim_{n\to\infty} \langle A_n f, g\rangle \quad \forall\, f, g \in H^2(\beta)\,.\end{aligned}$$

Now, A_n maps $H^2(\beta)$ to $H^2(\beta)$.
Therefore, $A_n e_j \in H^2(\beta)\ \forall\, j \geq 0$.

Also,

$$\begin{aligned}& \langle A_n e_j, e_i\rangle = \langle A e_j, e_i\rangle \quad \forall\, i, j \geq 0. \\ \Rightarrow \qquad & A_n e_j = A e_j.\end{aligned}$$

This is true for all j.

Thus $A_n = A$ on $H^2(\beta)$.

Hence

$$\begin{aligned}\langle PA_\infty f, g\rangle &= \lim_{n\to\infty} \langle A_n f, g\rangle \\ &= \langle Af, g\rangle \quad \forall\, f, g \in H^2(\beta) \\ \Rightarrow \qquad PA_\infty f &= Af \quad \forall\, f\,.\end{aligned}$$

Thus A is the compression of A_∞ on $H^2(\beta)$. Therefore, A is a weighted Toeplitz operator. ∎

If the weight sequence $w_n = \dfrac{\beta_{n+1}}{\beta_n}$ is known, the Fourier coefficients of ϕ can be obtained from the matrix of T_ϕ by the following set of equations.

$$\begin{aligned}a_0 &= \lambda_{0,0} \\ a_k &= \lambda_{k,0}\frac{\beta_0}{\beta_k} = \frac{\lambda_{k,0}}{\beta_k} \\ a_{-k} &= \lambda_{0,k}\frac{\beta_k}{\beta_0} = \lambda_{0,k}\beta_k\,.\end{aligned}$$

Let the compression of the bilateral weighted shift operator M on $H^2(\beta)$ be denoted by U. Then $U : H^2(\beta) \to H^2(\beta)$ and

$$Ue_j = w_j e_{j+1}\,, \quad j = 0, 1, 2, \ldots\,.$$

It may be recalled that $\{w_n\}$ is the weight sequence $w_n = \dfrac{\beta_{n+1}}{\beta_n}$, $n = 0, 1, 2, \ldots$.

Also, then

$$U^* e_j = w_{j-1} e_{j-1} \,.$$

Theorem 2.3. *A necessary and sufficient condition that an operator T on $H^2(\beta)$ be a weighted Toeplitz operator is that $TU = UT$; that is it commutes with the unilateral shift U.*

Proof. Let T be a weighted Toeplitz operator on $H^2(\beta)$.

Then $\langle Te_{j+1}, e_{i+1}\rangle = \dfrac{w_i}{w_j}\langle Te_j, e_j\rangle$.

Now,

$$\begin{aligned}\langle TUe_j, e_i\rangle &= \langle Tw_j e_{j+1}, e_i\rangle \\ &= w_j\langle Te_{j+1}, e_i\rangle \\ &= w_j \frac{w_{i-1}}{w_j}\langle Te_j, e_{i-1}\rangle \\ &= w_{i-1}\langle Te_j, e_{i-1}\rangle \\ &= \langle Te_j, U^* e_i\rangle \\ &= \langle UTe_j, e_i\rangle\end{aligned}$$

Thus $TU = UT$.

Conversely, let $TU = UT$.

Then

$$\begin{aligned} & \langle TUe_j, e_i\rangle = \langle UTe_j, e_i\rangle \\ \Rightarrow\quad & \langle Ue_j, T^* e_i\rangle = \langle Te_j, U^* e_i\rangle \\ \Rightarrow\quad & \langle w_j e_{j+1}, T^* e_i\rangle = \langle Te_j, w_{i-1} e_{i-1}\rangle \\ \Rightarrow\quad & \langle Te_{j+1}, e_i\rangle = \frac{w_{i-1}}{w_j}\langle Te_j, e_{i-1}\rangle\end{aligned}$$

Changing i to $i + 1$ on both sides we get

$$\langle Te_{j+1}, e_{i+1}\rangle = \frac{w_i}{w_j}\langle Te_j, e_i\rangle$$

This shows that the matrix of T is a weighted Toeplitz matrix. Hence by above theorem, T is a weighted Toeplitz operator. ∎

3. EIGENVALUES OF SOME WEIGHTED TOEPLITZ OPERATORS

Now we try to find the eigenvalues of some weighted Toeplitz operators.

