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ON A HILBERT-TYPE INEQUALITY WITH THE POLYGAMMA FUNCTION

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ABSTRACT. By applying the method of weight function and the technique of real analysis, a Hilbert-type inequality with a best constant factor is established, where the best constant factor is made of the polygamma function. Furthermore, the inverse form is given.

Key words and phrases: Hilbert's inequality, Weight coefficient, Hölder's inequality.

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1. INTRODUCTION

If $a_n, b_n \geq 0, 0 < \sum\limits_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum\limits_{n=1}^{\infty} b_n^2 < \infty$, then(see [1])

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left\{\sum_{n=1}^{\infty}a_n^2\sum_{n=1}^{\infty}b_n^2\right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well known as Hilbert's inequality. Soon after, inequality (1.1) had been generalized by Hardy-Riesz as(see [1]): If $a_n, b_n \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \sum_{n=1}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Inequality (1.2) is named of Hardy-Hilbert's inequality (see [1]). It is important in analysis and its applications. It was studied extensively and refinements, generalizations and numerous variants appeared in the literature (see [1]- [6]). Under the same condition of (1.2), we obtained the Hardy-Hilbert's type inequality (see [1], Th. 341, Th. 342)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < pq\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}; \tag{1.3}$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\log(m/n)}{m-n}a_m b_n < \pi^2\csc^2\frac{\pi}{p}\left\{\sum_{n=1}^{\infty}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}b_n^q\right\}^{1/q}, \tag{1.4}$$

where the constant factors pq and $\pi^2\csc^2\frac{\pi}{p}$ are both the best possible.

In 2008, Yang (see [7]) gave a bilateral inequality as follows: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < \lambda \leq 2, a, b, c \geq 0, a + bc > 0$, $a_n, b_n \geq 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q < \infty$, then

$$\begin{aligned} H &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda, n^\lambda\}+bm^\lambda+cn^\lambda} \\ &< C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{1/q}, \end{aligned} \tag{1.5}$$

where the constant factor $C_\lambda(a, b, c)$ is the best possible. In addition, for $0 < p < 1$, Yang got the reverse inequality as follows

$$\begin{aligned} H &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{a\max\{m^\lambda, n^\lambda\}+bm^\lambda+cn^\lambda} \\ &> C_\lambda(a,b,c)\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(a,b,c,n)]n^{p(1-\frac{\lambda}{2})-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda}{2})-1}b_n^q\right\}^{1/q}, \end{aligned} \tag{1.6}$$

where $\theta_\lambda(a, b, c, m) := \frac{1}{C_1(a,b,c)}\int_0^{1/m^\lambda}\frac{1}{a+b+cu}u^{-1/2}\mathrm{d}u = O(\frac{1}{m^{\lambda/2}}) \in (0, 1)$, and the constant factor $C_\lambda(a, b, c)$ is the best possible. By the way, in recent years, the reverse form of the Hardy-Hilbert's inequality has been studied by Zhong(see [8]), Zhao(see [9]) and so on.

The main purpose of this article is to attempt investigation for the bilateral form of the Hilbert's type inequality concerning series with the mixed homogeneous kernel of 0-degree.

2. SOME LEMMAS

Polygamma function is a special function mostly commonly denoted $\psi_n(z)$ or $\psi^{(n)}(z)$, which is given by the $(n+1)$st derivative of the logarithm of the gamma function $\Gamma(z)$ $(\Gamma(z) := \int_0^\infty \mathrm{e}^{-x}x^{z-1}\mathrm{d}x)$. This is equivalent to the nth normal derivative of the logarithmic derivative of $\Gamma(z)$ and, in the former case, to the nth normal derivative of the digamma function $\psi_0(z)$ which is given by the logarithmic derivative of the gamma function $\Gamma(z)$, i.e. $\psi_0(z) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Hence[10] (see also [11])

$$\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\ln\Gamma(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\psi_0(z). \tag{2.1}$$

Furthermore, polygamma function may also defined as[12]

$$\psi_n(z) = \frac{\mathrm{d}^{n+1}}{\mathrm{d}z^{n+1}}\ln\Gamma(z) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}\mathrm{d}t, \Re z > 0, n = 1, 2, 3, \cdots. \tag{2.2}$$

Let $x = e^{-t}$, then for $n = 1$, we have

$$\psi_1(z) = \int_0^\infty \frac{te^{-zt}}{1-e^{-t}}\mathrm{d}t = \int_0^1 \frac{t^{z-1}\ln x}{x-1}\mathrm{d}x, \tag{2.3}$$

