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MAXIMAL INEQUALITIES FOR MULTIDIMENSIONALLY INDEXED DEMIMARTINGALES AND THE HÁJEK-RÉNYI INEQUALITY FOR ASSOCIATED RANDOM VARIABLES

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ABSTRACT. Demimartingales and demisubmartingales introduced by Newman and Wright (1982) generalize the notion of martingales and submartingales respectively. In this paper we define multidimensionally indexed demimartingales and demisubmartingales and prove a maximal inequality for this general class of random variables. As a corollary we obtain a Hájek-Rényi inequality for multidimensionally indexed associated random variables, the bound of which, when reduced to the case of single index, is sharper than the bounds already known in the literature.

Key words and phrases: Multidimensionally Indexed Random Variables, Demimartingales, Demisubmartingales, Hájek-Rényi Inequality, Associated Random Variables, Strong Law of Large Numbers.

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1. INTRODUCTION

Newman and Wright (1982) introduced the concept of a demimartingale and demisubmartingale as a generalization of the notion of martingales and submartingales. The definition is a rather technical one and serves, among other things, the purpose of studying the behavior of the partial sum of mean zero associated random variables.

In this paper we define the class of multidimensionally indexed demimartingales and demisubmartingales as a natural generalization of the notion of Newman and Wright (1982) and prove a Chow type maximal inequality. Since the partial sum of mean zero associated random variables is a demimartingale, we obtain as a corollary a Hájek-Rényi inequality for multidimensionally indexed associated random variables. It is worth pointing out that this Hájek-Rényi inequality, when reduced to the case of single index, gives a bound which is half the (best) bound known in the literature.

Let d be a positive integer. We denote by $\mathbb{N}^d$ the d-dimensional positive integer lattice. For $\mathbf{n}, \mathbf{m} \in \mathbb{N}^d$ with $\mathbf{n} = (n_1, \ldots, n_d)$ and $\mathbf{m} = (m_1, \ldots, m_d)$ the notation $\mathbf{n} \le \mathbf{m}$ means that $n_i \le m_i \ \forall\ i = 1, \ldots, d$ while the notation $\mathbf{n} < \mathbf{m}$ means that $n_i \le m_i \ \forall\ i = 1, \ldots, d$ with at least one inequality strict. The notation $\mathbf{k} \to \infty$ means that $\min_{1 \le j \le d} k_j \to \infty$.

From now on, all random variables are defined on a probability space $(\Omega,\ \mathfrak{A},\ P)$.

Definition 1.1. A collection of multidimensionally indexed random variables $\{X_{\mathbf{i}}, \mathbf{i} \le \mathbf{n}\}$ is said to be associated if for any two coordinatewise nondecreasing functions f and g

$$Cov\left(f(X_{\mathbf{i}}, \mathbf{i} \le \mathbf{n}), g(X_{\mathbf{i}}, \mathbf{i} \le \mathbf{n})\right) \ge 0,$$

provided that the covariance is defined. An infinite collection is associated if every finite subcollection is associated.

The above definition is just the classical definition of association stated for the case of multidimensionally indexed random variables. The index of the variables in no way affects the qualitative property of association, i.e., that nondecreasing functions of all (or some) of the variables are nonnegatively correlated.

The concept of association is related to the idea of a demimartingale, as defined in [3]. Various authors have produced results for demimartingales and demisubmartingales, mainly maximal inequalities. See for example [1], [4], [5], and [6]. In this paper we extend the idea of a demimartingale and demisubmartingale to the case of multiple index as follows:

Definition 1.2. An array of random variables $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is called a multidimensionally indexed demimartingale if:

$$E\left\{(X_{\mathbf{j}} - X_{\mathbf{i}}) f(X_{\mathbf{k}}, \mathbf{k} \le \mathbf{i})\right\} \ge 0,\ \forall\ \mathbf{i},\ \mathbf{j} \in \mathbb{N}^d \text{ with } \mathbf{i} \le \mathbf{j},$$

and for all componentwise nondecreasing functions f. If in addition f is required to be nonnegative then $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is said to be a multidimensionally indexed demisubmartingale.

It is easy to verify that the partial sum of mean zero associated multidimensionally indexed random variables is a multidimensionally indexed demimartingale.

