The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 8, Issue 1, Article 1, pp. 1-9, 2011

COMMON FIXED POINT RESULTS FOR BANACH OPERATOR PAIRS AND APPLICATIONS TO BEST APPROXIMATION

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Received 16 July, 2007; accepted 26 May, 2008; published 8 September, 2011.

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ABSTRACT. The common fixed point results for Banach operator pair with generalized nonexpansive mappings in q-normed space have been obtained in the present work. As application, some more general best approximation results have also been determined without the assumption of linearity or affinity of mappings. These results unify and generalize various existing known results with the aid of more general class of noncommuting mappings.

Key words and phrases: Common fixed point, Banach operator pair, compatible maps, best approximation, q-normed space.

2000 *Mathematics Subject Classification.* Primary 41A50, 47H10. Secondary , 54H25.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Existence of fixed point has been used at many places in approximation theory. Number of results exist in the literature where fixed point theorems are used to prove the existence of best approximation(see in [1, 2, 4, 13, 16, 19, 21]).

Meinardus [13] was the first to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [2] obtained a celebrated result and generalized the Meinardus's result. Later, several results [4, 19, 21] have been proved in the direction of Brosowski [2]. In the year 1988, Sahab et al. [16] extended the result of Hicks and Humpheries [4] and Singh [19] by using two mappings, one linear and the other nonexpansive mappings for commuting mappings.

Al-Thagafi [1] extended the result of Sahab et al. [16] and proved some results on invariant approximations for commuting mappings. The introduction of non-commuting maps to this area, Shahzad [17, 18] further extended Al-Thagafi's results and obtained some results regarding invariant approximation. Afterwards, numbers of results by changing the nature of mappings for convex domain within various space structures appeared. Main contributors in this direction are Shahzad [17], Hussain et al. [6], Jungck and Hussain [9] and O'Regan and Hussain [14] for $\mathcal{R}$-subweakly commuting, compatible and $\mathcal{C}_q$-commuting maps. All the above mentioned results are obtained on starshaped domain and linearity or affinness condition of mappings.

Recently, Chen and Li [3] introduced the notion of Banach operator pair as a new class of noncommuting maps. Using this concept, common fixed-point theorems are obtained without the assumption of linearity or affinity of mappings and which is further applied to prove best approximation results in normed space.

Attempt has been made to show the validity of results of Chen and Li [3] for more general class of g-nonexpansive maps in q-normed space. Also, more general best approximation results have been determined as application of common fixed point theorem; incidently, the work of Al-Thagafi [1], Jungck and Hussain [9], O'Regan and Hussain [14] and Shahzad [17, 18] have also been extended and unified.

2. PRELIMINARIES

In the material to be produced here, the following definitions have been used:

Let $\mathcal{X}$ be a linear space. A q-norm on $\mathcal{X}$ is a real-valued function $\|.\|_q$ on $\mathcal{X}$ with $0 < q \leq 1$, satisfying the following conditions :

(a) $\|x\|_q \geq 0$ and $\|x\|_q = 0$ iff $x = 0,$

(b) $\|\lambda x\|_q = \mid \lambda \mid^q \|x\|_q,$

(c) $\|x + y\|_q \leq \|x\|_q + \|y\|_q,$

for all $x, y \in \mathcal{X}$ and all scalars λ. The pair $(\mathcal{X}, \|.\|_q)$ is called a q-normed space. It is a metric space with $d_q(x, y) = \|x - y\|_q$ for all $x, y \in \mathcal{X}$, defining a translation invariant metric d_q on $\mathcal{X}.$ If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some q-norm, $0 < q \leq 1.$ The spaces l_q and $\mathcal{L}_q[0, 1]$, $0 < q \leq 1$ are q-normed space. A q-normed space is not necessarily a locally convex space. Recall that, if $\mathcal{X}$ is a topological linear space, then its continuous dual space $\mathcal{X}^*$ is said to separate the points of $\mathcal{X}$, if for each $x \neq 0$ in $\mathcal{X},$ there exists an $\mathcal{I} \in \mathcal{X}^*$ such that $\mathcal{I}x \neq 0.$ In this case the weak topology on $\mathcal{X}$ is well-defined. We mention that, if $\mathcal{X}$ is not locally convex, then $\mathcal{X}^*$ need not separates the points of $X.$ For example, if $\mathcal{X} = \mathcal{L}_q[0, 1],$

$0 < q < 1$, then $\mathcal{X}^* = \{0\}$ ([15], page 36 and 37). However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [12].

