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TWO REMARKS ON COMMUTATORS OF HARDY OPERATORS

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ABSTRACT. Fu and Lu showed that the commutator of multiplication operator by b and the n-dimensional Hardy operator is bounded on L^p if b is in some CMO space. We shall prove the converse of this theorem and also prove that their result is optimal by giving a counterexample.

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1. INTRODUCTION

Since Coifman, Rochberg and Weiss [2] introduced the commutator of multiplication operator and singular integral operator, many studies have been done for this commutator (see the references in [5]). Long and Wang [5] considered the commutator of multiplication operator by b and Hardy operator $Hf(x) = x^{-1}\int_0^x f(t)dt$. Fu and Lu [3] generalized their results on $\mathbb{R}^n$. They showed that if $b \in CMO^p(\mathbb{R}^n) \cap CMO^{p'}(\mathbb{R}^n)$, the commutator of multiplication operator by b and the n-dimensional Hardy operator is bounded on $L^p(\mathbb{R}^n)$.

In this paper we show the converse of their theorem and also prove that the condition $CMO^p(\mathbb{R}^n) \cap CMO^{p'}(\mathbb{R}^n)$ is optimal by giving a counterexample in Section 5.

The following notation is used: For a set $E \subset \mathbb{R}^n$ we denote the Lebesgue measure of E by $|E|$. We denote the characteristic function of E by χ_E. We write a ball of radius r centered at the origin by $B(0,r) = \{x; |x| < r\}$.

2. DEFINITIONS

First we define n-dimensional fractional Hardy operators. Let $0 \le \beta < n$.

Definition 2.1.

$$H_\beta f(x) := \frac{1}{|x|^{n-\beta}} \int_{B(0,|x|)} f(y)dy, \quad x \in \mathbb{R}^n \setminus \{0\},$$

and the adjoint operator

$$H_\beta^* f(x) := \int_{\complement B(0,|x|)} \frac{f(y)}{|y|^{n-\beta}}dy, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

When $\beta = 0$, H_0 is the n-dimensional Hardy operator.

Let b be a locally integrable function on $\mathbb{R}^n$. We define the commutator operator of multiplication by b and the fractional Hardy operator as follows.

Definition 2.2.

$$H_{\beta,b}f(x) := b(x)H_\beta f(x) - H_\beta(bf)(x), \qquad H_{\beta,b}^* f(x) := b(x)H_\beta^* f(x) - H_\beta^*(bf)(x).$$

Chen and Lau [1] and García-Cuerva [4] introduced CMO^p spaces and Herz-Hardy spaces HA^p, and proved the next duality theorem. Let $1 < p < \infty$.

Definition 2.3. A function $f \in L^p_{loc}(\mathbb{R}^n)$ is said to belong to $CMO^p(\mathbb{R}^n)$, if

$$\|f\|_{CMO^p} := \sup_{r>0} \inf_c \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - c|^p dx \right)^{1/p} < \infty.$$

Remark 2.1. When $p_1 > p_2$, $CMO^{p_1} \subset CMO^{p_2}$ and the John-Nirenberg space BMO is contained in CMO^p.

Definition 2.4. We say a is a centered p-atom if there exists $r > 0$ such that

$$\mathrm{supp}(a) \subset B(0,r), \quad \|a\|_{L^p} \le |B(0,r)|^{1/p-1} \quad \text{and} \quad \int a(x)dx = 0.$$

Definition 2.5. We say f is in $HA^p(\mathbb{R}^n)$ if f can be written as

$$f(x) = \sum_{j=1}^\infty c_j a_j(x),$$

where a_j are centered p-atoms and $\sum_{j=1}^{\infty} |c_j| < \infty$, and we define

$$\|f\|_{HA^p} := \inf \sum_{j=1}^{\infty} |c_j|,$$

where the infimum is taken over all representations of f.

Remark 2.2. $HA^p(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$ where $H^1(\mathbb{R}^n)$ is the ordinary Hardy space.

