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UNIFORM CONVERGENCE OF SCHWARZ METHOD FOR NONCOERCIVE VARIATIONAL INEQUALITIES SIMPLE PROOF

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ABSTRACT. In this paper we study noncoercive variational inequalities, using the Schwarz method. The main idea of this method consists in decomposing the domain in two subdomains. We give a simple proof for the main result concerning L^∞ error estimates, using the Zhou geometrical convergence and the L^∞ approximation given for finite element methods by Courty-Dumont.

Key words and phrases: Schwarz method, Variational inequalities, L^∞-error estimates.

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1. INTRODUCTION

We are interested in the following noncoercive variational inequality
Find $u \in H_0^1(\Omega)$ solution of

$$\begin{cases} a(u, v-u) \geq (f, v-u) \\ u \leq \Psi, v \leq \Psi \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^2$with boundary $\partial\Omega$.
and the noncoercive bilinear form $a(u, v)$.
Or equivalently
Find $u \in H_0^1(\Omega)$ solution of

$$\begin{cases} b(u, v-u) \geq (f + \lambda u, v-u) \\ u \leq \Psi, v \leq \Psi \end{cases} \tag{1.2}$$

where

$$b(u, v) = a(u, v) + \lambda(u, v) \tag{1.3}$$

and $\lambda > 0$ large enough such that $\forall v \in H_0^1(\Omega)$ we have

$$b(v, v) \geq \mu \|v\|_{H^1(\Omega)}^2, \mu > 0 \tag{1.4}$$

In Section 2, we give the continuous V.I problem, we study the existence and the uniqueness of the solution, then we introduce the continuous Schwarz method. In Section 3, we consider the discrete problem and we establish a survey similar to the one of the continuous case. In Section 4, we give a simple proof for the main result concerning error estimates in the L^∞ norm for the problem studied, while taking as a basis on the combination of the Zhou [16] geometrical convergence and the the L^∞ approximation given for finite element methods by Courty-Dumont for variational inequalities.

2. THE CONTINUOUS PROBLEM

2.1. **Notations and assumptions.** Let's consider the functions

$$a_{i,j}(x), a_i(x), a_0(x) \in C^2\left(\overline{\Omega}\right), x \in \overline{\Omega}, 1 \leq i, j \leq n \tag{2.1}$$

such that

$$\sum_{1 \leq i,j \leq n} a_{ij}(x)\xi_i\xi_j \geq \alpha |\xi|^2 ; \xi \in \mathbb{R}^n, \alpha > 0 \tag{2.2}$$

$$a_{ij}(x) = a_{ji}(x); a_0(x) \geq \beta > 0 \tag{2.3}$$

We define the bilinear form, $\forall u, v \in H_0^1(\Omega)$

$$a(u, v) = \int_\Omega \left(\sum_{1 \leq i,j \leq n} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{1 \leq i \leq n} a_i(x) \frac{\partial u}{\partial x_i} v + a_0(x) uv \right) dx \tag{2.4}$$

Let f be

$$f \in L^\infty(\Omega) \cap C^2\left(\overline{\Omega}\right); f \geq 0 \tag{2.5}$$

and

$$K_{(\Psi,g)} = \left\{ v \in H^1(\Omega), v - g \in H_0^1(\Omega), 0 \leq v \leq \Psi \text{ on } \Omega \right\} \tag{2.6}$$

with the obstacle Ψ and g is a regular function defined on $\partial\Omega$.

$$\Psi, g \in W^{2,p}(\Omega), p > 2. \text{ such that } o \leq g \leq \Psi \text{ on } \partial\Omega \tag{2.7}$$

2.2. **The continuous problem.** Find $u \in K_{(\Psi,g)}$ the solution of

$$b\left(u, v-u\right) \geq \left(f+\lambda u, v-u\right), \forall v \in K_{(\Psi,g)} \tag{2.8}$$

