The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 8, Issue 1, Article 10, pp. 1-22, 2011

ERROR ESTIMATES FOR APPROXIMATIONS OF THE LAPLACE TRANSFORM OF FUNCTIONS IN L_p SPACES

ANDREA AGLIĆ ALJINOVIĆ

Received 8 April, 2009; accepted 29 August, 2009; published 24 November, 2011.

DEPARTMENT OF APPLIED MATHEMATICS, FACULTY OF ELECTRICAL ENGINEERING AND COMPUTING, UNIVERSITY OF ZAGREB, UNSKA 3, 10 000 ZAGREB, CROATIA
andrea.aglic@fer.hr

ABSTRACT. In this paper error estimates of approximations in complex domain for the Laplace transform are given for functions which vanish beyond a finite domain and whose derivatives belongs to L_p spaces. New inequalities concerning the Laplace transform, as well as estimates of the difference between the two Laplace transforms are presented and used to obtain two associated numerical rules and error bounds of their remainders.

Key words and phrases: Laplace transform, Montgomery identity, Quadrature formula.

2000 *Mathematics Subject Classification.* Primary 44A10, 26D15. Secondary 30E10, 41A55.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The Laplace transform $\mathcal{L}(f)$ of Lebegue integrable mapping $f:[a,b]\to\mathbb{R}$ where $[a,b]\subset[0,\infty\rangle$ is defined by

$$\mathcal{L}(f)(z)=\int_a^b f(t)\,e^{-zt}dt. \tag{1.1}$$

for every $z\in\mathbb{C}$ for which the integral on the right hand side of (1.1) exists, i.e. $\left|\int_a^b f(t)\,e^{-zt}dt\right|<\infty$.

In the recent paper [1] the following theorems involving the Fourier transform $\mathcal{F}(g)(x)=\int_a^b g(t)\,e^{-2\pi ixt}dt$ of the function g were proved:

Theorem 1.1. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous such that $g'\in L_p[a,b]$. Then for $1<p\le\infty$, and for all $x\neq 0$ we have the inequality*

$$\left|E(-2\pi ixa,-2\pi ixb)\int_a^b g(t)\,dt-\mathcal{F}(g)(x)\right|\le 2(b-a)^{1+\frac{1}{q}}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left\|g'\right\|_p,$$

while for $p=1$ we have

$$\left|E(-2\pi ixa,-2\pi ixb)\int_a^b g(t)\,dt-\mathcal{F}(g)(x)\right|\le 2(b-a)\left\|g'\right\|_1.$$

Here $E(z,w)$ is exponential mean of z and w

$$E(z,w)=\begin{cases}\frac{e^z-e^w}{z-w}, & \text{if } z\neq w,\\ e^w, & \text{if } z=w.\end{cases}\qquad z,w\in\mathbb{C} \tag{1.2}$$

Theorem 1.2. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $g:[a,b]\to\mathbb{R}$ be absolutely continuous such that $g'\in L_p[a,b]$. Then for $1<p\le\infty$, and for all $x\neq 0$ we have the inequality*

$$\left|E(-2\pi ixa,-2\pi ixb)\int_a^b g(t)\,dt-\mathcal{F}(g)(x)\right|\le\frac{(b-a)^{\frac{1}{q}}}{\pi|x|}\left\|g'\right\|_p, \tag{1.3}$$

while for $p=1$ we have

$$\left|E(-2\pi ixa,-2\pi ixb)\int_a^b g(t)\,dt-\mathcal{F}(g)(x)\right|\le\frac{1}{\pi|x|}\left\|g'\right\|_1. \tag{1.4}$$

where $\mathcal{F}(g)$ is a Fourier transform of the function g and $E(z,w)$ is given by (1.2).

In this paper error estimates of the analogue approximations in complex domain for the Laplace transform $\mathcal{L}(f)(x+iy)$ are given for functions which vanish beyond a finite domain $[a,b]\subset[0,\infty\rangle$ and such that $f'\in L_p[a,b]$. The estimate of difference between $\mathcal{L}(f)((x+iy))$ and $E(-(x+iy)a,-(x+iy)b)\int_a^b f(s)\,ds$ is given in the Section 2. Some new inequalities concerning the estimate of difference between two Laplace transforms are presented in the Section 3. Two associated numerical quadrature rules and error bounds of their remainders are derived in the Section 4.

2. ESTIMATES OF DIFFERENCE BETWEEN $\mathcal{L}(f)(x+iy)$ AND $E(-(x+iy)a, -(x+iy)b)\int_a^b f(s)\,ds$

Theorem 2.1. *Assume* (p,q) *is a pair of conjugate exponents, that is* $\frac{1}{p}+\frac{1}{q}=1$. *Let* $f:[a,b]\to\mathbb{R}$ *be absolutely continuous such that* $f'\in L_p[a,b]$. *Then for* $x+iy\neq 0$, $1<p\leq\infty$, *and for* $x\geq 0$ *we have the inequality*

$$\left|\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_a^b f(s)\,ds\right|\leq\frac{2e^{-xa}(b-a)^{\frac{1}{q}}}{x^2+y^2}\left\|f'\right\|_p,$$

and for $x<0$

$$\left|\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_a^b f(s)\,ds\right|\leq\frac{2e^{-xb}(b-a)^{\frac{1}{q}}}{x^2+y^2}\left\|f'\right\|_p,$$

while for $p=1$ *and* $x\geq 0$ *we have*

$$\left|\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_a^b f(s)\,ds\right|\leq\frac{2e^{-xa}}{x^2+y^2}\left\|f'\right\|_1.$$

and for $x<0$

$$\left|\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_a^b f(s)\,ds\right|\leq\frac{2e^{-xb}}{x^2+y^2}\left\|f'\right\|_1.$$

Proof. **Montgomery identity** states (see [3]):

$$f(t)=\frac{1}{b-a}\int_a^b f(s)\,ds+\int_a^b P(t,s)\,f'(s)\,ds,$$

where $P(t,s)$ is the Peano kernel, defined by

$$P(t,s)=\begin{cases}\frac{s-a}{b-a}, & a\leq s\leq t,\\[2ex] \frac{s-b}{b-a} & t<s\leq b.\end{cases}$$

Using Montgomery identity we have

$$\begin{aligned}\mathcal{L}(f)(x+iy)&=\int_a^b f(t)\,e^{-(x+iy)t}dt\\ &=\frac{1}{b-a}\int_a^b\left[\int_a^b f(s)\,ds+\int_a^t(s-a)\,f'(s)\,ds+\int_t^b(s-b)\,f'(s)\,ds\right]e^{-(x+iy)t}dt.\end{aligned}$$

By an interchange of the order of integration we get

$$\begin{aligned}\int_a^b\left(\int_a^b f(s)\,ds\right)e^{-(x+iy)t}dt&=\int_a^b\left(\int_a^b e^{-(x+iy)t}dt\right)f(s)\,ds\\ &=\int_a^b\left(\frac{e^{-(x+iy)b}-e^{-(x+iy)a}}{-(x+iy)}\right)f(s)\,ds\\ &=E(-(x+iy)a,-(x+iy)b)(b-a)\int_a^b f(s)\,ds,\end{aligned}$$

$$\begin{aligned}\int_a^b \left(\int_a^t (s-a) f'(s)\, ds\right) e^{-(x+iy)t} dt &= \int_a^b \left(\int_s^b e^{-(x+iy)t} dt\right) (s-a) f'(s)\, ds \\ &= \int_a^b \left(\frac{e^{-(x+iy)b} - e^{-(x+iy)s}}{-(x+iy)}\right) (s-a) f'(s)\, ds,\end{aligned}$$

$$\begin{aligned}\int_a^b \left(\int_t^b (s-b) f'(s)\, ds\right) e^{-(x+iy)t} dt &= \int_a^b \left(\int_a^s e^{-(x+iy)t} dt\right) (s-b) f'(s)\, ds \\ &= \int_a^b \left(\frac{e^{-(x+iy)s} - e^{-(x+iy)a}}{-(x+iy)}\right) (s-b) f'(s)\, ds.\end{aligned}$$