Theorem 3.1. *If $\phi = \alpha z$, then $\lambda \in \mathbb{C}$ is an eigenvalue of T_ϕ only if it satisfies the relation $a_n = \left(\dfrac{\alpha}{\lambda}\right)^n a_0 \;\forall\, n$ where $f = \sum_{n=0}^{\infty} a_n z^n$ is the corresponding eigenvector.*

Proof. Let λ be an eigenvalue of T_ϕ. Then, for some $0 \neq f \in H^2(\beta)$, we must have

$$\begin{aligned}
&\Rightarrow \quad T_\phi f = \lambda f \\
&\Rightarrow \quad \alpha z f = \lambda f \\
&\Rightarrow \quad \alpha \sum_{n=0}^{\infty} a_n z^{n+1} = \lambda \sum_{n=0}^{\infty} a_n z^n \\
&\Rightarrow \quad \alpha \sum_{n=0}^{\infty} a_{n-1} z^n = \lambda \sum_{n=0}^{\infty} a_n z^n \\
&\Rightarrow \quad \alpha \sum_{n=0}^{\infty} \beta_n e^n = \lambda \sum_{n=0}^{\infty} a_n \beta_n e_n
\end{aligned}$$

$$\Rightarrow \quad \alpha a_{n-1} = \lambda a_n, \quad \forall\, n \tag{3.1}$$

Taking $n = 1, 2, \ldots$, we get

$$a_1 = \frac{\alpha}{\lambda} a_0, \quad a_2 = \frac{\alpha}{\lambda} a_1 = \left(\frac{\alpha}{\lambda}\right)^2 a_0 \ldots a_0 \text{ so on.}$$

In general, $a_n = \left(\frac{\alpha}{\lambda}\right)^n a_0$. ∎

Observation 1. *From equation (3.1) above, we get that $\lambda = \frac{a_n}{a_{n-1}}\alpha$. Hence the eigenspace of T_ϕ consists of all functions f such that $\sum a_n$ is a geometric series.*

Observation 2. *For the weighted Toeplitz operator T_ϕ induced by $\phi = \alpha z$, zero can not be an eigenvalue.*

Theorem 3.2. *Zero can not be an eigenvalue of a weighted Toeplitz operator induced by $\phi(z) = z^k$.*

Proof. Suppose λ is an eigenvalue of T_ϕ. Then $\exists\, 0 \neq f$ such that

$$\begin{aligned}
&\Rightarrow \quad T_\phi f = \lambda f \\
&\Rightarrow \quad z^k f = \lambda f \\
&\Rightarrow \quad \sum a_{n-k} \beta_n e_n = \lambda \sum a_n \beta_n e_n
\end{aligned}$$

$$\Rightarrow \quad \lambda = \frac{a_{n-k}}{a_n} \quad \forall\, n \tag{3.2}$$

so $\lambda = 0$ gives that $a_n = 0\ \forall\, n$. Hence $f = 0$ which is a contradiction. ∎

In [4], Lauric has discussed in detail the weighted Toeplitz operator induced by the function $\phi(z) = az + \frac{b}{z}$. We now investigate the nature of the eigenvalues of this operator.

Theorem 3.3. *If $\phi(z) = az + \frac{b}{z}$, then λ is an eigenvalue of T_ϕ if it satisfies $aa_{n-1} + ba_{n+1} = \lambda a_n$ $\forall\, n$.*

Proof. Clearly, for a given eigenvalue $\lambda \in \mathbb{C}$, we must have $0 \neq f$ satisfying $T_\phi f = \lambda f$.

$$\begin{aligned}
&\Rightarrow \quad \left(az + \frac{b}{z}\right) \sum a_n z^n = \lambda \sum a_n z^n \\
&\Rightarrow \quad \left(az + \frac{b}{z}\right) \sum a_n \beta_n e_n = \lambda \sum a_n \beta_n e_n
\end{aligned}$$

This gives us the relation

$$aa_{n-1} + ba_{n+1} = \lambda a_n \quad \forall\, n\,. \tag{3.3}$$

∎

Observation. *If* $a = b = 1$ *then* $\phi(z) = z + \frac{1}{z}$ *and from equation (3.3)we get* $\lambda = \frac{a_0 + a_2}{a_1}$ *and so on.*

Further, if we choose $\lambda = 2$*, then corresponding eigenvectors constitute the set of all functions* $f = \sum a_n z^n$ *such that* $\langle a_n \rangle$ *is an arithmetic progression.*

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