Lemma 2.1. *Let $\alpha \in \mathbb{R}$ and $\lambda > |\alpha|$, define the weight function $\widetilde{\varphi}_\lambda(\alpha, x)$ and $\widetilde{\psi}_\lambda(\alpha, y)$ as*

$$\widetilde{\varphi}_\lambda(\alpha, x) := \int_0^\infty \frac{(\min\{x,y\})^\lambda \ln(x/y)}{x^\lambda - y^\lambda}\cdot\frac{x^\alpha}{y^{1+\alpha}}\mathrm{d}y, x \in (0, \infty), \tag{2.4}$$

$$\widetilde{\psi}_\lambda(\alpha, y) := \int_0^\infty \frac{(\min\{x,y\})^\lambda \ln(x/y)}{x^\lambda - y^\lambda}\cdot\frac{y^{-\alpha}}{x^{1-\alpha}}\mathrm{d}x, y \in (0, \infty), \tag{2.5}$$

then we obtain

$$\widetilde{\varphi}_\lambda(\alpha, x) = \widetilde{\psi}_\lambda(\alpha, y) = C_\lambda(\alpha), \tag{2.6}$$

where $C_\lambda(\alpha) = \frac{1}{\lambda^2}\left[\psi_1\left(1-\frac{\alpha}{\lambda}\right) + \psi_1\left(1+\frac{\alpha}{\lambda}\right)\right]$.

Proof. Let $t = y/x$, in view of (2.3), then

$$\begin{aligned}
\widetilde{\varphi}_\lambda(\alpha, x) &= \int_0^\infty \frac{(\min\{x,y\})^\lambda \ln(x/y)}{x^\lambda - y^\lambda}\cdot\frac{x^\alpha}{y^{1+\alpha}}\mathrm{d}y \\
&= \int_0^\infty \frac{(\min\{1,t\})^\lambda \ln t}{t^\lambda - 1}\cdot t^{-\alpha-1}\mathrm{d}t \\
&= \int_0^1 \frac{t^\lambda \ln t}{t^\lambda - 1}\cdot t^{-\alpha-1}\mathrm{d}t + \int_1^\infty \frac{\ln t}{t^\lambda - 1}\cdot t^{-\alpha-1}\mathrm{d}t \\
&= \int_0^1 \frac{t^{\lambda-\alpha-1}\ln t}{t^\lambda - 1}\mathrm{d}t + \int_0^1 \frac{t^{\lambda+\alpha-1}\ln t}{t^\lambda - 1}\mathrm{d}t \ (setting\ t^\lambda = x) \\
&= \frac{1}{\lambda^2}\int_0^1 \frac{x^{1-\frac{\alpha}{\lambda}-1}\ln x}{x-1}\mathrm{d}x + \frac{1}{\lambda^2}\int_0^1 \frac{x^{1+\frac{\alpha}{\lambda}-1}\ln x}{x-1}\mathrm{d}x \\
&= \frac{1}{\lambda^2}\left[\psi_1\left(1-\frac{\alpha}{\lambda}\right) + \psi_1\left(1+\frac{\alpha}{\lambda}\right)\right] = C_\lambda(\alpha).
\end{aligned}$$

Similarly, we can calculate that

$$\widetilde{\psi}_{\lambda}(\alpha, y) = C_{\lambda}(\alpha).$$

The Lemma is proved. ∎

Lemma 2.2. *Let* $|\alpha| \le 1$ *and* $|\alpha| < \lambda \le 1 + |\alpha|$*, define* $\varphi_{\lambda}(\alpha, m)$ *and* $\psi_{\lambda}(\alpha, n)$ *as*

$$\varphi_{\lambda}(\alpha, m) := \sum_{n=1}^{\infty} \frac{(\min\{m,n\})^{\lambda} \ln(m/n)}{m^{\lambda} - n^{\lambda}} \cdot \frac{m^{\alpha}}{n^{1+\alpha}}, m \in \mathbb{N}, \tag{2.7}$$

$$\psi_{\lambda}(\alpha, n) := \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda} \ln(m/n)}{m^{\lambda} - n^{\lambda}} \cdot \frac{n^{-\alpha}}{m^{1-\alpha}}, n \in \mathbb{N}, \tag{2.8}$$

then

$$C_{\lambda}(\alpha)\,([1 - \theta_{\lambda}(\alpha, m)] < \varphi_{\lambda}(\alpha, m) < C_{\lambda}(\alpha), \tag{2.9}$$

$$\psi_{\lambda}(\alpha, n) < C_{\lambda}(\alpha), \tag{2.10}$$

where

$$0 < \theta_{\lambda}(\alpha, m) := \frac{1}{C_{\lambda}(\alpha)} \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1} \ln t}{t^{\lambda} - 1} \mathrm{d}t = O(\frac{1}{m^{\lambda-\alpha}}) \in (0,1), m \to \infty.$$