So we have

$$\begin{aligned}&\mathcal{L}(f)((x+iy)) - E(-(x+iy)a, -(x+iy)b) \int_a^b f(s)\, ds = \int_a^b \frac{e^{-(x+iy)s}}{(x+iy)} f'(s)\, ds \\ &+ \left[\int_a^b \frac{e^{-(x+iy)b}}{-(x+iy)} \left(\frac{s-a}{b-a}\right) f'(s)\, ds + \int_a^b \left(\frac{e^{-(x+iy)a}}{-(x+iy)}\right) \left(\frac{b-s}{b-a}\right) f'(s)\, ds\right].\end{aligned}$$

For $1 < p \leq \infty$, by applying Hölder inequality we obtain

$$\begin{aligned}&\left|\mathcal{L}(f)(x+iy) - E(-(x+iy)a, -(x+iy)b) \int_a^b f(s)\, ds\right| \\ &= \left|\int_a^b \left[\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right) \frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right) \frac{e^{-(x+iy)a}}{x+iy}\right] f'(s)\, ds\right| \\ &\leq \left\|\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right) \frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right) \frac{e^{-(x+iy)a}}{x+iy}\right\|_q \|f'\|_p.\end{aligned}$$

Now, if $x \geq 0$ we have

$$\begin{aligned}&\left\|\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right) \frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right) \frac{e^{-(x+iy)a}}{x+iy}\right\|_q \\ &\leq \left\|\frac{e^{-(x+iy)s}}{x+iy}\right\|_q + \left\|\left(\frac{s-a}{b-a}\right) \frac{e^{-(x+iy)b}}{x+iy} + \left(\frac{b-s}{b-a}\right) \frac{e^{-(x+iy)a}}{x+iy}\right\|_q \\ &\leq \left\|\frac{e^{-xs}}{x+iy}\right\|_q + \left\|\left(\frac{s-a}{b-a}\right) \frac{e^{-xb}}{x+iy} + \left(\frac{b-s}{b-a}\right) \frac{e^{-xa}}{x+iy}\right\|_q \\ &\leq \frac{e^{-xa}}{|x+iy|} \left(\|1\|_q + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q\right) = \frac{2e^{-xa} (b-a)^{\frac{1}{q}}}{|x+iy|},\end{aligned}$$

and if $x < 0$ we have

$$\begin{aligned}&\left\|\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right) \frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right) \frac{e^{-(x+iy)a}}{x+iy}\right\|_q \\ &\leq \frac{e^{-xb}}{|x+iy|} \left(\|1\|_q + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_q\right) = \frac{2e^{-xb} (b-a)^{\frac{1}{q}}}{|x+iy|}.\end{aligned}$$

Similarly for $p = 1$ we have

$$\left|\mathcal{L}(g)(x+iy) - E(-(x+iy)a, -(x+iy)b)\int_a^b f(s)\,ds\right|$$
$$\leq \left\|\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right)\frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right)\frac{e^{-(x+iy)a}}{x+iy}\right\|_\infty \|f'\|_1 .$$

If $x \geq 0$

$$\left\|\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right)\frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right)\frac{e^{-(x+iy)a}}{x+iy}\right\|_\infty$$
$$\leq \frac{e^{-xa}}{|x+iy|}\left(\|1\|_\infty + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_\infty\right) = \frac{2e^{-xa}}{|x+iy|},$$

and if $x < 0$ we have

$$\left\|\frac{e^{-(x+iy)s}}{x+iy} - \left(\frac{s-a}{b-a}\right)\frac{e^{-(x+iy)b}}{x+iy} - \left(\frac{b-s}{b-a}\right)\frac{e^{-(x+iy)a}}{x+iy}\right\|_\infty$$
$$\leq \frac{e^{-xb}}{|x+iy|}\left(\|1\|_\infty + \left\|\left(\frac{s-a}{b-a} + \frac{b-s}{b-a}\right)\right\|_\infty\right) = \frac{2e^{-xb}}{|x+iy|},$$

and the proof is done. ∎

Remark 2.1. We have

$$\mathcal{L}(f)(0) = \int_a^b f(t)\,dt$$

and for $x = y = 0$ the left-hand side of the inequalities from the previous Theorem reduces to

$$\left|\mathcal{L}(f)(0) - E(0,0)\int_a^b f(s)\,ds\right| = \left|\int_a^b f(s)\,ds - \int_a^b f(s)\,ds\right| = 0.$$

3. ESTIMATES OF THE DIFFERENCE BETWEEN TWO LAPLACE TRANSFORMS

Let weighted function $w : [a,b] \to \mathbb{R}$ be integrable such that $\int_a^b w(t)\,dt \neq 0$ and $W(x) = \int_a^x w(t)\,dt$, $x \in [a,b]$. Then **weighted Montgomery identity** states (given by Pečarić in [4])

$$f(x) - \frac{1}{\int_a^b w(t)\,dt}\int_a^b f(t)\,w(t)\,dt = \int_a^b P_w(x,t)\,f'(t)\,dt \tag{3.1}$$

where $P_w(t,s)$ the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \leq s \leq x, \\ \\ \frac{W(t)}{W(b)} - 1, & x < s \leq b. \end{cases} \tag{3.2}$$

By subtracting two weighted Montgomery identities, one for the interval $[a,b]$ and the other for $[c,d]$, the next result is obtained (see [1]).

Lemma 3.1. *Let $f:[a,b]\cup[c,d]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]\cup[c,d]$, $w:[a,b]\to\mathbb{R}$ and $u:[c,d]\to\mathbb{R}$ some weighted functions, such that $\int_a^b w(t)\,dt\neq 0$, $\int_c^d u(t)\,dt\neq 0$ and*

$$W(x)=\begin{cases}0, & t<a,\\ \int_a^x w(t)\,dt, & a\le t\le b,\\ \int_a^b w(t)\,dt, & t>b,\end{cases}\quad U(x)=\begin{cases}0, & t<c,\\ \int_c^x u(t)\,dt & c\le t\le d,\\ \int_c^d u(t)\,dt, & t>d,\end{cases}$$

and $[a,b]\cap[c,d]\neq\emptyset$. Then, for both cases $[c,d]\subseteq[a,b]$ and $[a,b]\cap[c,d]=[c,b]$, (and also for $[a,b]\subseteq[c,d]$ and $[a,b]\cap[c,d]=[a,d]$) the next formula is valid

$$\frac{1}{\int_a^b w(t)\,dt}\int_a^b w(t)f(t)\,dt-\frac{1}{\int_c^d u(t)\,dt}\int_c^d u(t)f(t)\,dt$$
$$=\int_{\min\{a,c\}}^{\max\{b,d\}}K(t)f'(t)\,dt \tag{3.3}$$

where

$$K(t)=P_u(x,t)-P_w(x,t),\ t\in[\min\{a,c\},\max\{b,d\}]$$

and $P_u(x,t)$, $P_w(x,t)$ are given by

$$P_w(x,t)=\begin{cases}\frac{W(t)}{W(b)}, & a\le s\le x,\\ \\ \frac{W(t)}{W(b)}-1, & x<s\le b,\end{cases},\quad P_u(x,t)=\begin{cases}\frac{U(t)}{U(b)}, & c\le s\le x,\\ \\ \frac{U(t)}{U(b)}-1, & x<s\le d.\end{cases}$$

thus

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\ \\ -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in\langle c,d\rangle, \quad \textit{if}\quad [c,d]\subseteq[a,b],\\ \\ 1-\frac{W(t)}{W(b)}, & t\in[d,b],\end{cases} \tag{3.4}$$

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\ \\ -\frac{W(t)}{W(b)}+\frac{U(t)}{U(d)}, & t\in\langle c,b\rangle, \quad \textit{if}\quad [a,b]\cap[c,d]=[c,b].\\ \\ \frac{U(t)}{U(d)}-1, & t\in[b,d].\end{cases} \tag{3.5}$$

Remark 3.1. Weighted Montgomery identity and the previous Lemma hold also for $w:[a,b]\to\mathbb{C}$ integrable and such that $\int_a^b w(t)\,dt\neq 0$.