Theorem 2.1. (cf. [11]) *Under the conditions (1.1) to (1.4) and (2.1) to (2.7), the problem (2.8) has an unique solution $u \in K_{(\Psi,g)}$. Moreover we have*

$$u \in W^{2,p}\left(\Omega\right), 2 < p < \infty \tag{2.9}$$

3. THE DISCRETE PROBLEM

3.1. **Discretization.** Let $V_{h_i} = V_{h_i}\left(\Omega_i\right)$ be the space of continuous piecewise linear functions on τ^{h_i} which vanish on $\partial\Omega \cap \Omega_i$.
For $w \in C\left(\overline{\Lambda}_i\right)$, we define the following space

$$V_{h_i}^{(w)} = \{v \in V_{h_i}/v = 0 \text{ on } \partial\Omega \cap \Omega_i; v = \pi_{h_i}(w) \text{ on } \Lambda_i\} \tag{3.1}$$

Where π_{h_i} denotes the interpolation operator on Λ_i. For $i = 1, 2$, let τ^{h_i} be a standard regular finite element triangulation in Ω_i, h_i being the meshsize. We suppose that the two triangulations are mutually independent on $\Omega_1 \cup \Omega_2$ A triangle belonging to one triangulation does not necessarily belong to the other. We assume that the corresponding matrices resulting from the discretizations of problem, are M-matrices. (Cf. [16]).

3.2. **Position of the discrete problem.** The discrete probleme is find $u_h \in H_0^1\left(\Omega\right)$ the solution of

$$\left\{ \begin{array}{c} b\left(u_h, v_h-u_h\right) \geq \left(f+\lambda u_h, v_h-u_h\right) \\ u_h \leq r_h\Psi, v_h \leq r_h\Psi \end{array} \right. \tag{3.2}$$

Let $\bar{u}_h$ be the solution of

$$b_h(\bar{u}_h, v_h) = b(u, v_h) \tag{3.3}$$

where u is the solution of the continuous variational inequality.
We proved that

$$\|u-\bar{u}_h\| \leq Ch^2|lnh|^2 \tag{3.4}$$

and

$$\|u-r_hu\| \leq Ch^2|lnh|^2 \tag{3.5}$$

We given assumption related to (2.1), we taken $\rho = \psi_{|B(x_0;Ch)}$.
Thus $\forall x \in B(x_0; Ch)$ such that $u(x_0) = \psi(x_0)$ then

$$|u(x)-\rho(x)| \leq Ch^2|lnh|^2 \tag{3.6}$$

Theorem 3.1. (Cf.[8]) *Under the conditions in (1.1) to (1.4),(2.1) to (2.7), (3.3) to (3.6) and the the maximum principle, there exists a constant C_1 independent of h such that*

$$\|u-u_h\|_{L^\infty(\Omega)} \leq C_1h^2\left|\ln h\right|^2 \tag{3.7}$$

Lemma 3.2. (Cf.[10]) *Under the conditions in (1.1) to (1.4) and (2.1), (2.6) to (2.9) and the maximum principle, there exists a constant C_2 independent of h such that*

$$\|u_h-r_h\Psi\| \leq C_2h^2\left|\ln h\right|^2 \tag{3.8}$$

3.3. **Domain Decomposition Method.** We decompose Ω into two overlapping polygonal subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2 \tag{3.9}$$

In Theorem **??**, the solution u satisfies the condition of the following local regularity

$$u \;/\; \Omega_i \in W^{2,p}\left(\Omega_i\right), 2 \leq p < \infty \tag{3.10}$$

We denote $\partial\Omega_i$ the boundary of Ω_i and

$$\Lambda_1 = \partial\Omega_1 \cap \Omega_2, \Lambda_2 = \partial\Omega_2 \cap \Omega_1 \tag{3.11}$$

We assume that

$$\overline{\Lambda}_1 \cap \overline{\Lambda}_2 = \emptyset \tag{3.12}$$

where $f_i = \left(f + \lambda u^n\right)/\Omega_i, i = 1, 2$
and $u_i = u/\Omega_i, b\left(u, v\right) = b\left(u, v\right)/\Omega_i; i = 1, 2.$