2. A CHOW TYPE MAXIMAL INEQUALITY

The following result is a Chow type maximal inequality for the collection $\{g(Y_{\mathbf{n}}), \mathbf{n} \in \mathbb{N}^d\}$ where $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ is a multidimensionally indexed demimartingale and g is a nondecreasing convex function. The monotonicity assumption of g will be relaxed later.

Lemma 2.1. *Let* $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be a multidimensionally indexed demimartingale and* $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *a nonincreasing array of positive numbers. Further let* g *be a nonnegative and nondecreasing convex function on* $\mathbb{R}$ *with* $g(0) = 0$*. Then* $\forall\, \varepsilon > 0$*:*

$$\varepsilon P\left(\max_{\mathbf{k}\le\mathbf{n}}\{c_{\mathbf{k}} g(Y_{\mathbf{k}})\} \ge \varepsilon\right) \quad\le\quad \min_{1\le s\le d}\left\{\sum_{\mathbf{k}\le\mathbf{n}} c_{\mathbf{k}} E[g(Y_{\mathbf{k};s;i}) - g(Y_{\mathbf{k};s;i-1})]\right\},$$

where $Y_{\mathbf{k};s;i} = Y_{k_1\ldots k_{s-1} i k_{s+1}\ldots k_d}$*, i.e., at the* s^{th} *position of the index* $\mathbf{k}$ *the component* k_s *is equal to* i*, and where* $Y_{\mathbf{k}}$ *should be taken to be zero if at least one of* $k_1, \ldots, k_d$ *is zero.*

Proof. For simplicity we give the proof for $d = 2$. The case $d > 2$ is similar.

Define the sets

$$\begin{aligned} A &= \left\{\max_{(i,j)\le(n_1,n_2)}\{c_{ij} g(Y_{ij})\} \ge \varepsilon\right\}, \\ B_{1j} &= \{c_{1j} g(Y_{1j}) \ge \varepsilon\},\ 1 \le j \le n_2, \\ B_{ij} &= \{c_{lj} g(Y_{lj}) < \varepsilon,\ 1 \le l < i,\ c_{ij} g(Y_{ij}) \ge \varepsilon\},\ 2 \le i \le n_1,\ 1 \le j \le n_2. \end{aligned}$$

By the definitions of the sets A and B_{ij} we have that $A = \bigcup_{i,j} B_{ij}$ and thus

$$\begin{aligned} \varepsilon P(A) &= \varepsilon P\left(\bigcup_{(i,j)\le(n_1,n_2)} B_{ij}\right) \\ &\le \varepsilon \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} P(B_{ij}) \\ &= \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} E\left(\varepsilon I_{B_{ij}}\right) \\ &\le \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\ &= \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j}) I_{B_{1j}}\right] + \sum_{j=1}^{n_2}\sum_{i=2}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\ &= \sum_{j=1}^{n_2} E[c_{1j} g(Y_{1j})] - \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j}) I_{B_{1j}^c}\right] + \sum_{j=1}^{n_2} E\left[c_{2j} g(Y_{2j}) I_{B_{2j}}\right] \\ &+ \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\ &\le \sum_{j=1}^{n_2} E[c_{1j} g(Y_{1j})] + \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{2j}} - g(Y_{1j}) I_{B_{1j}^c}\right] \\ &+ \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right], \end{aligned} \tag{2.1}$$

where the last inequality follows from the monotonicity of the array $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$.
Since $B_{2j} \subseteq B_{1j}^c \Rightarrow I_{B_{2j}} = I_{B_{1j}^c} - I_{B_{1j}^c \bigcap B_{2j}^c}$ then:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j})) I_{B_{1j}^c}\right] \\
&- \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] + \sum_{j=1}^{n_2} \sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&= \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] \\
&- \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j})) I_{B_{1j}}\right] - \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\
&+ \sum_{j=1}^{n_2} \sum_{i=3}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right]. \qquad (2.2)
\end{aligned}$$

Since g is nondecreasing convex, we can write

$$g(y) - g(x) \geq (y - x) h(x)$$

where

$$h(y) = \lim_{x \to y^-} \frac{g(x) - g(y)}{x - y}$$

is the left derivative of g. Observe that $I_{B_{1j}} h(Y_{1j})$ is a nonnegative and nondecreasing function of Y_{1j} and by the demimartingale property of $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$ we have that

$$E\left[(g(Y_{2j}) - g(Y_{1j})) I_{B_{1j}}\right] \geq E\left[(Y_{2j} - Y_{1j}) h(Y_{1j}) I_{B_{1j}}\right] \geq 0, \text{ for } j = 1, 2, \ldots, n_2.$$