Proof. On one hand, for $\lambda > 0$, the function $f_1(t) := \frac{\ln t}{t^{\lambda}-1}$ is strictly decreasing in $(0,1)$ and $(1,\infty)$. For $|\alpha| \le 1$ and $|\alpha| < \lambda \le 1 + |\alpha|$, the function $f_2(t) := t^{\lambda-\alpha-1}$ is monotonically decreasing in $(0,1)$ and the function $f_3(t) := t^{-\alpha-1}$ is monotonically decreasing in $(1,\infty)$. Let $t = y/m$, by monotonicity and in view of (2.6), then

$$\begin{aligned} \varphi_{\lambda}(\alpha, m) &< \widetilde{\varphi}_{\lambda}(\alpha, m) = \int_0^{\infty} \frac{(\min\{m,y\})^{\lambda} \ln(m/y)}{m^{\lambda} - y^{\lambda}} \cdot \frac{m^{\alpha}}{y^{1+\alpha}} \mathrm{d}y \\ &= \int_0^{\infty} \frac{(\min\{1,t\})^{\lambda} \ln t}{t^{\lambda} - 1} \cdot t^{-\alpha-1} \mathrm{d}t \\ &= \int_0^{1} \frac{t^{\lambda} \ln t}{t^{\lambda} - 1} \cdot t^{-\alpha-1} \mathrm{d}t + \int_1^{\infty} \frac{\ln t}{t^{\lambda} - 1} \cdot t^{-\alpha-1} \mathrm{d}t = C_{\lambda}(\alpha). \end{aligned}$$

Similarly, we obtain

$$\psi_{\lambda}(\alpha, n) < C_{\lambda}(\alpha),$$

thus (2.10) is valid.

On the other hand, letting $t = y/m$ gives

$$\begin{aligned}\varphi_\lambda(\alpha, m) &> \int_1^\infty \frac{(\min\{m, y\})^\lambda \ln(m/y)}{m^\lambda - y^\lambda} \cdot \frac{m^\alpha}{y^{1+\alpha}} \mathrm{d}y \\ &= \int_{\frac{1}{m}}^\infty \frac{(\min\{1, t\})^\lambda \ln t}{t^\lambda - 1} \cdot t^{-\alpha-1} \mathrm{d}t \\ &= C_\lambda(\alpha) - \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1} \ln t}{t^\lambda - 1} \mathrm{d}t \\ &= C_\lambda(\alpha) \left[1 - \frac{1}{C_\lambda(\alpha)} \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1} \ln t}{t^\lambda - 1} \mathrm{d}t\right] \\ &= C_\lambda(\alpha)[1 - \theta_\lambda(\alpha, m)].\end{aligned}$$

Obvious, $0 < \theta_\lambda(\alpha, m) := \frac{1}{C_\lambda(\alpha)} \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1} \ln t}{t^\lambda - 1} \mathrm{d}t < 1$. Since

$$\begin{aligned}0 &< \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1} \ln t}{t^\lambda - 1} \mathrm{d}t = \int_0^{\frac{1}{m}} t^{\lambda-\alpha-1} \sum_{k=0}^\infty (t^\lambda)^k (-\ln t) \mathrm{d}t \\ &= \sum_{k=0}^\infty \int_{\frac{1}{m}}^\infty \frac{-\ln t}{\lambda + \lambda k - \alpha} \mathrm{d}t^{\lambda+\lambda k-\alpha} \\ &= \frac{1}{m^{\lambda-\alpha}} \sum_{k=0}^\infty \frac{1}{\lambda + \lambda k - \alpha} \left[\frac{\ln m}{m^{\lambda k}} + \frac{1}{\lambda + \lambda k - \alpha} \cdot \frac{1}{m^{\lambda k}}\right] = O(\frac{1}{m^{\lambda-\alpha}}).\end{aligned}$$