Let $\mathcal{M}$ be a subset of a normed space $(\mathcal{X}, \|, \|)$. The set $\mathcal{P}_{\mathcal{M}}(\widehat{x}) = \{x \in \mathcal{M} : \|x - \widehat{x}\| = dist(\widehat{x}, \mathcal{M})\}$ is called the set of best approximants to $\widehat{x} \in \mathcal{X}$ out of $\mathcal{M}$, where $dist(\widehat{x}, \mathcal{M}) = inf\{\|y - \widehat{x}\| : y \in \mathcal{M}\}$. We denote by $\Im_0$ the class of closed convex subsets of $\mathcal{X}$ containing 0. For $\mathcal{M} \in \Im_0$, we define $\mathcal{M}_{\widehat{x}}$= $\{x \in \mathcal{M} : \|x\| \leq 2\|\widehat{x}\|\}$. It is clear that $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \subset \mathcal{M}_{\widehat{x}} \in \Im_0$ [1]. We shall use $\mathbb{N}$ to denote the set of positive integers, $cl(\mathcal{M})$ to denote the closure of a set $\mathcal{M}$ and $wcl(\mathcal{M})$ to denote the weak closure of a set $\mathcal{M}$. Let $f : \mathcal{M} \to \mathcal{M}$ be a mapping. A mapping $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is called an f-contraction if, for any $x, y \in \mathcal{M}$, there exists $0 \leq k < 1$ such that $\|\mathcal{T}x - \mathcal{T}y\| \leq k\|fx - fy\|$. If $k = 1$, then $\mathcal{T}$ is called f-nonexpansive. The set of fixed points of $\mathcal{T}$(resp. f) is denoted by $\mathcal{F}(\mathcal{T})$(resp. $\mathcal{F}(f)$). A point $x \in \mathcal{M}$ is a coincidence point (common fixed point) of f and $\mathcal{T}$ if $fx = \mathcal{T}x$ $(x = fx = \mathcal{T}x)$. The set of coincidence points of f and $\mathcal{T}$ is denoted by $\mathcal{C}(f, \mathcal{T})$. The pair $(f, \mathcal{T})$ is called (1) commuting if $\mathcal{T}fx = f\mathcal{T}x$ for all $x \in \mathcal{M}$, (2) $\mathcal{R}$-weakly commuting [17] if for all $x \in \mathcal{M}$, there exists $\mathcal{R} > 0$ such that $\|f\mathcal{T}x - \mathcal{T}fx\| \leq \mathcal{R}\|fx - \mathcal{T}x\|$. If $\mathcal{R} = 1$, then the maps are called weakly commuting; (3) compatible [8] if $\lim_n \|\mathcal{T}fx_n - f\mathcal{T}x_n\| = 0$ when $\{x_n\}$ is a sequence such that $\lim_n \mathcal{T}x_n = \lim_n fx_n = t$ for some t in $\mathcal{M}$; (4) weakly compatible if they commute at their coincidence points, i.e.,if $f\mathcal{T}x = \mathcal{T}fx$ whenever $fx = \mathcal{T}x$. The set $\mathcal{M}$ is called p-starshaped with $p \in \mathcal{M}$, if the segment $[p, x] = \{(1 - k)p + kx : 0 \leq k \leq 1\}$ joining q to x, is contained in $\mathcal{M}$ for all $x \in \mathcal{M}$. Suppose that $\mathcal{M}$ is p-starshaped with $p \in \mathcal{F}(f)$ and is both $\mathcal{T}$- and f-invariant. Then $\mathcal{T}$ and f are called (5) $\mathcal{R}$-subweakly commuting on $\mathcal{M}$ (see [17]) if for all $x \in \mathcal{M}$, there exists a real number $\mathcal{R} > 0$ such that $\|f\mathcal{T}x - \mathcal{T}fx\| \leq \mathcal{R}dist(fx, [p, \mathcal{T}x])$. It is well known that $\mathcal{R}$-subweakly commuting maps are $\mathcal{R}$-weakly commuting and $\mathcal{R}$-weakly commuting maps are compatible but not conversely in does not hold general (see for examples [17, 18]).