Then,

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] - \sum_{j=1}^{n_2} c_{2j} E\left[g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\
&+ \sum_{j=1}^{n_2} E\left[c_{3j} g(Y_{3j}) I_{B_{3j}}\right] + \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] \\
&+ \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{3j}} - g(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] + \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \qquad (2.3)
\end{aligned}$$

where (2.3) follows from the monotonicity of the array $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$.

Since $B_{3j} \subseteq B_{1j}^c \bigcap B_{2j}^c$ then $I_{B_{3j}} = I_{B_{1j}^c \bigcap B_{2j}^c} - I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}$ and we further have:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] \\
&+ \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j})) I_{B_{1j}^c \bigcap B_{2j}^c}\right] - \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] \\
&+ \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right] \\
&= \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] + \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j}))\right] \\
&- \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j})) I_{B_{1j} \bigcup B_{2j}}\right] - \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] \\
&+ \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}$$

Using the same arguments as before regarding the demimartingale property of $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$ it can be shown that, since $I_{B_{1j} \bigcup B_{2j}}$ is a nonnegative nondecreasing function of Y_{1j} and Y_{2j},

$$E\left[(g(Y_{3j}) - g(Y_{2j}))\, I_{B_{1j} \bigcup B_{2j}}\right] \geq 0, \text{ for } j = 1, 2, \ldots, n_2.$$

Therefore:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{j=1}^{n_2} E\left[c_{1j} g(Y_{1j})\right] + \sum_{j=1}^{n_2} E\left[c_{2j}(g(Y_{2j}) - g(Y_{1j}))\right] + \sum_{j=1}^{n_2} c_{3j} E\left[(g(Y_{3j}) - g(Y_{2j}))\right] \\
&- \sum_{j=1}^{n_2} c_{3j} E\left[g(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] + \sum_{j=1}^{n_2} \sum_{i=4}^{n_1} E\left[c_{ij} g(Y_{ij}) I_{B_{ij}}\right].
\end{aligned}$$

Continuing in the same manner and since by definition $Y_{0j} = 0$ we finally have:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{i-1j})\right] - \sum_{j=1}^{n_2} c_{n_1 j} E\left[g(Y_{n_1 j}) I_{\bigcap_{i=1}^{n_1} B_{ij}^c}\right] \\
&\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{i-1j})\right]. \qquad (2.4)
\end{aligned}$$

Similarly it can be shown that:

$$\begin{aligned}
\varepsilon P(A) &\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{ij-1})\right] - \sum_{i=1}^{n_1} c_{i n_2} E\left[g(Y_{i n_2}) I_{\bigcap_{j=1}^{n_2} B_{ij}^c}\right] \\
&\leq \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} c_{ij} E\left[g(Y_{ij}) - g(Y_{ij-1})\right]. \qquad (2.5)
\end{aligned}$$

Inequalities (2.4) and (2.5) give the desired result. ∎

Remark 2.1. Lemma 2.1 was proved under the assumption that g is nondecreasing. However, as the next result shows, the assumption is not necessary. The proof of Theorem 2.2 uses Lemma 2.1 as an auxiliary result.

Theorem 2.2. *Let $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ be a multidimensionally indexed demimartingale and $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ a nonincreasing array of positive numbers. Let g be a nonnegative convex function on $\mathbb{R}$ with $g(0) = 0$.*
Then $\forall\, \varepsilon > 0$:

$$\varepsilon P\left(\max_{\mathbf{k}\leq\mathbf{n}}\{c_{\mathbf{k}}g(Y_{\mathbf{k}})\} \geq \varepsilon\right) \leq \min_{1\leq s\leq d}\left\{\sum_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}} E\left[g(Y_{\mathbf{k};s;i}) - g(Y_{\mathbf{k};s;i-1})\right]\right\},$$

where $Y_{\mathbf{k};s;i} = Y_{k_1\ldots k_{s-1} i k_{s+1}\ldots k_d}$, i.e., at the s^{th} position of the index $\mathbf{k}$ the component k_s is equal to i, and where $Y_{\mathbf{k}}$ should be taken to be zero if at least one of $k_1, \ldots, k_d$ is zero.