Theorem 3.2. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $f'\in L_p[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $x\ge 0$ and*

$1 < p \leq \infty$ *we have inequalities*

$$\left| \frac{d-c}{b-a} \mathcal{L}(f)(x+iy) - E(-(x+iy)a, -(x+iy)b) \int_c^d f(t)\,dt \right|$$
$$\leq e^{-ax}(d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \|f'\|_p$$
$$\leq (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \|f'\|_p ,$$

while for $p = 1$ *we have*

$$\left| \frac{d-c}{b-a} \mathcal{L}(f)(x+iy) - E(-(x+iy)a, -(x+iy)b) \int_c^d f(t)\,dt \right|$$
$$\leq 2e^{-ax}(d-c)\|f'\|_1$$
$$\leq 2(d-c)\|f'\|_1 ,$$

where $E(z,w)$ *is given by (1.2).*

Proof. If we apply identity (3.3) with $w(t) = e^{-(x+iy)t}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$, we have $W(t) = (t-a)E(-(x+iy)a, -(x+iy)t)$, $t \in [a,b]$; $U(t) = \frac{t-c}{d-c}$, $t \in [c,d]$ and

$$\frac{1}{(b-a)E(-(x+iy)a, -(x+iy)b)} \mathcal{L}(f)(x+iy) - \frac{1}{d-c} \int_c^d f(t)\,dt$$
$$= \int_a^b K(t) f'(t)\,dt.$$

Since $[c,d] \subseteq [a,b]$ we use (3.4) so

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{t-c}{d-c}, & t \in \langle c,d \rangle, \\ 1 - \frac{W(t)}{W(b)}, & t \in [d,b]. \end{cases}$$

Thus

$$\frac{d-c}{b-a} \mathcal{L}(f)(x+iy) - E(-(x+iy)a, -(x+iy)b) \int_c^d f(t)\,dt$$
$$= \frac{d-c}{b-a} W(b) \int_a^b K(t) f'(t)\,dt$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left| \frac{d-c}{b-a} \mathcal{L}(f)(x+iy) - E(-(x+iy)a, -(x+iy)b) \int_c^d f(t)\,dt \right|$$
$$\leq \left\| \frac{d-c}{b-a} W(b) K(t) \right\|_q \|f'\|_p .$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\begin{aligned}
&\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_q = \left(\int_a^c \left| \frac{d-c}{b-a} W(t) \right|^q dt \right. \\
&\left. + \int_c^d \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|^q dt + \int_d^b \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right|^q dt \right)
\end{aligned}$$

and since $x \ge 0$ we have $|W(t)| = \left| \int_a^t e^{-(x+iy)s} ds \right| \le \int_a^t \left| e^{-(x+iy)s} \right| ds = \int_a^t \left| e^{-xs} \right| ds \le (t-a) e^{-ax}$ for $t \in [a,b]$, thus

$$\int_a^c \left| \frac{d-c}{b-a} W(t) \right|^q dt \le \int_a^c \left(e^{-ax} \frac{d-c}{b-a} (t-a) \right)^q dt = e^{-axq} \left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned}
&\int_c^d \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|^q dt \le \int_c^d \left(\left| \frac{d-c}{b-a} W(t) \right| + \left| \frac{t-c}{b-a} W(b) \right| \right)^q dt \\
&\le e^{-axq} \int_c^d \left(\frac{d-c}{b-a} (t-a) + t - c \right)^q dt \\
&= \left(\frac{e^{-ax}}{b-a} \right)^q \int_c^d ((b-a+d-c) t - c(b-a) - a(d-c))^q dt.
\end{aligned}$$

If we denote

$$\lambda(t) = (b-a+d-c) t - c(b-a) - a(d-c) \tag{3.6}$$

we have $\lambda(c) = (d-c)(c-a)$ and $\lambda(d) = (d-c)(b+d-2a)$ so

$$\begin{aligned}
&\left(\frac{e^{-ax}}{b-a} \right)^q \int_c^d ((b-a+d-c) t - c(b-a) - a(d-c))^q dt \\
&= \frac{e^{-axq} \left(\lambda(d)^{q+1} - \lambda(c)^{q+1} \right)}{(b-a)^q (q+1)(b-a+d-c)} \\
&= \frac{e^{-axq} (d-c)^{q+1} \left((b+d-2a)^{q+1} - (c-a)^{q+1} \right)}{(b-a)^q (q+1)(b-a+d-c)} \le \frac{e^{-axq} 2^q (d-c)^q (b-a)}{(q+1)}.
\end{aligned}$$

Also

$$\begin{aligned}
&\int_d^b \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right|^q dt = \int_d^b \left| \frac{d-c}{b-a} \int_t^b e^{-(x+iy)s} ds \right|^q dt \\
&\le e^{-axq} \int_d^b \left(\frac{d-c}{b-a} (b-t) \right)^q dt = e^{-axq} \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left\|\frac{d-c}{b-a}W\left(b\right)K\left(t\right)\right\|_{q}\\
&\leq e^{-ax}\left(\left(\frac{d-c}{b-a}\right)^{q}\frac{\left(c-a\right)^{q+1}}{\left(q+1\right)}+\frac{2^{q}\left(d-c\right)^{q}\left(b-a\right)}{\left(q+1\right)}+\left(\frac{d-c}{b-a}\right)^{q}\frac{\left(b-d\right)^{q+1}}{\left(q+1\right)}\right)^{\frac{1}{q}}\\
&\leq e^{-ax}\left(\left(\frac{d-c}{b-a}\right)^{q}\frac{\left(b-a\right)^{q+1}}{\left(q+1\right)}+\frac{2^{q}\left(d-c\right)^{q}\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}\\
&=e^{-ax}\left(d-c\right)\left(\frac{\left(2^{q}+1\right)\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}
\end{aligned}$$

and since $e^{-ax}\leq 1$ inequalities in case $1<p\leq\infty$ are proved. For $p=1$ we have

$$\begin{aligned}
\left\|\frac{d-c}{b-a}W\left(b\right)K\left(t\right)\right\|_{\infty}&=\max\left\{\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W\left(t\right)\right|,\right.\\
&\left.\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W\left(t\right)-\frac{t-c}{b-a}W\left(b\right)\right|,\ \sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W\left(t\right)-\frac{d-c}{b-a}W\left(b\right)\right|\right\}
\end{aligned}$$

and

$$\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W\left(t\right)\right|\leq e^{-ax}\frac{\left(d-c\right)\left(c-a\right)}{\left(b-a\right)},$$

$$\begin{aligned}
&\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W\left(t\right)-\frac{t-c}{b-a}W\left(b\right)\right|\leq\sup_{t\in[c,d]}\left\{\left|\frac{d-c}{b-a}W\left(t\right)\right|+\left|\frac{t-c}{b-a}W\left(b\right)\right|\right\}\\
&\leq e^{-ax}\frac{d-c}{b-a}\left(d-a\right)+e^{-ax}\left(d-c\right)=e^{-ax}\left(d-c\right)\frac{b+d-2a}{b-a},
\end{aligned}$$

$$\sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W\left(t\right)-\frac{d-c}{b-a}W\left(b\right)\right|\leq e^{-ax}\frac{\left(d-c\right)\left(b-d\right)}{\left(b-a\right)}.$$

Thus

$$\begin{aligned}
\left\|\frac{d-c}{b-a}W\left(b\right)K\left(t\right)\right\|_{\infty}&\leq e^{-ax}\frac{d-c}{b-a}\max\left\{\left(c-a\right),\left(b+d-2a\right),\left(b-d\right)\right\}\\
&\leq e^{-ax}2\left(d-c\right)
\end{aligned}$$

and since $e^{-ax}\leq 1$ the proof is completed. ∎

Remark 3.2. The inequalities from the previous Theorem hods for $x\geq 0$. Similarly it can be proved that in case $x<0$ and $1<p\leq\infty$ we have the inequality

$$\begin{aligned}
&\left|\frac{d-c}{b-a}\mathcal{L}\left(f\right)\left(x+iy\right)-E\left(-\left(x+iy\right)a,-\left(x+iy\right)b\right)\int_{c}^{d}f\left(t\right)dt\right|\\
&\leq e^{-bx}\left(d-c\right)\left(\frac{\left(2^{q}+1\right)\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}\left\|f'\right\|_{p},
\end{aligned}$$

while for $x<0$ and $p=1$ we have

$$\left|\frac{d-c}{b-a}\mathcal{L}\left(f\right)\left(x+iy\right)-E\left(-\left(x+iy\right)a,-\left(x+iy\right)b\right)\int_{c}^{d}f\left(t\right)dt\right|\leq e^{-bx}2\left(d-c\right)\left\|f'\right\|_{1}.$$