Further, definition providing the notion of Banach operator pair introduced by Chen and Li [3] may be written as:

Definition 1. ***Banach Operator Pair***. *The ordered pair (f, g) of two self-maps of a metric space $(\mathcal{X}, d)$ is called a Banach operator pair, if the set $\mathcal{F}(g)$ is f-invariant, namely $f(\mathcal{F}(g)) \subseteq \mathcal{F}(g)$. Obviously commuting pair (f, g) is Banach operator pair but not conversely in general, see* [3]. *If (f, g) is Banach operator pair then (g, f) need not be Banach operator pair(see* [3, Example 1]*).*

If the self-maps f and g of $\mathcal{X}$ satisfy

$$d(gfx, fx) \leq kd(gx, x) \tag{2.1}$$

for all $x \in \mathcal{X}$ and $k \geq 0$, then (f, g) is Banach operator pair. In particular, when $f = g$ and $\mathcal{X}$ is a normed space, (2.1) can be rewritten as

$$\|f^2x - fx\| \leq k\|fx - x\| \tag{2.2}$$

for all $x \in \mathcal{X}$. Such f is called Banach operator of type k in [21].

The following result would also be used in the sequel.

Theorem 2.1. [14, Corollary 2.2]. *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$, and f and g be self-maps of $\mathcal{M}$. Assume that $clf(\mathcal{M}) \subset g(\mathcal{M})$, $clf(\mathcal{M})$ is complete, f*

is g-continuous and f and g satisfy for all $x, y \in \mathcal{M}$ and $0 \le h < 1$,

$$d(fx, fy) \le hmax\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \tag{2.3}$$

Then $\mathcal{C}(f, g) \ne \emptyset$.

3. MAIN RESULTS

A lemma is presented below, which extends and improves Lemma 3.1 of Chen and Li [3]:

Lemma 1. *Let $\mathcal{M}$ be a nonempty closed subset of a metric space $(\mathcal{X}, d)$, and (f, g) be Banach operator pair on $\mathcal{M}$. Assume that $clf(\mathcal{M})$ is complete, and f and g satisfy for all $x, y \in \mathcal{M}$ and $0 \le h < 1$,*

$$d(fx, fy) \le h \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}. \tag{3.1}$$

If g is continuous, $\mathcal{F}(g)$ is nonempty and f is g-continuous, then there is unique common fixed point of f and g.

Proof. According to assumptions, $f(\mathcal{F}(g)) \subseteq \mathcal{F}(g)$ and $\mathcal{F}(g)$ is nonempty closed and $clf(\mathcal{F}(g)) \subseteq clf(\mathcal{M})$ is complete. Also (3.1) implies that

$$d(fx, fy) \le h \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(fy, gx), d(fx, gy)\}$$

$$= h \max\{d(x, y), d(x, fx), d(y, fy), d(fy, x), d(fx, y)\}$$

for all $x,\ y \in \mathcal{F}(g)$. Hence f is generalized contraction on $\mathcal{F}(g)$ and $clf(\mathcal{F}(g)) \subseteq cl\mathcal{F}(g) = \mathcal{F}(g)$. Thus, Theorem 2.1 guarantees that, f has a unique fixed point w in $\mathcal{F}(g)$ and consequently $\mathcal{F}(f, g)$ is singleton. ∎