Proof. (For $d = 2$.)
Following a standard argument (see for example [6]) let $u(x) = g(x)I\{x \geq 0\}$ and $v(x) = g(x)I\{x < 0\}$. Clearly u is a nonnegative nondecreasing convex function while v a nonnegative nonincreasing convex function. From the definition of $u(x)$ and $v(x)$ we have:

$$g(x) = u(x) + v(x) = \max\{u(x), v(x)\}.$$

Then,

$$\begin{aligned}\varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} g(Y_{ij}) \geq \varepsilon\right) &= \varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij}\max\{u(Y_{ij}), v(Y_{ij})\} \geq \varepsilon\right)\\ &\leq \varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)}\{c_{ij}u(Y_{ij})\} \geq \varepsilon\right)\\ &+ \varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)}\{c_{ij}v(Y_{ij})\} \geq \varepsilon\right). \end{aligned} \tag{2.6}$$

Since u is nonnegative nondecreasing convex, by Lemma 2.1 we have:

$$\begin{aligned}\varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)}\{c_{ij}u(Y_{ij})\} \geq \varepsilon\right) &\leq \min\left\{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[u(Y_{ij}) - u(Y_{i-1j})\right],\right.\\ &\qquad \left.\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[u(Y_{ij}) - u(Y_{ij-1})\right]\right\}. \end{aligned} \tag{2.7}$$

We will show that

$$\begin{aligned}\varepsilon P\left(\max_{(i,j)\leq(n_1,n_2)}\{c_{ij}v(Y_{ij})\} \geq \varepsilon\right) &\leq \min\left\{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[v(Y_{ij}) - v(Y_{i-1j})\right],\right.\\ &\qquad \left.\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}E\left[v(Y_{ij}) - v(Y_{ij-1})\right]\right\}. \end{aligned} \tag{2.8}$$

Define the sets

$$\begin{aligned} A &= \left\{ \max_{(i,j)\le(n_1,n_2)} \{c_{ij} v(Y_{ij})\} \ge \varepsilon \right\}, \\ B_{1j} &= \{c_{1j} v(Y_{1j}) \ge \varepsilon\},\ 1 \le j \le n_2, \\ B_{ij} &= \{c_{lj} v(Y_{ij}) < \varepsilon,\ 1 \le l < i,\ c_{ij} v(Y_{ij}) \ge \varepsilon\},\ 2 \le i \le n_1,\ 1 \le j \le n_2. \end{aligned}$$

Then,

$$\begin{aligned} \varepsilon P(A) &\le \sum_{j=1}^{n_2} c_{1j} E\left[v(Y_{1j})\right] + \sum_{j=1}^{n_2} c_{2j} E\left[v(Y_{2j}) - v(Y_{1j})\right] - \sum_{j=1}^{n_2} c_{2j} E\left[(v(Y_{2j}) - v(Y_{1j})) I_{B_{1j}}\right] \\ &- \sum_{j=1}^{n_2} c_{2j} E\left[v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] + \sum_{j=1}^{n_2}\sum_{i=3}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right], \end{aligned} \tag{2.9}$$

where (2.9) is obtained by following all the steps presented between (2.1) and (2.2) but for the function v instead of the function g.

Since $v(x)$ is a nonnegative nonincreasing convex function, the function

$$h(y) = \lim_{x \to y^-} \frac{v(x) - v(y)}{x - y}$$

is a nonpositive nondecreasing function. By the convexity of the function v,

$$v(Y_{2j}) - v(Y_{1j}) \ge (Y_{2j} - Y_{1j}) h(Y_{1j}).$$

Since $h(Y_{1j})$ is a nonpositive nondecreasing function, the function $-h(Y_{1j})$ is nonnegative nonincreasing and $-h(Y_{1j}) I_{B_{1j}}$ is a nonincreasing function of Y_{1j}, since by definition the indicator function $I_{B_{1j}}$ is a nonincreasing function of Y_{1j}. Then $h(Y_{1j}) I_{B_{1j}}$ is a nondecreasing function of Y_{1j}. Further, by the demimartingale property of $\{Y_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$ we have:

$$E\left[(v(Y_{2j}) - v(Y_{1j})) I_{B_{ij}}\right] \ge E\left[(Y_{2j} - Y_{1j}) I_{B_{ij}} h(Y_{1j})\right] \ge 0.$$