Proposition 2.1 ([4])**.** *Let $1 < p < \infty$. The dual space of $HA^p(\mathbb{R}^n)$ is $CMO^{p'}(\mathbb{R}^n)$ where $1/p + 1/p' = 1$.*

$$(HA^p(\mathbb{R}^n))^* = CMO^{p'}(\mathbb{R}^n).$$

3. Theorems

Fu and Lu [3] showed the following.

Theorem 3.1 ([3])**.** *Let $1 < p < \infty, 0 \le \beta < n$ and $1/q = 1/p - \beta/n > 0$. If $b \in CMO^{p'}(\mathbb{R}^n) \cap CMO^q(\mathbb{R}^n)$, then $H_{\beta,b}$ and $H^*_{\beta,b}$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

$$\|H_{\beta,b}\|_{L^q} \le C(\|b\|_{CMO^{p'}} + \|b\|_{CMO^q})\|f\|_{L^p},$$
$$\|H^*_{\beta,b}\|_{L^q} \le C(\|b\|_{CMO^{p'}} + \|b\|_{CMO^q})\|f\|_{L^p}.$$

Throughout this paper, C is a positive constant which is independent of essential parameters and not necessarily same at each occurrence.

We obtain the converse of this theorem.

Theorem 3.2. *Let $1 < p < \infty, 0 \le \beta < n$ and $1/q = 1/p - \beta/n > 0$. If $H_{\beta,b}$ and $H^*_{\beta,b}$ are bounded operators from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $b \in CMO^{p'}(\mathbb{R}^n) \cap CMO^q(\mathbb{R}^n)$. Furthermore*

$$\|b\|_{CMO^{p'}} + \|b\|_{CMO^q} \le C\big(\|H_{\beta,b}\|_{L^p\to L^q} + \|H^*_{\beta,b}\|_{L^p\to L^q}\big).$$

We also show that the both conditions $b \in CMO^{p'}(\mathbb{R}^n)$ and $b \in CMO^q(\mathbb{R}^n)$ are necessary to obtain $H_{\beta,b} : L^p \to L^q$ only. We shall prove this by giving a counterexample in Section 5.

4. Proof of Theorem

To prove Theorem 3.2 we shall prove the following theorem.

Theorem 4.1. *Let $1 < p < \infty, 0 \le \beta < n$ and $1/q = 1/p - \beta/n > 0$. If $H_{\beta,b}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, then $b \in CMO^{p'}(\mathbb{R}^n)$. Furthermore*

$$\|b\|_{CMO^{p'}} \le C\|H_{\beta,b}\|_{L^p\to L^q}.$$

By using this theorem, we can prove Theorem 3.2.

Proof of Theorem 3.2. By the assumption, $H^*_{\beta,b}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, therefore $H_{\beta,b}$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$. Note that $1/p' = 1/q' - \beta/n$. By Theorem 4.1 we obtain $b \in CMO^q(\mathbb{R}^n)$, since $(q')' = q$. ∎

Now we prove Theorem 4.1.

Proof of Theorem 4.1. By the duality between HA^p and $CMO^{p'}$ (see Proposition 2.1 in Section 2), it suffices to show the following: For any centered p-atom a,

$$\left|\int a(x)b(x)dx\right| \le C\|H_{\beta,b}\|_{L^p\to L^q}. \tag{4.1}$$

To prove (4.1) we need the next lemma.

Lemma 4.2. *Let $1 < p < \infty, 0 \le \beta < n$ and $1/q = 1/p - \beta/n$. For any centered p-atom a, there exist f and g such that*

$$a(x) = f(x)H_\beta^* g(x) - g(x)H_\beta f(x) \quad \text{and} \quad \|f\|_{L^p} \cdot \|g\|_{L^{q'}} \le C.$$

Proof of Lemma 4.2. Let a be a centered p-atom supported in $B(0, r)$. We set

$$f(x) = (\log \tfrac{3}{2} \cdot \omega_{n-1})^{-1} a(x) \quad \text{and} \quad g(x) = |x|^{-\beta}\chi_{\{2r \le |x| \le 3r\}}(x),$$

where ω_{n-1} is the surface of the unit sphere S^{n-1}.