Theorem 3.1. *Let $\mathcal{M}$ be a nonempty closed subset of a q-normed space $\mathcal{X}$ which is starshaped with respect to $p \in \mathcal{M}$, and f and g be self-maps of $\mathcal{M}$. Suppose that g is continuous, $\mathcal{F}(g)$ is p-starshaped with $p \in \mathcal{F}(g)$ and f is g-continuous. If (f, g) is Banach operator pair and satisfies, for all $x, y \in \mathcal{M}$,*

$$\begin{aligned}\|fx - fy\|_q \le \max\{&\|gx - gy\|_q, dist(gx, [fx, p]), dist(gy, [fy, p]),\\ &dist(gx, [fy, p]), dist(gy, [fx, p])\},\end{aligned} \tag{3.2}$$

then $\mathcal{M} \cap \mathcal{F}(f, g) \ne \emptyset$, provided one of the following conditions holds;

(i) *$clf(\mathcal{M})$ is compact,*

(ii) *$\mathcal{X}$ is complete, $wclf(\mathcal{M})$ is weakly compact, g is weakly continuous and $g - f$ is demiclosed at 0,*

Proof. Choose a sequence $\{k_n\} \subset (0, 1)$ with $k_n \to 1$ as $n \to \infty$. Define for each $n \ge 1$ and for all $x \in \mathcal{M}$, a mapping f_n by

$$f_n x = (1 - k_n)p + k_n fx.$$

Then each n, f_n is a self-mapping of $\mathcal{M}$, since $\mathcal{M}$ is p-starshaped with $p \in \mathcal{M}$. Again by (3.2),

$$\begin{aligned}\|f_nx - f_ny\|_q &= (k_n)^q\|fx - fy\|_q \\ &\leq (k_n)^q \max\{\|gx - gy\|_q, dist(gx, [fx, p]), dist(gy, [fy, p]), \\ &\qquad dist(gx, [fy, p]), dist(gy, [fx, p])\} \\ &\leq (k_n)^q\, max\{\|gx - gy\|_q, \|gx - f_nx\|_q, \|gy - f_ny\|_q, \|gx - f_ny\|_q, \\ &\qquad \|gy - f_nx\|_q\},\end{aligned}$$

for each $x, y \in \mathcal{M}$ and $0 < k_n < 1$. Since (f, g) is Banach operator pair, for $x \in \mathcal{F}(g)$, we have $fx \in \mathcal{F}(g)$, and hence $f_nx = (1 - k_n)p + k_nfx \in \mathcal{F}(g)$ by the fact that $\mathcal{F}(g)$ is p-starshaped with $p \in \mathcal{F}(g)$. Thus (f_n, g) is Banach operator pair on $\mathcal{M}$ for each n.

(i) As $clf(\mathcal{M})$ is compact, for each $n \in \mathbb{N}$, $clf_n(\mathcal{M})$ is compact and hence complete. By Lemma 1, for each $n \geq 1$, there exists $y_n \in \mathcal{M}$ such that $y_n = gy_n = f_ny_n$. The compactness of $cl(f(\mathcal{M}))$ implies that there exists a subsequence $\{fy_m\}$ of $\{fy_n\}$ such that $fy_m \to z \in cl(f(\mathcal{M}))$ as $m \to \infty$. Since $k_m \to 1$, $y_m = f_my_m = (1-k_m)p+k_mfy_m \to z$. As g is continuous, then gy_m converges to y and hence $y = gy$. The g continuity of f implies that fy_m converges to fy. Consequently, $y = fy = gy$. Thus $\mathcal{M} \cap \mathcal{F}(f, g) \neq \emptyset$.