Thus,

$$\begin{aligned} \varepsilon P(A) &\le \sum_{j=1}^{n_2} c_{1j} E\left[v(Y_{1j})\right] + \sum_{j=1}^{n_2} c_{2j} E\left[(v(Y_{2j}) - v(Y_{1j}))\right] - \sum_{j=1}^{n_2} c_{2j} E\left[v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\ &+ \sum_{j=1}^{n_2} c_{3j} E\left[v(Y_{3j}) I_{B_{3j}}\right] + \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right] \\ &\le \sum_{j=1}^{n_2}\sum_{i=1}^{2} c_{ij} E\left[(v(Y_{ij}) - v(Y_{i-1j}))\right] + \sum_{j=1}^{n_2} c_{3j} E\left[v(Y_{3j}) I_{B_{3j}} - v(Y_{2j}) I_{B_{1j}^c \bigcap B_{2j}^c}\right] \\ &+ \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right] \\ &= \sum_{j=1}^{n_2}\sum_{i=1}^{3} c_{ij} E\left[(v(Y_{ij}) - v(Y_{i-1j}))\right] - \sum_{j=1}^{n_2} c_{3j} E\left[(v(Y_{3j}) - v(Y_{2j})) I_{B_{1j} \bigcup B_{2j}}\right] \\ &- \sum_{j=1}^{n_2} c_{3j} E\left[v(Y_{3j}) I_{B_{1j}^c \bigcap B_{2j}^c \bigcap B_{3j}^c}\right] + \sum_{j=1}^{n_2}\sum_{i=4}^{n_1} E\left[c_{ij} v(Y_{ij}) I_{B_{ij}}\right]. \end{aligned}$$

The indicator $I_{B_{1j}\bigcup B_{2j}}$ is a nonincreasing function of Y_{1j}, Y_{2j}, so by using the same arguments as before we have:

$$E[(v(Y_{3j}) - v(Y_{2j}))I_{B_{1j}\bigcup B_{2j}}] \geq 0.$$

Continuing in the same way we finally have:

$$\begin{aligned} \varepsilon P(A) &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{i-1j})\right] - \sum_{j=1}^{n_2} c_{n_1 j} E\left[v(Y_{n_1 j}) I_{\bigcap_{i=1}^{n_1} B_{ij}^c}\right] \\ &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{i-1j})\right]. \end{aligned} \tag{2.10}$$

By symmetry it can be shown that:

$$\begin{aligned} \varepsilon P(A) &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{ij-1})\right] - \sum_{i=1}^{n_1} c_{i n_2} E\left[v(Y_{i n_2}) I_{\bigcap_{j=1}^{n_2} B_{ij}^c}\right] \\ &\leq \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} c_{ij} E\left[v(Y_{ij}) - v(Y_{i-1j})\right]. \end{aligned} \tag{2.11}$$

(2.10) and (2.11) together yield (2.8) and finally combining (2.6), (2.7) and (2.8) we obtain the desired result. ∎

Using Theorem 2.2 as a source result, one can obtain various maximal probability and maximal moment inequalities, as well as asymptotic results. For example, we can have the following strong law of large numbers whose proof is established using similar arguments to those found in the proof of Corollary 2.7 in [2].

Corollary 2.3. *Assume that* $\{Y_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$, $\{c_{\mathbf{k}}, \mathbf{k} \in \mathbb{N}^d\}$ *and the function* g *are as in Theorem 2.2. Further assume that there exists a number* $p \geq 1$ *such that* $E[g(Y_{\mathbf{k}})]^p < \infty$ *and for some* $1 \leq s \leq d$, $\sum_{\mathbf{k}} c_{\mathbf{k}}^p E([g(Y_{\mathbf{k}})]^p - [g(Y_{\mathbf{k};s;k_s-1})]^p) < \infty$ *and* $\sum_{k_i, i\neq s} c_{\mathbf{k};s;N}^p E[g(Y_{\mathbf{k};s;N})]^p < \infty$ *for each* $N \in \mathbb{N}$. *Then*

$$c_{\mathbf{k}} g(Y_{\mathbf{k}}) \to 0 \text{ a.s , as } \mathbf{k} \to \infty.$$

3. THE HÁJEK-RÉNYI INEQUALITY FOR ASSOCIATED RANDOM VARIABLES

Using Theorem 2.2 we derive the Hájek-Rényi inequality for arrays of mean zero associated random variables.