Theorem 3.3. *Assume* (p,q) *is a pair of conjugate exponents, that is* $\frac{1}{p}+\frac{1}{q}=1$. *Let* $f:[a,b]\to\mathbb{R}$ *be absolutely continuous,* $f'\in L_p[a,b]$ *and* $c,d\in[a,b]$, $c<d$. *Then for* $x+iy\neq 0$, $x\geq 0$ *and* $1<p\leq\infty$, *we have inequalities*

$$\begin{aligned}&\left|\frac{d-c}{b-a}\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_c^d f(t)\,dt\right|\\&\leq e^{-ax}(d-c)\frac{2(b-a)^{\frac{1}{q}-1}}{x^2+y^2}\left\|f'\right\|_p\\&\leq (d-c)\frac{2(b-a)^{\frac{1}{q}-1}}{x^2+y^2}\left\|f'\right\|_p,\end{aligned}$$

while for $p=1$ *we have*

$$\begin{aligned}&\left|\frac{d-c}{b-a}\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_c^d f(t)\,dt\right|\\&\leq e^{-ax}\frac{2(d-c)}{(b-a)(x^2+y^2)}\left\|f'\right\|_1\\&\leq \frac{2(d-c)}{(b-a)(x^2+y^2)}\left\|f'\right\|_1,\end{aligned}$$

where $E(z,w)$ *is given by (1.2).*

Proof. If we apply identity (3.3) with $w(t)=e^{-(x+iy)t}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$ again we have $W(t)=(t-a)E(-(x+iy)a,-(x+iy)t)$, $t\in[a,b]$; $U(t)=\frac{t-c}{d-c}$, $t\in[c,d]$ and

$$\begin{aligned}&\frac{1}{(b-a)E(-(x+iy)a,-(x+iy)b)}\mathcal{L}(f)(x+iy)-\frac{1}{d-c}\int_c^d f(t)\,dt\\&=\int_a^b K(t)f'(t)\,dt.\end{aligned}$$

Since $[c,d]\subseteq[a,b]$ we use (3.4) so

$$K(t)=\begin{cases}-\frac{t-a}{b-a}\frac{E(-(x+iy)a,-(x+iy)t)}{E(-(x+iy)a,-(x+iy)b)}, & t\in[a,c],\\ -\frac{t-a}{b-a}\frac{E(-(x+iy)a,-(x+iy)t)}{E(-(x+iy)a,-(x+iy)b)}+\frac{t-c}{d-c}, & t\in\langle c,d\rangle,\\ 1-\frac{t-a}{b-a}\frac{E(-(x+iy)a,-(x+iy)t)}{E(-(x+iy)a,-(x+iy)b)}, & t\in[d,b].\end{cases}$$

Thus

$$\begin{aligned}&\frac{d-c}{b-a}\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_c^d f(t)\,dt\\&=(d-c)E(-(x+iy)a,-(x+iy)b)\int_a^b K(t)f'(t)\,dt\end{aligned}$$

and by taking the modulus and applying Hölder inequality we obtain

$$\begin{aligned}&\left|\frac{d-c}{b-a}\mathcal{L}(f)(x+iy)-E(-(x+iy)a,-(x+iy)b)\int_c^d f(t)\,dt\right|\\&\leq (d-c)\left\|E(-(x+iy)a,-(x+iy)b)K(t)\right\|_q\left\|f'\right\|_p.\end{aligned}$$

Now, for $1<p\leq\infty$ (for $1\leq q<\infty$) we have

$$\begin{aligned}\left\|E(-(x+iy)a,-(x+iy)b)K(t)\right\|_q&=\left(\int_a^c\left|\frac{t-a}{b-a}E(-(x+iy)a,-(x+iy)t)\right|^q dt\right.\\&+\int_c^d\left|\frac{t-a}{b-a}E(-(x+iy)a,-(x+iy)t)-\frac{t-c}{d-c}E(-(x+iy)a,-(x+iy)b)\right|^q dt\\&\left.+\int_d^b\left|\frac{t-a}{b-a}E(-(x+iy)a,-(x+iy)t)-E(-(x+iy)a,-(x+iy)b)\right|^q dt\right)\end{aligned}$$

and since $|E(-(x+iy)r,-(x+iy)s)|=\left|\frac{e^{-(x+iy)r}-e^{-(x+iy)s}}{-(x+iy)(r-s)}\right|\leq\frac{2e^{-ax}}{|(x+iy)||r-s|}$ for $r,s\in[a,b]$, we have

$$\begin{aligned}\int_a^c\left|\frac{t-a}{b-a}E(-(x+iy)a,-(x+iy)t)\right|^q dt&\leq\int_a^c\left(\frac{2e^{-ax}}{(b-a)|x+iy|}\right)^q dt\\&=(c-a)\left(\frac{2e^{-ax}}{(b-a)|x+iy|}\right)^q,\end{aligned}$$

$$\begin{aligned}&\int_c^d\left|\frac{t-a}{b-a}E(-(x+iy)a,-(x+iy)t)-\frac{t-c}{d-c}E(-(x+iy)a,-(x+iy)b)\right|^q dt\\&=\frac{1}{((b-a)|x+iy|)^q}\int_c^d\left|\frac{d-t}{d-c}e^{-(x+iy)a}+\frac{t-c}{d-c}e^{-(x+iy)b}-e^{-(x+iy)t}\right|^q dt\\&\leq\frac{1}{((b-a)|x+iy|)^q}\left(\int_c^d\left|\frac{d-t}{d-c}e^{-(x+iy)a}+\frac{t-c}{d-c}e^{-(x+iy)b}\right|^q dt+\int_c^d\left|e^{-(x+iy)t}\right|^q dt\right)\\&\leq\frac{e^{-xaq}}{((b-a)|x+iy|)^q}\left(\int_c^d\left|\frac{d-t}{d-c}+\frac{t-c}{d-c}\right|^q dt+\int_c^d|1|^q\,dt\right)\leq\frac{2(d-c)e^{-xaq}}{((b-a)|x+iy|)^q},\end{aligned}$$

$$\begin{aligned}&\int_d^b\left|\frac{t-a}{b-a}E(-(x+iy)a,-(x+iy)t)-E(-(x+iy)a,-(x+iy)b)\right|^q dt\\&=\frac{1}{((b-a)|x+iy|)^q}\int_d^b\left|e^{-(x+iy)b}-e^{-(x+iy)t}\right|^q dt\\&\leq\frac{1}{((b-a)|x+iy|)^q}\int_d^b\left(2e^{-ax}\right)^q dt=\frac{\left(2e^{-xa}\right)^q(b-d)}{((b-a)|x+iy|)^q}.\end{aligned}$$