(ii) By weak compactness of $wclf(\mathcal{M})$, $wclf_n(\mathcal{M})$ is weakly compact[9] and hence complete for each n. By Lemma 1, there exists $y_n \in \mathcal{M}$ such that $y_n = gy_n = f_ny_n$. Since $wclf(\mathcal{M})$ is weakly compact, there exists a subsequence $\{y_m\}$ of $\{y_n\}$ and $y \in \mathcal{M}$ such that $y_m \to y$ weakly. The weak continuity of g implies that $y = gy$. Further, $\|gx_m - fx_m\|_q = \|((1 - k_m)p + k_mfx_m) - fx_m\|_q = (1 - k_m)^q\|p - fy_m\|_q$ converges to 0, as y_m is bounded and $k_m \to 1$. The demiclosedness of $g - f$ at 0 implies that $gy = fy$. Thus $\mathcal{M} \cap \mathcal{F}(f, g) \neq \emptyset$.

∎

Remark 1. *Theorem 3.1 extends and improves Theorem* 2.2 *of* [1], *Theorems 3.2 - Theorem 3.3 of* [3] *and Theorem* 6 *of* [10].

Following is a more general result in best approximation theory with the aid of Banach operator pair, a generalized class of noncommuting mappings in q-normed space.

Theorem 3.2. *Let $\mathcal{M}$ be subset of a q-normed space $\mathcal{X}$ and $f, g : \mathcal{X} \to \mathcal{X}$ be mappings such that $\widehat{x} \in \mathcal{F}(f, g)$ for some $\widehat{x} \in \mathcal{X}$ and $f(\partial\mathcal{M}) \subset \mathcal{M}$. Suppose that g is continuous on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(\widehat{x})$, $\mathcal{D} \cap \mathcal{F}(g)$ is nonempty closed p-starshaped, $g(\mathcal{D}) = \mathcal{D}$ and f is g-continuous. If the pair (f, g) is a Banach operator pair on $\mathcal{D}$ and satisfies*

$$\|fx - fy\|_q \leq \begin{cases} \|gx - g\widehat{x}\|_q \ \textit{if} \ \ y = \widehat{x}, \\ \max\{\|gx - gy\|_q, dist(gx, [p, fx]), dist(gy, [p, fy]), \\ dist(gx, [p, fy]), dist(gy, [p, fx])\}, \ \textit{if}\, y \in \mathcal{D}, \end{cases} \tag{3.3}$$

then $\mathcal{D} \cap \mathcal{F}(f,g) \neq \emptyset$, *provided one of the following conditions is satisfied;*

(i) $cl(f(\mathcal{D}))$ *is compact,*

(ii) $\mathcal{X}$ *is complete,* $wcl(f(\mathcal{D}))$ *is weakly compact,* g *is weakly continuous and* $g - f$ *is demi-closed at* 0*,*

Proof. First, we show that f is self-map on $\mathcal{D}$, i.e., $f : \mathcal{D} \to \mathcal{D}$. Let $y \in \mathcal{D}$, then $gy \in \mathcal{D}$, since $g(\mathcal{D}) = \mathcal{D}$. Also, if $y \in \partial\mathcal{M}$, then $gy \in \mathcal{M}$, since $g(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $g\widehat{x} = \widehat{x} = f\widehat{x}$, one may have from (3.3)

$$\|fy - \widehat{x}\|_q = \|fy - f\widehat{x}\|_q \leq \|gy - g\widehat{x}\|_q = \|gy - \widehat{x}\|_q = dist(\widehat{x}, \mathcal{M}).$$

Thus, $fy \in \mathcal{D}$. Consequently f and g are self-maps on $\mathcal{D}$. The conditions of Theorem 3.1 ((i) and (ii)) are satisfied and hence, there exists a $z \in \mathcal{D}$ such that $fz = z = gz$. ∎

Defines $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(\widehat{x})$, where $\mathcal{D}^{\mathcal{I}}_{\mathcal{M}}(\widehat{x}) = \{x \in \mathcal{M} : \mathcal{I}x \in \mathcal{P}_{\mathcal{M}}(\widehat{x})\}$.