When $|x| > r$, $H_\beta f(x) = 0$, therefore $g \cdot H_\beta f \equiv 0$.

When $|x| \le r$,

$$H_\beta^* g(x) = \omega_{n-1} \int_{2r}^{3r} \frac{1}{t} dt = \log \tfrac{3}{2} \cdot \omega_{n-1}.$$

Therefore we have $f(x) \cdot H_\beta^* g(x) = a(x)$, and obtain $a = fH_\beta^* g - gH_\beta f$.

Furthermore we have

$$\|f\|_{L^p} \le C\|a\|_{L^p} \le Cr^{n(1/p-1)} \quad \text{and} \quad \|g\|_{L^{q'}} \le Cr^{-\beta+n/q'} = Cr^{-\beta+n-n/q},$$

and obtain $\|f\|_{L^p}\|g\|_{L^{q'}} \le C$. ∎

By using this Lemma, we prove (4.1).

$$\left|\int a(x)b(x)dx\right| = \left|\int \big(f(x)H_\beta^* g - g(x)H_\beta f(x)\big)b(x)dx\right| = \left|\int f(x)H_{\beta,b}^* g(x)dx\right|$$
$$\le \|f\|_{L^p}\|H_{\beta,b}^* g\|_{L^{p'}} \le \|H_{\beta,b}^*\|_{L^{q'} \to L^{p'}}\|f\|_{L^p}\|g\|_{L^{q'}} \le C\|H_{\beta,b}\|_{L^p \to L^q}.$$

We obtain the desired result. ∎

5. COUNTEREXAMPLE

In Theorem 4.1 we have already showed that the condition $b \in CMO^{p'}(\mathbb{R}^n)$ is necessary to obtain the boundedness of $H_{\beta,b}$ from L^p to L^q. In this section we shall show that the condition $b \in CMO^q(\mathbb{R}^n)$ is optimal by giving a counterexample. When $p_1 > p_2$, $CMO^{p_1} \subset CMO^{p_2}$. Therefore we need to consider the case $p' < q$ where $1/q = 1/p - \beta/n$. We prove the following: For any $p' < r < q$, there exists a function b such that $b \in CMO^r \setminus CMO^q$ and $H_{\beta,b}$ is not bounded from L^p to L^q.

Counterexample 1. *Suppose that $1/q = 1/p - \beta/n$ and $p' < r < q$. Let $A_j = \{x \in \mathbb{R}^n; 2^j < |x| < 2^j + 1\}$, $A = \cup_{j=2}^\infty A_j$ and define*

$$b(x) = \sum_{j=2}^\infty 2^{j/r}\chi_{A_j}(x),$$
$$f(x) = \big(|x|^n(\log|x|)^2\big)^{-1/p}\chi_{\{|x|>2\}\setminus A}(x).$$

Then $b \in CMO^r(\mathbb{R}^n) \setminus CMO^q(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, but $H_{\beta,b}f \notin L^q(\mathbb{R}^n)$.

Proof. It suffices to show that $H_{\beta,b}f \notin L^q(\mathbb{R}^n)$. Since the supports of b and f are disjoint, $H_{\beta,b}f(x) = b(x)H_\beta f(x)$. For $x \in A_j$, we have

$$H_\beta f(x) \ge \frac{C}{2^{j(n-\beta)}} \int_{2^{j-1}+1<|y|<2^j} \big(|y|^n(\log|y|)^2\big)^{-1/p} dy \ge C2^{j(\beta-n/p)}j^{-2/p},$$

and

$$b(x)H_\beta f(x) \ge C2^{j(1/r+\beta-n/p)}j^{-2/p} = C2^{j(1/r-n/q)}j^{-2/p}.$$

Therefore we obtain

$$\int_{\mathbb{R}^n} (H_{\beta,b} f(x))^q dx \geq C \sum_{j=2}^{\infty} 2^{jq(1/r-n/q)} 2^{j(n-1)} j^{-2q/p} = \infty,$$

since $q(1/r - n/q) + n - 1 > q/r - 1 > 0$.
∎

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