Thus

$$\begin{aligned}\left\|E(-(x+iy)a,-(x+iy)b)K(t)\right\|_q&\leq e^{-xa}\left(\frac{2^q(c-a)+2(d-c)+2^q(b-d)}{((b-a)|x+iy|)^q}\right)^{\frac{1}{q}}\\&=e^{-xa}\frac{2(b-a)^{\frac{1}{q}-1}}{|x+iy|}\end{aligned}$$

and since $e^{-ax} \leq 1$ inequalities in case $1 < p \leq \infty$ are proved. For $p = 1$ we have

$$\begin{aligned}
&\left\| E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right) K\left(t\right)\right\|_{\infty} \\
&= \max\left\{ \sup_{t\in[a,c]} \left| \frac{t-a}{b-a} E\left(-\left(x+iy\right)a, -\left(x+iy\right)t\right)\right|, \right. \\
&\sup_{t\in[c,d]} \left| \frac{t-a}{b-a} E\left(-\left(x+iy\right)a, -\left(x+iy\right)t\right) - \frac{t-c}{d-c} E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right)\right|, \\
&\left. \sup_{t\in[d,b]} \left| \frac{t-a}{b-a} E\left(-\left(x+iy\right)a, -\left(x+iy\right)t\right) - E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right)\right| \right\}
\end{aligned}$$

and

$$\sup_{t\in[a,c]} \left| \frac{t-a}{b-a} E\left(-\left(x+iy\right)a, -\left(x+iy\right)t\right)\right| \leq \frac{2e^{-xa}}{\left(b-a\right)\left|\left(x+iy\right)\right|},$$

$$\begin{aligned}
&\sup_{t\in[c,d]} \left| \frac{t-a}{b-a} E\left(-\left(x+iy\right)a, -\left(x+iy\right)t\right) - \frac{t-c}{d-c} E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right)\right| \\
&= \frac{1}{\left(b-a\right)\left|\left(x+iy\right)\right|} \sup_{t\in[c,d]} \left| \frac{d-t}{d-c} e^{-(x+iy)a} + \frac{t-c}{d-c} e^{-(x+iy)b} - e^{-(x+iy)t}\right| \\
&\leq \frac{e^{-xa}}{\left(b-a\right)\left|\left(x+iy\right)\right|} \sup_{t\in[c,d]} \left| \frac{d-t}{d-c} + \frac{t-c}{d-c} + 1\right| = \frac{2e^{-xa}}{\left(b-a\right)\left|\left(x+iy\right)\right|},
\end{aligned}$$

$$\begin{aligned}
&\sup_{t\in[d,b]} \left| \frac{t-a}{b-a} E\left(-\left(x+iy\right)a, -\left(x+iy\right)t\right) - E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right)\right| \\
&= \frac{1}{\left(b-a\right)\left|\left(x+iy\right)\right|} \sup_{t\in[d,b]} \left| e^{-|(x+iy)|b} - e^{-|(x+iy)|t}\right| \leq \frac{2e^{-xa}}{\left(b-a\right)\left|\left(x+iy\right)\right|}.
\end{aligned}$$

Thus

$$\left\| E\left(-xa, -xb\right) K\left(t\right)\right\|_{\infty} \leq \frac{2e^{-xa}}{\left(b-a\right)\left|\left(x+iy\right)\right|}$$

and since $e^{-ax} \leq 1$ the proof is completed. ∎

Remark 3.3. The inequalities from the previous Theorem hods for $x \geq 0$. Similarly it can be proved that in case $x < 0$ and $1 < p \leq \infty$ we have the inequality

$$\begin{aligned}
&\left| \frac{d-c}{b-a} \mathcal{L}\left(f\right)\left(x+iy\right) - E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right) \int_c^d f\left(t\right)dt\right| \\
&\leq e^{-bx}\left(d-c\right) \frac{2\left(b-a\right)^{\frac{1}{q}-1}}{x^2+y^2} \left\| f'\right\|_p,
\end{aligned}$$

while for $x < 0$ and $p = 1$ we have

$$\begin{aligned}
&\left| \frac{d-c}{b-a} \mathcal{L}\left(f\right)\left(x+iy\right) - E\left(-\left(x+iy\right)a, -\left(x+iy\right)b\right) \int_c^d f\left(t\right)dt\right| \\
&\leq e^{-bx} \frac{2\left(d-c\right)}{\left(b-a\right)\left(x^2+y^2\right)} \left\| f'\right\|_1.
\end{aligned}$$

Theorem 3.4. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $f'\in L_p[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $x\geq 0$ and $1<p\leq\infty$, we have inequalities*

$$\begin{aligned}&\left|\frac{d-c}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\int_a^b f\left(t\right)dt-\int_c^d e^{-(x+iy)t}f\left(t\right)dt\right|\\&\leq e^{-xc}\left(d-c\right)\left(\frac{\left(2^q+1\right)\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}\left\|f'\right\|_p\\&\leq\left(d-c\right)\left(\frac{\left(2^q+1\right)\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}\left\|f'\right\|_p,\end{aligned}$$

while for $p=1$ we have

$$\begin{aligned}&\left|\frac{d-c}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\int_a^b f\left(t\right)dt-\int_c^d e^{-(x+iy)t}f\left(t\right)dt\right|\\&\leq e^{-xc}2\left(d-c\right)\left\|f'\right\|_1\\&\leq 2\left(d-c\right)\left\|f'\right\|_1,\end{aligned}$$

where $E(z,w)$ is given by (1.2).

Proof. If we apply identity (3.3) with $w(t)=\frac{1}{b-a}$, $t\in[a,b]$ and $u(t)=e^{-(x+iy)t}$, $t\in[c,d]$, we have $W(t)=\frac{t-a}{b-a}$, $t\in[a,b]$; $U(t)=(t-c)E(-(x+iy)c,-(x+iy)t)$, $t\in[c,d]$ and

$$\begin{aligned}&\frac{1}{(b-a)}\int_a^b f\left(t\right)dt-\frac{1}{\left(d-c\right)E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)}\int_c^d e^{-(x+iy)t}f\left(t\right)dt\\&=\int_a^b K\left(t\right)f'\left(t\right)dt.\end{aligned}$$

Since $[c,d]\subseteq[a,b]$ we use (3.4) so

$$K\left(t\right)=\begin{cases}-\frac{t-a}{b-a}, & t\in\left[a,c\right],\\ \frac{U(t)}{U(d)}-\frac{t-a}{b-a}, & t\in\left\langle c,d\right\rangle,\\ \frac{b-t}{b-a}, & t\in\left[d,b\right].\end{cases}$$

Thus

$$\begin{aligned}&\frac{d-c}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\int_a^b f\left(t\right)dt-\int_c^d e^{-(x+iy)t}f\left(t\right)dt\\&=U\left(d\right)\int_a^b K\left(t\right)f'\left(t\right)dt\end{aligned}$$

and by taking the modulus and applying Hölder inequality we obtain

$$\begin{aligned}&\left|\frac{d-c}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\int_a^b f\left(t\right)dt-\int_c^d e^{-(x+iy)t}f\left(t\right)dt\right|\\&\leq\left\|U\left(d\right)K\left(t\right)\right\|_q\left\|f'\right\|_p.\end{aligned}$$

Now, for $1 < p \leq \infty$ (for $1 \leq q < \infty$) we have

$$\begin{aligned} \|U(d)K(t)\|_q = &\left(\int_a^c \left| \frac{t-a}{b-a} U(d) \right|^q dt \right. \\ &\left. + \int_c^d \left| U(t) - \frac{t-a}{b-a} U(d) \right|^q dt + \int_d^b \left| \frac{b-t}{b-a} U(d) \right|^q dt \right) \end{aligned}$$

and since $|U(t)| = \left| \int_c^t e^{-(x+iy)s} ds \right| \leq \int_c^t |e^{-xs}| \, ds \leq e^{-xc} \int_c^t ds = e^{-xc} (t-c)$ for $t \in [c,d]$, we have

$$\int_a^c \left| \frac{t-a}{b-a} U(d) \right|^q dt \leq e^{-xcq} \int_a^c \left(\frac{t-a}{b-a} (d-c) \right)^q dt = e^{-xcq} \left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned} &\int_c^d \left| U(t) - \frac{t-a}{b-a} U(d) \right|^q dt \leq \int_c^d \left(|U(t)| + \left| \frac{t-a}{b-a} U(d) \right| \right)^q dt \\ &\leq e^{-xcq} \int_c^d \left(t - c + \frac{d-c}{b-a} (t-a) \right)^q dt \\ &\leq \frac{e^{-xcq}}{(b-a)^q} \int_c^d ((b-a+d-c)\,t - c(b-a) - a(d-c))^q \, dt \\ &= e^{-xcq} \frac{\left(\lambda(d)^{q+1} - \lambda(c)^{q+1} \right)}{(b-a)^q (q+1)(b-a+d-c)} \\ &= e^{-xcq} \frac{(d-c)^{q+1} \left((b+d-2a)^{q+1} - (c-a)^{q+1} \right)}{(b-a)^q (q+1)(b-a+d-c)} \leq e^{-xcq} \frac{2^q (d-c)^q (b-a)}{(q+1)}, \end{aligned}$$