Theorem 3.3. *Let* $\mathcal{M}$ *be subset of a* q*-normed space* $\mathcal{X}$ *and* $f, g : \mathcal{X} \to \mathcal{X}$ *be mappings such that* $\widehat{x} \in \mathcal{F}(f,g)$ *for some* $\widehat{x} \in \mathcal{X}$ *and* $f(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$*. Suppose that* g *is nonexpansive on* $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cup \{\widehat{x}\}$*,* $\mathcal{D} \cap \mathcal{F}(g)$ *is nonempty closed* p*-starshaped,* $g(\mathcal{D}(\widehat{x})) = \mathcal{D}$ *and* f *is* g*-continuous. If the pair* (f,g) *is a Banach operator pair on* $\mathcal{D}$ *and satisfies*

$$(3.4)\qquad \|fx - fy\|_q \leq \begin{cases} \|gx - g\widehat{x}\|_q \ \textit{if} \ \ y = \widehat{x}, \\ \\ \max\{\|gx - gy\|_q, dist(gx, [p, fx]), dist(gy, [p, fy]), \\ \\ dist(gx, [p, fy]), dist(gy, [p, fx])\}, \ \textit{if}\ y \in \mathcal{D}, \end{cases}$$

then $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{F}(f,g) \neq \emptyset$, *provided one of the following conditions is satisfied;*

(i) $cl(f(\mathcal{D}))$ *is compact,*

(ii) $\mathcal{X}$ *is complete,* $wcl(f(\mathcal{D}))$ *is weakly compact,* g *is weakly continuous and* $g - f$ *is demi-closed at* 0*,*

Proof. Let $x \in \mathcal{D}$. Then, $x \in \mathcal{P}_{\mathcal{M}}(\widehat{x})$ and hence $\|x - \widehat{x}\|_q = dist(x_0, \mathcal{M})$. Note that for any $t \in (0,1)$,

$$\|t\widehat{x} + (1-t)x - \widehat{x}\|_q = (1-t)^q\|x - \widehat{x}\|_q < dist(\widehat{x}, \mathcal{M}).$$

It follows that the line segment $\{t\widehat{x} + (1-t)x : 0 < t < 1\}$ and the set $\mathcal{M}$ are disjoint. Thus x is not in the interior of $\mathcal{M}$ and so $x \in \partial\mathcal{M} \cap \mathcal{M}$. Since $f(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$, fx must be in $\mathcal{M}$. Also, proceeding as in the proof of Theorem 3.2, we have $fx \in \mathcal{P}_{\mathcal{M}}(\widehat{x})$. As g is nonexpansive on $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cup \{\widehat{x}\}$, we have

$$\|gfx - \widehat{x}\|_q \leq \|fx - f\widehat{x}\|_q \leq \|gx - g\widehat{x}\|_q = \|gx - \widehat{x}\|_q = dist(\widehat{x}, \mathcal{M}).$$

Thus $gfx \in \mathcal{P}_{\mathcal{M}}(\widehat{x})$ and so $fx \in \mathcal{D}^{g}_{\mathcal{M}}(\widehat{x})$. Hence $fx \in \mathcal{D}$. Consequently, $f(\mathcal{D}) \subset \mathcal{D} = g(\mathcal{D})$. Now Theorem 3.1 ((i) and (ii)) guarantee that $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{F}(f,g) \neq \phi$. ∎

Remark 2. *It is remark that the Theorem 3.3 is trivial if $\widehat{x} \in \mathcal{M}$, because the statement in the proof that $\mathcal{M}$ and the line segment $t\widehat{x} + (1-t)x$ are disjoint is no longer necessarily true if $\widehat{x} \in \mathcal{M}$.*

For $h \geq 0$, let $\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})$=$\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{G}_{\mathcal{M}}{}^{h,g}(\widehat{x})$, where $\mathcal{G}_{\mathcal{M}}{}^{h,g}(\widehat{x})$=$\{x \in \mathcal{M} : \|gx - \widehat{x}\|_q \leq (2h+1)dist(\widehat{x}, \mathcal{M})\}$.