Hence (2.9) is valid. The Lemma is proved. ∎

Lemma 2.3. *If* $p > 0, p \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $|\alpha| \leq 1$ *and* $|\alpha| < \lambda \leq 1 + |\alpha|$, *define* $J(\varepsilon)$ *as*

$$J(\varepsilon) := \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{m, n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda} \cdot m^{\alpha-1-\frac{\varepsilon}{p}} n^{-\alpha-1-\frac{\varepsilon}{q}}, \tag{2.11}$$

where ε *is sufficiently small and positive, then*

$$[C_\lambda(\alpha) - o(1)] \sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} < J(\varepsilon) < [C_\lambda(\alpha) + \widetilde{o}(1)] \sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}, \varepsilon \to 0^+. \tag{2.12}$$

Proof. Let $t = \frac{x}{n}$ in the following, in view of Lemma 2.3, then

$$\begin{aligned}J(\varepsilon) &< \sum_{n=1}^\infty n^{-\alpha-1-\frac{\varepsilon}{q}} \left(\int_0^\infty \frac{(\min\{x, n\})^\lambda \ln(x/n)}{x^\lambda - n^\lambda} \cdot x^{\alpha-1-\frac{\varepsilon}{p}} \mathrm{d}x\right) \\ &= \sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} \left(\int_0^\infty \frac{(\min\{t, 1\})^\lambda \ln t}{t^\lambda - 1} \cdot t^{\alpha-1-\frac{\varepsilon}{p}} \mathrm{d}t\right) \\ &= [C_\lambda(\alpha) + \widetilde{o}(1)] \sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} \quad (\varepsilon \to 0^+);\end{aligned}$$

$$\begin{aligned}
J(\varepsilon) &> \sum_{n=1}^{\infty} n^{-\alpha-1-\frac{\varepsilon}{q}} \left(\int_{1}^{\infty} \frac{(\min\{x,n\})^{\lambda} \ln(x/n)}{x^{\lambda}-n^{\lambda}} \cdot x^{\alpha-1-\frac{\varepsilon}{p}} \mathrm{d}x \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\int_{\frac{1}{n}}^{\infty} \frac{(\min\{t,1\})^{\lambda} \ln t}{t^{\lambda}-1} \cdot t^{\alpha-1-\frac{\varepsilon}{p}} \mathrm{d}t \right) \\
&> \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left(\int_{0}^{\infty} \frac{(\min\{t,1\})^{\lambda} \ln t}{t^{\lambda}-1} \cdot t^{\alpha-1-\frac{\varepsilon}{p}} \mathrm{d}t \right) \\
&\quad - \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_{0}^{\frac{1}{n}} \frac{t^{\lambda+\alpha-1-\frac{\varepsilon}{p}} \ln t}{t^{\lambda}-1} \mathrm{d}t \right) \\
&= [C_{\lambda}(\alpha) + \widetilde{o}(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} - \sum_{n=1}^{\infty} \left(\frac{1}{n} \int_{0}^{\frac{1}{n}} \frac{t^{\lambda+\alpha-1-\frac{\varepsilon}{p}} \ln t}{t^{\lambda}-1} \mathrm{d}t \right) \quad (\varepsilon \to 0^{+}).
\end{aligned}$$

Since

$$\begin{aligned}
0 &< \sum_{n=1}^{\infty} \left(\frac{1}{n} \int_{0}^{\frac{1}{n}} \frac{t^{\lambda+\alpha-1-\frac{\varepsilon}{p}} \ln t}{t^{\lambda}-1} \mathrm{d}t \right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} \int_{0}^{\frac{1}{n}} t^{\lambda+\alpha-1-\frac{\varepsilon}{p}} \sum_{k=0}^{\infty} (t^{\lambda})^{k} (-\ln t) \mathrm{d}t \right) \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=0}^{\infty} \int_{0}^{\frac{1}{n}} \frac{-\ln t}{\lambda+\lambda k+\alpha-\frac{\varepsilon}{p}} \mathrm{d}t^{\lambda+\lambda k+\alpha-\frac{\varepsilon}{p}} \right) = \sum_{n=1}^{\infty} O\left(\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}} \right).
\end{aligned}$$