where $\lambda(t)$ is given by (3.6) and

$$\int_d^b \left| \frac{b-t}{b-a} U(d) \right|^q dt \leq e^{-xcq} \int_d^b \left(\frac{b-t}{b-a} (d-c) \right)^q dt = e^{-xcq} \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\begin{aligned} &\|U(d)K(t)\|_q \\ &\leq e^{-xc} \left(\left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} + \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)} \right)^{\frac{1}{q}} \\ &\leq e^{-xc} \left(\left(\frac{d-c}{b-a} \right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} \right)^{\frac{1}{q}} \\ &= e^{-xc} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \end{aligned}$$

and since $e^{-xc} \leq 1$ inequalities in case $1 < p \leq \infty$ are proved. For $p = 1$ we have

$$\begin{aligned} &\|U(d)K(t)\|_\infty \\ &= \max \left\{ \sup_{t \in [a,c]} \left| \frac{t-a}{b-a} U(d) \right|, \sup_{t \in [c,d]} \left| U(t) - \frac{t-a}{b-a} U(d) \right|, \sup_{t \in [d,b]} \left| \frac{b-t}{b-a} U(d) \right| \right\} \end{aligned}$$

and

$$\sup_{t \in [a,c]} \left| \frac{t-a}{b-a} U(d) \right| \leq e^{-xc} \frac{(c-a)(d-c)}{(b-a)},$$

$$\sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right| = \sup_{t\in[c,d]}\left\{|U(t)|+\left|\frac{t-a}{b-a}U(d)\right|\right\}$$
$$\leq e^{-xc}\sup_{t\in[c,d]}\left|d-c+\frac{d-a}{b-a}(d-c)\right| = e^{-xc}(d-c)\frac{b+d-2a}{b-a},$$
$$\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right|\leq e^{-xc}\frac{(b-d)(d-c)}{(b-a)}.$$

Thus

$$\|U(d)K(t)\|_\infty \leq e^{-xc}\frac{d-c}{b-a}\max\{(c-a),(b+d-2a),(b-d)\}\leq e^{-xc}2(d-c)$$

and since $e^{-xc}\leq 1$ the proof is completed. ∎

Remark 3.4. The inequalities from the previous Theorem hods for $x\geq 0$. Similarly it can be proved that in case $x<0$ and $1<p\leq\infty$ we have the inequality

$$\left|\frac{d-c}{b-a}E(-(x+iy)c,-(x+iy)d)\int_a^b f(t)\,dt-\int_c^d e^{-(x+iy)t}f(t)\,dt\right|$$
$$\leq e^{-dx}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\|f'\|_p,$$

while for $x<0$ and $p=1$ we have

$$\left|\frac{d-c}{b-a}E(-(x+iy)c,-(x+iy)d)\int_a^b f(t)\,dt-\int_c^d e^{-(x+iy)t}f(t)\,dt\right|$$
$$\leq e^{-dx}2(d-c)\|f'\|_1.$$

Theorem 3.5. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $f'\in L_p[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for $x+iy\neq 0$, $x\geq 0$ and $1<p\leq\infty$, we have inequalities*

$$\left|\frac{d-c}{b-a}E(-(x+iy)c,-(x+iy)d)\int_a^b f(t)\,dt-\int_c^d e^{-(x+iy)t}f(t)\,dt\right|$$
$$\leq e^{-xc}\frac{2(b-a)^{\frac{1}{q}}}{x^2+y^2}\|f'\|_p$$
$$\leq \frac{2(b-a)^{\frac{1}{q}}}{x^2+y^2}\|f'\|_p,$$

while for $p=1$ we have

$$\left|\frac{d-c}{b-a}E(-(x+iy)c,-(x+iy)d)\int_a^b f(t)\,dt-\int_c^d e^{-(x+iy)t}f(t)\,dt\right|$$
$$\leq e^{-xc}\frac{2}{x^2+y^2}\|f'\|_1$$
$$\leq \frac{2}{x^2+y^2}\|f'\|_1,$$

where $E(z,w)$ is given by (1.2).

Proof. We apply identity (3.3) again with $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = e^{-(x+iy)t}$, $t \in [c,d]$, so we have $W(t) = \frac{t-a}{b-a}$, $t \in [a,b]$;

$U(t) = (t-c)\, E(-(x+iy)\,c, -(x+iy)\,t)$, $t \in [c,d]$ and

$$\frac{1}{(b-a)} \int_a^b f(t)\,dt - \frac{1}{(d-c)\,E(-(x+iy)\,c, -(x+iy)\,d)} \int_c^d e^{-(x+iy)t} f(t)\,dt$$
$$= \int_a^b K(t)\, f'(t)\,dt.$$

Since $[c,d] \subseteq [a,b]$ we use (3.4) so

$$K(t) = \begin{cases} -\frac{t-a}{b-a}, & t \in [a,c]\,, \\ \frac{t-c}{d-c}\frac{E(-(x+iy)c,-(x+iy)t)}{E(-(x+iy)c,-(x+iy)d)} - \frac{t-a}{b-a}, & t \in \langle c,d\rangle\,, \\ \frac{b-t}{b-a}, & t \in [d,b]\,. \end{cases}$$

Thus

$$\frac{d-c}{b-a} E(-(x+iy)\,c, -(x+iy)\,d) \int_a^b f(t)\,dt - \int_c^d e^{-(x+iy)t} f(t)\,dt$$
$$= (d-c)\, E(-(x+iy)\,c, -(x+iy)\,d) \int_a^b K(t)\, f'(t)\,dt$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a} E(-(x+iy)\,c, -(x+iy)\,d) \int_a^b f(t)\,dt - \int_c^d e^{-(x+iy)t} f(t)\,dt\right|$$
$$\leq (d-c)\, \|E(-(x+iy)\,c, -(x+iy)\,d)\, K(t)\|_q \, \|f'\|_p\,.$$

Now, for $1 < p \leq \infty$ (for $1 \leq q < \infty$) we have

$$\|E(-(x+iy)\,c, -(x+iy)\,d)\, K(t)\|_q = \left(\int_a^c \left|\frac{t-a}{b-a} E(-(x+iy)\,c, -(x+iy)\,d)\right|^q dt\right.$$
$$+ \int_c^d \left|\frac{t-c}{d-c} E(-x(x+iy)\,, -(x+iy)\,t) - \frac{t-a}{b-a} E(-(x+iy)\,c, -(x+iy)\,d)\right|^q dt$$
$$\left. + \int_d^b \left|\frac{b-t}{b-a} E(-(x+iy)\,c, -(x+iy)\,d)\right|^q dt\right)$$

and since $|E(-(x+iy)\,r, -(x+iy)\,s)| \leq \left|\frac{e^{-(x+iy)r} - e^{-(x+iy)s}}{-x(r-s)}\right| \leq \frac{2e^{-xc}}{|x+iy||r-s|}$ for $r, s \in [c,d]$ we have

$$\int_a^c \left|\frac{t-a}{b-a} E(-(x+iy)\,c, -(x+iy)\,d)\right|^q dt$$
$$= |E(-(x+iy)\,c, -(x+iy)\,d)|^q \int_a^c \left(\frac{t-a}{b-a}\right)^q dt$$
$$= |E(-(x+iy)\,c, -(x+iy)\,d)|^q \frac{(c-a)^{q+1}}{(q+1)\,(b-a)^q} \leq \frac{2^q e^{-xcq}\,(c-a)^{q+1}}{(q+1)\,(|x+iy|\,(d-c)\,(b-a))^q},$$