Theorem 3.4. *Let $\mathcal{M}$ be subset of a q-normed space $\mathcal{X}$ and $f, g : \mathcal{X} \to \mathcal{X}$ be mappings such that $\widehat{x} \in \mathcal{F}(f,g)$ for some $\widehat{x} \in \mathcal{X}$ and $f(\partial\mathcal{M} \cap \mathcal{M}) \subset \mathcal{M}$. Suppose that g is continuous on $\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})$, $\mathcal{D}_{\mathcal{M}}{}^{h,g}(u) \cap \mathcal{F}(g)$ is nonempty closed p-starshaped, $g(\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x}))$ = $\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})$ and f is g-continuous. If the pair (f,g) satisfies*

(a) $\|gfx - fx\|_q \leq h\|gx - x\|_q$ for all $x \in \mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})$ and $h \geq 0$

(b) for all $x \in \mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x}) \cup \{\widehat{x}\}$,

$$\|fx - fy\|_q \leq \begin{cases} \|gx - g\widehat{x}\|_q \ \ \textit{if} \ \ y = \widehat{x}, \\ \max\{\|gx - gy\|_q, dist(gx, [p, fx]), dist(gy, [p, fy]), \\ dist(gx, [p, fy]), dist(gy, [p, fx])\}, \ \ \textit{if}\ y \in \mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x}), \end{cases} \tag{3.5}$$

then $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{F}(f,g) \neq \emptyset$, provided one of the following conditions is satisfied;

(i) *$cl(f(\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})))$ is compact,*

(ii) *$\mathcal{X}$ is complete, $wcl(f(\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})))$ is weakly compact, g is weakly continuous and $g - f$ is demiclosed at 0,*

Proof. Let $x \in \mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})$. Then, along in the line of the proof of Theorem 3.3, $fx \in \mathcal{P}_{\mathcal{M}}(\widehat{x})$. From inequality in (a) and (3.5), it follow that,

$$\begin{aligned}
\|gfx - \widehat{x}\|_q &= \|gfx - fx + fx - \widehat{x}\|_q \\
&\leq \|gfx - fx\|_q + \|fx - \widehat{x}\|_q \\
&\leq h\|gx - x\|_q + \|fx - \widehat{x}\|_q \\
&= h\|gx - \widehat{x} + \widehat{x} - x\|_q + \|fx - \widehat{x}\|_q \\
&\leq h(\|gx - u\|_q + \|x - \widehat{x}\|_q) + \|fx - \widehat{x}\|_q \\
&\leq h(dist(\widehat{x}, \mathcal{M}) + dist(\widehat{x}, \mathcal{M})) + dist(\widehat{x}, \mathcal{M}) \\
&\leq (2h+1)dist(\widehat{x}, \mathcal{M}).
\end{aligned}$$

Thus $fx \in \mathcal{G}_{\mathcal{M}}{}^{h,g}(\widehat{x})$. Consequently, $f(\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})) \subset \mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x})$=$g(\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x}))$. Inequality in (a) also implies that (f,g) is a Banach operator pair. Now by Theorem 3.1 ((i) and (ii)) we obtain, $\mathcal{P}_{\mathcal{M}}(\widehat{x}) \cap \mathcal{F}(f,g) \neq \emptyset$ in each of the cases (i) and (ii). ∎

Remark 3. *If we take* $\mathcal{C}_{\mathcal{M}}{}^{g}(\widehat{x}) = \{x \in \mathcal{M} : gx \in \mathcal{P}_{\mathcal{M}}(\widehat{x})\}$. *Then* $g(\mathcal{P}_{\mathcal{M}}(\widehat{x})) \subset \mathcal{P}_{\mathcal{M}}(\widehat{x})$ *implies* $\mathcal{P}_{\mathcal{M}}(u) \subset \mathcal{C}_{\mathcal{M}}{}^{g}(\widehat{x}) \subset \mathcal{G}_{\mathcal{M}}{}^{h,g}(\widehat{x})$ *and hence* $\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x}) = \mathcal{P}_{\mathcal{M}}(\widehat{x})$. *Consequently, Theorem 3.4 remains valid when* $\mathcal{D}_{\mathcal{M}}{}^{h,g}(\widehat{x}) = \mathcal{P}_{\mathcal{M}}(\widehat{x})$ *and the pair* (f, g) *is Banach operator on* $\mathcal{P}_{\mathcal{M}}(\widehat{x})$ *instead of satisfying* (a)*, which in turn extends many results (see* [1, 10, 11, 13, 16, 19, 21]*).*