In view of the above inequalities, we obtain

$$\begin{aligned}
J(\varepsilon) &> [C_{\lambda}(\alpha) + \widetilde{o}(1)] \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} - \sum_{n=1}^{\infty} O\left(\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left[(C_{\lambda}(\alpha) + \widetilde{o}(1)) - \sum_{n=1}^{\infty} O\left(\frac{1}{n^{\lambda+\alpha+1-\frac{\varepsilon}{p}}} \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right)^{-1} \right] \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} [C_{\lambda}(\alpha) - o(1)] \ (\varepsilon \to 0^{+}).
\end{aligned}$$

The Lemma is proved. ∎

3. MAIN RESULTS

Theorem 3.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $|\alpha| \leq 1, |\alpha| < \lambda \leq 1 + |\alpha|$, $a_n, b_n \geq 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\alpha)-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1+\alpha)-1} b_n^q < \infty$, *then we obtain the following inequality*

$$I := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{m,n\})^{\lambda} \ln(m/n)}{m^{\lambda}-n^{\lambda}} a_m b_n$$

$$< C_{\lambda}(\alpha) \left\{ \sum_{n=1}^{\infty} n^{p(1-\alpha)-1} a_n^p \right\}^{1/p} \left\{ \sum_{n=1}^{\infty} n^{q(1+\alpha)-1} b_n^q \right\}^{1/q}, \tag{3.1}$$

where the constant factor $C_\lambda(\alpha) = \frac{1}{\lambda^2}\left[\psi_1\left(1-\frac{\alpha}{\lambda}\right)+\psi_1\left(1+\frac{\alpha}{\lambda}\right)\right]$ *is the best possible. In particular, for* $\alpha = 0$, (3.1) *reduces to*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}a_m b_n < \frac{\pi^2}{3\lambda^2}\left\{\sum_{n=1}^{\infty} n^{p-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q-1}b_n^q\right\}^{1/q}, \tag{3.2}$$

Proof. By Hölder's inequality with weight[13], we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}a_m b_n \\
= &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}\left[\frac{m^{(1-\alpha)/q}}{n^{(1+\alpha)/p}}a_m\right]\left[\frac{n^{(1+\alpha)/p}}{m^{(1-\alpha)/q}}b_n\right] \\
\leq &\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}\frac{m^{(1-\alpha)(p-1)}}{n^{1+\alpha}}a_m^p\right\}^{1/p} \\
&\times\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}\frac{n^{(1+\alpha)(q-1)}}{m^{1-\alpha}}b_n^q\right\}^{1/q} \\
= &\left\{\sum_{m=1}^{\infty}\varphi_\lambda(\alpha,m)m^{p(1-\alpha)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\psi_\lambda(\alpha,n)n^{q(1+\alpha)-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

In view of (2.9) and (2.10), we have (3.1).

Suppose that ε is positive and sufficiently small, let $\widetilde{a}_m = m^{\alpha-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-\alpha-1-\frac{\varepsilon}{q}}$ $(m,n \in \mathbb{N})$, then by (2.9)

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}\cdot m^{\alpha-1-\frac{\varepsilon}{p}}n^{-\alpha-1-\frac{\varepsilon}{q}} = J(\varepsilon),$$

Assuming that there exists a positive number k with $0 < k \leq C_\lambda(\alpha)$, such that (3.1) is still valid by changing $C_\lambda(\alpha)$ to k, then, in particular, by (2.12), we have

$$[C_\lambda(\alpha) - o(1)]\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < J(\varepsilon)$$

$$< k\left\{\sum_{n=1}^{\infty} n^{p(1-\alpha)-1}\widetilde{a}_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q(1+\alpha)-1}\widetilde{b}_n^q\right\}^{1/q} = k\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}.$$

It follows that $C_\lambda(\alpha) - o(1) < k$, so $C_\lambda(\alpha) \leq k(\varepsilon \to 0^+)$. Hence the constant factor $k = C_\lambda(\alpha)$ in (3.1) is the best possible. In particular, for $\alpha = 0$, we get $C_\lambda(0) = \frac{2}{\lambda^2}\psi_1(1) = \frac{\pi^2}{3\lambda^2}$, thus (3.2) is valid. This completes the proof. ∎