$$\begin{aligned}
&\int_c^d \left|\frac{t-c}{d-c}E\left(-\left(x+iy\right)c,-\left(x+iy\right)t\right)-\frac{t-a}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|^q dt\\
&=\frac{1}{\left(\left(d-c\right)\left|x+iy\right|\right)^q}\int_c^d\left|\frac{b-t}{b-a}e^{-(x+iy)c}+\frac{t-a}{b-a}e^{-(x+iy)d}-e^{-(x+iy)t}\right|^q dt\\
&\le\frac{1}{\left(\left(d-c\right)\left|x+iy\right|\right)^q}\left(\int_c^d\left|\frac{b-t}{b-a}e^{-(x+iy)c}+\frac{t-a}{b-a}e^{-(x+iy)d}\right|^q+\int_c^d\left|e^{-(x+iy)t}\right|^q\right)dt\\
&=\frac{2e^{-xcq}}{\left(\left(d-c\right)\left|x+iy\right|\right)^q}\int_c^d\left|1\right|^q dt\le\frac{2e^{-xcq}\left(d-c\right)}{\left(\left(d-c\right)\left|x+iy\right|\right)^q},
\end{aligned}$$

$$\begin{aligned}
&\int_d^b\left|\frac{b-t}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|^q dt\\
&=\left|E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|^q\int_d^b\left(\frac{b-t}{b-a}\right)^q dt\\
&=\left|E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|^q\frac{\left(b-d\right)^{q+1}}{\left(q+1\right)\left(b-a\right)^q}\le\frac{2^q e^{-xcq}\left(b-d\right)^{q+1}}{\left(q+1\right)\left(\left|x+iy\right|\left(d-c\right)\left(b-a\right)\right)^q}.
\end{aligned}$$

Thus

$$\begin{aligned}
\left\|E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)K\left(t\right)\right\|_q&\le e^{-xc}\left(\frac{\frac{2^q(c-a)^{q+1}}{q+1}+2\left(d-c\right)\left(b-a\right)^q+\frac{2^q(b-d)^{q+1}}{q+1}}{\left(\left(d-c\right)\left(b-a\right)\left|x+iy\right|\right)^q}\right)^{\frac{1}{q}}\\
&\le e^{-xc}\frac{2\left(b-a\right)^{\frac{1}{q}}}{\left(d-c\right)\left|x+iy\right|}
\end{aligned}$$

and since $e^{-xc}\le 1$ inequalities in case $1<p\le\infty$ are proved. For $p=1$ we have

$$\begin{aligned}
&\left\|E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)K\left(t\right)\right\|_\infty\\
&=\max\left\{\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|,\right.\\
&\sup_{t\in[c,d]}\left|\frac{t-c}{d-c}E\left(-\left(x+iy\right)c,-\left(x+iy\right)t\right)-\frac{t-a}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|,\\
&\left.\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|\right\}
\end{aligned}$$

and

$$\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|\le\frac{e^{-xc}2\left(c-a\right)}{\left(d-c\right)\left(b-a\right)\left|x+iy\right|},$$

$$\begin{aligned}
&\sup_{t\in[c,d]}\left|\frac{t-c}{d-c}E\left(-\left(x+iy\right)c,-\left(x+iy\right)t\right)-\frac{t-a}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|\\
&=\frac{1}{\left(d-c\right)x}\sup_{t\in[c,d]}\left|\frac{b-t}{b-a}e^{-(x+iy)c}+\frac{t-a}{b-a}e^{-(x+iy)d}-e^{-(x+iy)t}\right|\\
&\le\frac{e^{-xc}}{\left(d-c\right)x}\sup_{t\in[c,d]}\left|\frac{d-t}{d-c}+\frac{t-c}{d-c}+1\right|=\frac{2e^{-xc}}{\left(d-c\right)\left|x+iy\right|},
\end{aligned}$$

$$\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\right|\leq\frac{e^{-xc}2\left(b-d\right)}{\left(d-c\right)\left(b-a\right)\left|x+iy\right|}.$$

Thus

$$\begin{aligned}\left\|E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)K\left(t\right)\right\|_{\infty}&\leq e^{-xc}\frac{\max\left\{2\left(c-a\right),2\left(b-a\right),2\left(b-d\right)\right\}}{\left(d-c\right)\left(b-a\right)\left|x+iy\right|}\\&=\frac{2e^{-xc}}{\left(d-c\right)\left|x+iy\right|}\end{aligned}$$

and since $e^{-xc}\leq 1$ the proof is completed. ∎

Remark 3.5. The inequalities from the previous Theorem hods for $x\geq 0$. Similarly it can be proved that in case $x<0$ and $1<p\leq\infty$ we have the inequality

$$\begin{aligned}&\left|\frac{d-c}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\int_a^b f\left(t\right)dt-\int_c^d e^{-\left(x+iy\right)t}f\left(t\right)dt\right|\\&\leq e^{-dx}\frac{2\left(b-a\right)^{\frac{1}{q}}}{x^2+y^2}\left\|f'\right\|_p,\end{aligned}$$

while for $x<0$ and $p=1$ we have

$$\begin{aligned}&\left|\frac{d-c}{b-a}E\left(-\left(x+iy\right)c,-\left(x+iy\right)d\right)\int_a^b f\left(t\right)dt-\int_c^d e^{-\left(x+iy\right)t}f\left(t\right)dt\right|\\&\leq e^{-dx}\frac{2}{x^2+y^2}\left\|f'\right\|_1.\end{aligned}$$

Corollary 3.6. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous and $f'\in L_p[a,b]$. Then for all $x\geq 0$ and $1<p\leq\infty$, we have the inequality*

$$\begin{aligned}&\left|E\left(-\left(x+iy\right)a,-\left(x+iy\right)b\right)\int_a^b f\left(t\right)dt-\mathcal{L}\left(f\right)\left(x+iy\right)\right|\\&\leq\left(b-a\right)^{1+\frac{1}{q}}\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\left\|f'\right\|_p,\end{aligned}$$

while for $p=1$ we have

$$\left|E\left(-\left(x+iy\right)a,-\left(x+iy\right)b\right)\int_a^b f\left(t\right)dt-\mathcal{L}\left(f\right)\left(x+iy\right)\right|\leq 2\left(b-a\right)\left\|f'\right\|_1.$$

Proof. By applying the proof of the Theorems 3.2 or 3.4 in the special case when $c=a$ and $d=b$. ∎

Remark 3.6. The results of the Theorems 3.3 and 3.5 in case $c=a$ and $d=b$ reduce to the results of the Theorem 2.1.

Corollary 3.7. *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous and $f'\in L_p[a,b]$. Then for all $x\geq 0$, for any $c\in[a,b]$ and $1<p\leq\infty$, we have the inequality*

$$\begin{aligned}&\left|\mathcal{L}(f)(x+iy)-(b-a)E(-(x+iy)a,-(x+iy)b)f(c)\right|\\ &\leq (b-a)^{1+\frac{1}{q}}\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\|f'\|_p,\end{aligned}$$

while for $p=1$ we have

$$\left|\mathcal{L}(f)(x+iy)-(b-a)E(-(x+iy)a,-(x+iy)b)f(c)\right|\leq 2(b-a)\|f'\|_1.$$

Proof. By applying the proof of the Theorem 3.2 in the special case when $c=d$. Since f is absolutely continuous, it is continuous, thus as a limit case we have $\lim_{c\to d}\frac{1}{d-c}\int_c^d f(t)\,dt=f(c)$. ∎

4. TWO NUMERICAL QUADRATURE FORMULAE

Let $I_n: a=t_0<t_1<\cdots<t_{n-1}<t_n=b$ be a division of the interval $[a,b]$, $h_k:=t_{k+1}-t_k$, $k=0,1,\ldots,n-1$ and $\nu(h):=\max_k\{h_k\}$. Define the sum

$$\mathcal{E}(f,I_n,x+iy)=\sum_{k=0}^{n-1}E(-(x+iy)t_k,-(x+iy)t_{k+1})\int_{t_k}^{t_{k+1}}f(t)\,dt \tag{4.1}$$

where $x\geq 0$.

The following approximation theorem holds.