Corollary 3.1. *Let* $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *be mean zero multidimensionally indexed associated random variables,* $\{c_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^d\}$ *a nonincreasing array of positive numbers and* $S_{\mathbf{n}} = \sum_{\mathbf{k}\leq\mathbf{n}} X_{\mathbf{k}}$. *Then* $\forall\, \varepsilon > 0$,

$$P\left(\max_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}} \mid S_{\mathbf{k}} \mid \geq \varepsilon\right) \leq \min_{1\leq s\leq d}\left\{\varepsilon^{-2}\sum_{\mathbf{k}\leq\mathbf{n}} c_{\mathbf{k}}^2\left[2Cov(S_{\mathbf{k};s;i-1}, S_{\mathbf{k}}^{(s)}) + E(S_{\mathbf{k}}^{(s)})^2\right]\right\}$$

where

$$S_{\mathbf{k}}^{(s)} = \sum_{i_1=1}^{k_1}\cdots\sum_{i_{s-1}=1}^{k_{s-1}}\sum_{i_{s+1}=1}^{k_{s+1}}\cdots\sum_{i_d=1}^{k_d} X_{i_1\ldots i_{s-1}k_s i_{s+1}\ldots i_r},$$

$$S_{\mathbf{k};s;k_s-1} = \sum_{l_1=1}^{k_1}\cdots\sum_{l_{s-1}=1}^{k_{s-1}}\sum_{l_s=1}^{k_s-1}\sum_{l_{s+1}=1}^{k_{s+1}}\cdots\sum_{l_d=1}^{k_d} X_{\mathbf{l}},$$

and where $S_{\mathbf{k}}$ *should be taken to be zero if at least one of* $k_1, \ldots, k_d$ *is zero.*

Proof. (For $d = 2$)
It can be easily verified that the array $\{S_{\mathbf{n}}, \mathbf{n} \in \mathbb{N}^2\}$ is a 2-indexed demimartingale. Let $g(x) = \mid x \mid^2$. Then g is a nonnegative convex function.

$$\begin{aligned} P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} \mid S_{ij} \mid \geq \varepsilon\right) &= P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij}^2 \mid S_{ij} \mid^2 \geq \varepsilon^2\right) \\ &\leq \varepsilon^{-2} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}^2 E\left(\mid S_{ij} \mid^2 - \mid S_{i-1j} \mid^2\right) \\ &= \varepsilon^{-2} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}^2 E\left[(S_{ij} + S_{i-1j})(S_{ij} - S_{i-1j})\right] \\ &= \varepsilon^{-2} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}^2 E\left[\sum_{m=1}^{j} X_{im}\left(2S_{i-1j} + \sum_{m=1}^{j} X_{im}\right)\right] \\ &= \varepsilon^{-2} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}^2 \left[2Cov\left(S_{i-1j}, \sum_{m=1}^{j} X_{im}\right) + E\left(\sum_{m=1}^{j} X_{im}\right)^2\right] \qquad (3.1) \end{aligned}$$

where the first inequality follows from Theorem 2.2. Similarly it can be shown that
(3.2)

$$P\left(\max_{(i,j)\leq(n_1,n_2)} c_{ij} \mid S_{ij} \mid \geq \varepsilon\right) \leq \varepsilon^{-2} \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} c_{ij}^2 \left[2Cov\left(S_{ij-1}, \sum_{m=1}^{i} X_{mj}\right) + E\left(\sum_{m=1}^{i} X_{mj}\right)^2\right].$$

The result now follows from (3.1)and (3.2). ∎

Remark 3.1. Observe that the results in this paper, although proved for $d \geq 2$ are also trivially valid for the case $d = 1$. It is easy to see that for the case $d = 1$ the bound derived by Corollary 3.1 is half the bound of the Hájek-Rényi inequality for associated random variables derived in [1]. For $d = 2$ the result compares favorably with the Hájek-Rényi inequality in [6] for various choices of the c_{ij}'s, for example for $c_{ij} = 1 \;\forall\; (i, j)$, or $c_{ij} = (ij)^{-1} \;\forall\; (i, j)$.

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