Remark 4. *Theorem 3.1 - Theorem 3.4 generalize Theorems 3.2-Theorem 4.2 in* [3] *in the sense that the more generalized relatively nonexpansive mappings in* q*-normed space have been used in place of relatively nonexpansive.*

Acknowledgement 1. *My deep sense of gratitude to the Professor Sever S. Dragormir and referee for accepting the paper.*

REFERENCES

[1] M. A. AL-THAGAFI, Common fixed points and best approximation, *J. Approx. Theory*, **85**(1996), No. 3, pp. 318–323.

[2] B. BROSOWSKI, Fixpunktsätze in der Approximationstheorie, *Mathematica(Cluj)*, **11**(1969), pp. 165–220.

[3] J. CHEN and Z. LI, Common fixed points for Banach operator pairs in best approximation, *J. Math. Anal. Appl.*, **336** (2007), No. 2, pp. 1466–1475.

[4] T. L. HICKS and M. D. HUMPHRIES, A note on fixed point theorems, *J. Approx. Theory*, **34**(1982), pp. 221–225.

[5] N. HUSSAIN and A. R. KHAN, Common fixed point results in best approximation theory, *Applied Math. Lett.*, **16**(2003), pp. 575–580.

[6] N. HUSSAIN, D. O'REGAN and R. P. AGARWAL, Common fixed point and invariant approximation results on non-starshaped domains, *Georgian Math. J.*, **12**(2005), pp. 659–669.

[7] G. JUNGCK, Common fixed points for commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, **103**(1988), pp. 977–983.

[8] G. JUNGCK, Common fixed point theorems for compatible self maps of Hausdorff topological spaces, *Fixed Point Theory and Appl.*, **3**(2005), 355–363.

[9] G. JUNGCK and N. HUSSAIN, Compatible maps and invariant approximations, *J. Math. Anal. Appl.*, **325**(2007), pp. 1003–1012.

[10] G. JUNGCK and S. SESSA, Fixed point theorems in best approximation theory, *Math. Japon.*, **42**(1995), pp. 249–252.

[11] L. A. KHAN and A. R. KHAN, An extention of Brosowski-Meinardus theorem on invariant approximations, *Approx. Theory Appl.*, **11**(1995), pp. 1–5.

[12] G. KÖTHE, *Topological vector spaces I*, Springer-Verlag, Berlin, 1969.

[13] G. MEINARDUS, Invarianze bei linearen approximationen, *Arch. Rational Mech. Anal.*, **14**(1963), pp. 301–303.

[14] D. O'REGAN and N. HUSSAIN, Generalized I-contractions and pointwise R-subweakly commuting maps, *Acta Math. Sinica*, **23**(2007), No. 8, pp. 1505–1508.

[15] W. RUDIN, *Functional Analysis*, 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.

[16] S. A. SAHAB, M. S. KHAN and S. SESSA, A result in best approximation theory, *J. Approx. Theory*, **55**(1988), pp. 349–351.

[17] N. SHAHZAD, Invariant approximations and R-subweakly commuting maps, *J. Math. Anal. Appl.*, **257**(2001), pp. 39–45.

[18] N. SHAHZAD, Remarks on invariant approximations, *Intern. J. Math. Game Theory and Algebra*, **13**(2003), pp. 143–145.

[19] S. P. SINGH, An application of fixed point theorem to approximation theory, *J. Approx. Theory*, **25**(1979), pp. 89–90.

[20] A. SMOLUK, Invariant approximations, *Mat. Stos.*, **17**(1981), 17–22.

[21] P. V. SUBRAHMANYAM, An application of a fixed point theorem to best approximation, *J. Approx. Theory*, **20**(1977), pp. 165–172.