Theorem 4.1. *Assume (p,q) is a pair of conjugate exponents. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous function on $[a,b]$, $f'\in L_p[a,b]$. Then we have the quadrature rule*

$$\mathcal{L}(f)(x+iy)=\mathcal{E}(f,I_n,x+iy)+R(f,I_n,x+iy)$$

where $x\geq 0$, $\mathcal{E}(f,I_n,x+iy)$ is given by (4.1) and for $1<p\leq\infty$ the reminder $R(f,I_n,x+iy)$ satisfies the estimate

$$|R(f,I_n,x+iy)|\leq\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}h_k^{q+1}\right]^{\frac{1}{q}}\|f'\|_p, \tag{4.2}$$

while for $p=1$

$$|R(f,I_n,x+iy)|\leq 2\nu(h)\|f'\|_1. \tag{4.3}$$

Proof. For $1<p\leq\infty$ by applying the Corollary 3.6 with $a=t_k$, $b=t_{k+1}$ we have

$$\begin{aligned}&\left|E(-(x+iy)t_k,-(x+iy)t_{k+1})\int_{t_k}^{t_{k+1}}f(t)\,dt-\int_{t_k}^{t_{k+1}}e^{-(x+iy)t}f(t)\,dt\right|\\ &\leq(t_{k+1}-t_k)^{1+\frac{1}{q}}\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\left(\int_{t_k}^{t_{k+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}}.\end{aligned}$$

Summing over k from 0 to $n-1$ and using generalized triangle inequality, we obtain

$$\begin{aligned}|R(f,I_n,x+iy)| &= |\mathcal{L}(f)(x+iy)-\mathcal{E}(f,I_n,x+iy)| \\ &\leq \sum_{k=0}^{n-1}(h_k)^{1+\frac{1}{q}}\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\left(\int_{t_k}^{t_{k+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}}.\end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned}&\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\sum_{k=0}^{n-1}(h_k)^{1+\frac{1}{q}}\left(\int_{t_k}^{t_{k+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}} \\ &\leq \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}\left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}\left(\left(\int_{t_k}^{t_{k+1}}|f'(t)|^p\,dt\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} \\ &= \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\left[\sum_{k=0}^{n-1}h_k^{q+1}\right]^{\frac{1}{q}}\|f'\|_p\end{aligned}$$

and the inequality (4.2) is proved. For $p=1$ we have

$$\begin{aligned}|R(f,I_n,x+iy)| &\leq \sum_{k=0}^{n-1}2h_k\left(\int_{t_k}^{k+1}|f'(t)|\,dt\right) \\ &\leq 2\nu(h)\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}}|f'(t)|\,dt\right)=2\nu(h)\|f'\|_1\end{aligned}$$

and the proof is completed. ∎

Corollary 4.2. *Suppose that all assumptions of Theorem 4.1 hold. Additionally suppose*

(4.4)

$$\begin{aligned}\mathcal{E}(f,I_n,x+iy) &= \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}}f(t)\,dt \\ &\cdot\sum_{k=0}^{n-1}E\left(-(x+iy)\left(a+k\cdot\frac{b-a}{n}\right),-(x+iy)\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right).\end{aligned}$$

Then we have the quadrature rule

$$\mathcal{L}(f)(x+iy)=\mathcal{E}(f,I_n,x+iy)+R(f,I_n,x+iy)$$

where $x\geq 0$ and for $1<p\leq\infty$ the reminder $R(f,I_n,x+iy)$ satisfies the estimate

$$|R(f,I_n,x+iy)|\leq\left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}}\frac{(b-a)^{1+\frac{1}{q}}}{n}\|f'\|_p, \tag{4.5}$$

while for $p=1$ we have

$$|R(g,I_n,x+iy)|\leq\frac{2(b-a)}{n}\|f'\|_1. \tag{4.6}$$

Proof. If we apply Theorem 4.1 with equidistant partition of $[a,b]$, $t_j = a + j \cdot \frac{b-a}{n}$, $j = 0,1,\ldots,n$, we have (4.4) and $h_k = \frac{b-a}{n}$, $k = 0,1,\ldots,n-1$. For $1 < p \le \infty$ we obtain

$$\begin{aligned}|R(f,I_n,x+iy)| &\le \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \|f'\|_p \\ &= \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|f'\|_p,\end{aligned}$$

while for $p = 1$, $\nu(h) = \frac{b-a}{n}$ and the claim immediately follows. ∎

Now, define the sum

$$\mathcal{A}(f,I_n,x+iy) = \sum_{k=0}^{n-1} (t_{k+1}-t_k)\, E(-(x+iy)t_k, -(x+iy)t_{k+1})\, f\left(\frac{t_{k+1}+t_k}{2}\right) \tag{4.7}$$

where $x \ge 0$.

Also the following approximation theorem holds.

Theorem 4.3. *Assume (p,q) is a pair of conjugate exponents. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous function on $[a,b]$, $f' \in L_p[a,b]$. Then we have the quadrature rule*

$$\mathcal{L}(f)(x+iy) = \mathcal{A}(f,I_n,x+iy) + R(f,I_n,x+iy)$$

where $x \ge 0$, $\mathcal{A}(f,I_n,x+iy)$ is given by (4.7) and for $1 < p \le \infty$ the reminder $R(f,I_n,x+iy)$ satisfies the estimate

$$|R(f,I_n,x+iy)| \le \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \|f'\|_p, \tag{4.8}$$

while for $p = 1$

$$|R(f,I_n,x+iy)| \le 2\nu(h)\|f'\|_1. \tag{4.9}$$

Proof. By applying the Corollary 3.7 with $a = t_k$, $b = t_{k+1}$, $c = \frac{t_{k+1}+t_k}{2}$ and then summing over k from 0 to $n-1$, we obtain results similarly as in the proof of the Theorem 4.1. ∎

Corollary 4.4. *Suppose that all assumptions of Theorem 4.3 hold. Additionally suppose*

$$\begin{aligned}\mathcal{A}(f,I_n,x+iy) = {}& \frac{b-a}{n} f\left(a + \frac{k(k+1)(b-a)}{2n}\right) \\ &\cdot \sum_{k=0}^{n-1} E\left(-(x+iy)\left(a + k\cdot\frac{b-a}{n}\right), -(x+iy)\left(a+(k+1)\cdot\frac{b-a}{n}\right)\right).\end{aligned}$$

Then we have the quadrature rule

$$\mathcal{L}(f)(x+iy) = \mathcal{A}(f,I_n,x+iy) + R(f,I_n,x+iy)$$

where $x \ge 0$ and for $1 < p \le \infty$ the reminder $R(f,I_n,x+iy)$ satisfies the estimate

$$|R(f,I_n,x+iy)| \le \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|f'\|_p, \tag{4.10}$$

while for $p = 1$ *we have*

$$|R(g, I_n, x + iy)| \leq \frac{2(b-a)}{n} \|f'\|_1 . \tag{4.11}$$

Proof. By applying Theorem 4.3 with equidistant partition of $[a, b]$. ∎

Remark 4.1. For both numerical quadrature formulae in case $x < 0$, for $1 < p \leq \infty$, the reminder $R(f, I_n, x + iy)$ satisfies the estimate

$$|R(f, I_n, x + iy)| \leq e^{-xb} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1} \right]^{\frac{1}{q}} \|f'\|_p ,$$

while for $p = 1$

$$|R(f, I_n, x + iy)| \leq e^{-xb} 2\nu(h) \|f'\|_1 .$$

For equidistant partition of $[a, b]$ and for $1 < p \leq \infty$ we have

$$|R(f, I_n, x + iy)| \leq e^{-xb} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|f'\|_p ,$$

while for $p = 1$

$$|R(f, I_n, x + iy)| \leq e^{-xb} \frac{2(b-a)}{n} \|f'\|_1 .$$

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ and J. PEČARIĆ, Error estimates for approximations of the Fourier transform of functions in L_p spaces, *AJMAA*, **1**(2004), Issue 2, Article 10.

[2] N. S. BARNETT, S. S. DRAGOMIR and G. HANNA, Error estimates for approximating the Fourier transform of functions of bounded variation, *RGMIA Res. Rep. Coll.*, **7**(2004), No. 1, Article 11.

[3] D. S. MITRINOVIĆ, J. E. PEČARIĆ and A. M. FINK, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.

[4] J. PEČARIĆ, On the Čebyšev inequality, *Bul. Inst. Politehn. Timisoara*, **25** (39), (1980), pp. 10-11.