Theorem 3.2. *If* $0 < p < 1, \frac{1}{p}+\frac{1}{q} = 1$, $|\alpha| \leq 1, |\alpha| < \lambda \leq 1 + |\alpha|$, $A > -1$, $a_n, b_n \geq 0$ *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\alpha)-1}a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1+\alpha)-1}b_n^q < \infty$, *then we obtain the following inverse inequality*

$$\begin{aligned}
I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}a_m b_n \\
&> C_\lambda(\alpha)\left\{\sum_{n=1}^{\infty}[1-\theta_\lambda(\alpha,n)]n^{p(1-\alpha)-1}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty} n^{q(1+\alpha)-1}b_n^q\right\}^{1/q}, \qquad (3.3)
\end{aligned}$$

where $0 < \theta_\lambda(\alpha, m) := \frac{1}{C_\lambda(\alpha)} \int_0^{\frac{1}{m}} \frac{t^{\lambda-\alpha-1}\ln t}{t^\lambda - 1} \mathrm{d}t = O(\frac{1}{m^{\lambda-\alpha}}) \in (0,1)(m \to \infty)$ *and the constant factor* $C_\lambda(\alpha, A) = \frac{1}{\lambda^2}\left[\psi_1\left(1-\frac{\alpha}{\lambda}\right) + \psi_1\left(1+\frac{\alpha}{\lambda}\right)\right]$ *is the best possible. In particular, for* $\alpha = 0$, (3.3) *reduces to*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda} a_m b_n$$
$$> \frac{\pi^2}{3\lambda^2}\left\{\sum_{n=1}^\infty [1-\theta_\lambda(0,n)] n^{p-1} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^\infty n^{q-1} b_n^q\right\}^{1/q}, \tag{3.4}$$

Proof. By the reverse Hölder's inequality with weight[13], in view of (2.7) and (2.8), we obtain

$$\begin{aligned} I &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda}\left[\frac{m^{(1-\alpha)/q}}{n^{(1+\alpha)/p}} a_m\right]\left[\frac{n^{(1+\alpha)/p}}{m^{(1-\alpha)/q}} b_n\right] \\ &\geq \left\{\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda} \frac{m^{(1-\alpha)(p-1)}}{n^{(1+\alpha)}} a_m^p\right\}^{1/p} \\ &\quad \times \left\{\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda} \frac{n^{(1+\alpha)(q-1)}}{m^{(1-\alpha)}} b_n^q\right\}^{1/q} \\ &= \left\{\sum_{m=1}^\infty \varphi_\lambda(\alpha, m) m^{p(1-\alpha)-1} a_m^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^\infty \psi_\lambda(\alpha, n) n^{q(1+\alpha)-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

By (2.9) and (2.10), in view of $q < 0$, we have (3.3).

Suppose that ε is positive and small enough, let $\widetilde{a}_m = m^{\alpha-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{-\alpha-1-\frac{\varepsilon}{q}}$ $(m, n \in \mathbb{N})$, then by (2.11)

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{m,n\})^\lambda \ln(m/n)}{m^\lambda - n^\lambda} \cdot m^{\alpha-1-\frac{\varepsilon}{p}} n^{-\alpha-1-\frac{\varepsilon}{q}} = J(\varepsilon).$$

Assuming that there exists a positive number k with $k \geq C_\lambda(\alpha)$, such that (3.3) is still correct by changing $C_\lambda(\alpha)$ to k, then, in particular, by (2.12), we have

$$(C_\lambda(\alpha) + \widetilde{o}(1)) \sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} > J(\varepsilon)$$

$$\begin{aligned} &> k\left\{\sum_{n=1}^\infty [1-\theta_\lambda(\alpha,n)] n^{p(1-\alpha)-1} \widetilde{a}_n^p\right\}^{1/p} \left\{\sum_{n=1}^\infty n^{q(1+\alpha)-1} \widetilde{b}_n^q\right\}^{1/q} \\ &= k\left\{\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} - \sum_{n=1}^\infty \left[O\left(\frac{1}{n^{\lambda-\alpha}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p} \left\{\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}\right\}^{1/q} \\ &= k\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}\left\{1 - \left(\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}\right)^{-1} \sum_{n=1}^\infty \left[O\left(\frac{1}{n^{\lambda-\alpha}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p}. \end{aligned}$$

It follows that

$$C_\lambda(\alpha) + \widetilde{o}(1) > k\left\{1 - \left(\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}}\right)^{-1} \sum_{n=1}^\infty \left[O\left(\frac{1}{n^{\lambda-\alpha}}\right)\frac{1}{n^{1+\varepsilon}}\right]\right\}^{1/p},$$

and then $C_\lambda(\alpha) \geq k(\varepsilon \to 0^+)$. Thus the constant factor $k = C_\lambda(\alpha)$ in (3.3) is the best possible. The theorem is proved. ∎

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