3.4. **The discrete Schwarz method.** We give the discrete Schwarz method as follows starting from

$$u_{1h}^0 = 0 \text{ and } u_{2h}^0 = \overline{u}_h \tag{3.13}$$

such that $\overline{u}_h$ is a solution of the following equation

$$b\left(\overline{u}_h, v\right) = \left(f + \lambda\overline{u}_h, v\right), \forall v \in K_{(\Psi,0)} \tag{3.14}$$

We define the discrete sequence of Schwarz $\left(u_h^n\right)_{n\in\mathbb{N}}$ such that

$$\left\{\begin{array}{c} u_{1h}^{n+1} \in V_{h_1}^{(u_{2h}^n)} \text{ is a solution of} \\ b_1\left(u_{1h}^{n+1}, v - u_{1h}^{n+1}\right) \geq \left(f_1 + \lambda u_{1h}^n, v - u_{1h}^{n+1}\right), \forall v \in V_{h_1}^{(u_{2h}^n)} \\ u_{1h}^{n+1} \leq r_h\Psi, v \leq r_h\Psi \end{array}\right. \tag{3.15}$$

and

$$\left\{\begin{array}{c} u_{2h}^{n+1} \in V_{h_2}^{(u_{1h}^{n+1})} \text{ is a solution of} \\ b_2\left(u_{2h}^{n+1}, v - u_{2h}^{n+1}\right) \geq \left(f_2 + \lambda u_{2h}^n, v - u_{2h}^{n+1}\right), \forall v \in V_{h_2}^{(u_{1h}^{n+1})} \\ u_{2h}^{n+1} \leq r_h\Psi, v \leq r_h\Psi \end{array}\right. \tag{3.16}$$

Zhou in [16] gives the algebraic form of the discrete algorithm and the geometrical convergence of the sequences.

Theorem 3.3. Cf. [16] *Under the conditions in (1.1) to (1.4) and (2.1), the sequence*

$$\left(u_{1h}^{n+1}\right), \left(u_{2h}^{n+1}\right); n \geq 0$$

converge geometrically to the unique solution u of the discrete problem, such that

$$\exists\theta \in \left]0, 1\right[, \forall n \geq 0.$$

$$\left\|u_{ih} - u_{ih}^{n+1}\right\|_{L^\infty(\Omega_i)} \leq (\theta)^n \left\|u_h - u_h^0\right\|_{L^\infty(\Lambda_i)}; i = 1, 2. \tag{3.17}$$

4. $L^{\infty}-$ERROR ESTIMATE

4.1. $L^{\infty}-$**error estimate.** We finish by $L^{\infty}-$ error estimate.

Theorem 4.1. *There exists a constant C independent of h such that*

$$\left\|u_i - u_{ih}^{n+1}\right\|_{L^{\infty}(\Omega_i)} \leq Ch^2 \left|\ln h\right|^2 = 1, 2. \tag{4.1}$$

Proof. We have

$$\left\|u_i - u_{ih}^{n+1}\right\|_{L^{\infty}(\Omega_i)} \leq \left\|u_i - u_{ih}\right\|_{L^{\infty}(\Omega_i)} + \left\|u_{ih} - u_{ih}^{n+1}\right\|_{L^{\infty}(\Omega_i)}$$

We used Theorems 3.1 and 3.3

$$\leq C_1 h^2 \left|\ln h\right|^2 + (\theta)^n \left\|u_h - u_h^0\right\|_{L^{\infty}(\Lambda_i)}$$

$$\leq C_1 h^2 \left|\ln h\right|^2 + (\theta)^n \left\|u_h - r_h \Psi\right\|_{L^{\infty}(\Lambda_i)}$$

and the Lemma 3.2

$$\leq C_1 h^2 \left|\ln h\right|^2 + (\theta)^n C_2 h^2 \left|\ln h\right|^2$$
$$\leq \left(C_1 + (\theta)^n C_2\right) h^2 \left|\ln h\right|^2$$

Therefore,

$$\left\|u_i - u_{ih}^{n+1}\right\|_{L^{\infty}(\Omega_i)} \leq Ch^2 \left|\ln h\right|^2$$

This completes the proof. ∎

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