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SOME REMARKS ON A RESULT OF BOUGOFFA

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ABSTRACT. Some new generalizations of the result of L. Bougoffa [J. Inequal. Pure Appl. Math. **7** (**2**) (2006), Art. 60] are derived and discussed.

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1. INTRODUCTION

Recently, L. Bougoffa [1] stated the following result :

Theorem 1.1. *Let $p > 1$ and let $f_1, f_2, ..., f_n$, $n \in \mathbb{Z}_+$, be nonnegative integrable functions. Define $F_k(x) = \frac{1}{x}\int\limits_0^x f_k(t)dt,\ k = 1, 2, ..., n$. Then*

$$\int\limits_0^\infty \left(\prod_{k=1}^n F_k(x)\right)^{\frac{p}{n}} dx \le \left(\frac{p}{p-1}\right)^p \int\limits_0^\infty \left(\frac{1}{n}\sum_{k=1}^n f_k(x)\right)^p dx. \tag{1.1}$$

We note that this result follows directly by using the classical Hardy inequality (see [4] or some of the books [6], [7], [8] and [10])

$$\int\limits_0^\infty \left(\frac{1}{x}\int\limits_0^x f(t)\, dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int\limits_0^\infty f^p(x)dx \tag{1.2}$$

and the Arithmetic-Geometric Mean inequality

$$\left(\prod_{k=1}^n F_k(x)\right)^{\frac{1}{n}} \le \frac{1}{n}\sum_{k=1}^n F_k(x) = \frac{1}{n}\int\limits_0^x \left(\sum_{k=1}^n f_k(t)\right) dt.$$

2. MAIN RESULTS

First we generalize the Bougoffa result in the following way:

Theorem 2.1. *Let $p > 0$, $p \ne 1$ and $n \in \mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^\infty$ be a positive sequence such $\sum\limits_{k=1}^\infty \alpha_k = 1$ and $\{f_k\}_{k=1}^\infty$ be a sequence of integrable functions and let*

$$F_k(x) = \int\limits_0^x f_k(t)dt,\ k = 1, 2, ...$$

Then the inequality

$$\int\limits_0^\infty \left(\prod_{k=1}^\infty \left[\frac{1}{x}F_k(x)\right]^{\alpha_k}\right)^p dx \le \left(\frac{p}{|p-1|}\right)^p \int\limits_0^\infty \left(\sum_{k=1}^\infty \alpha_k f_k(x)\right)^p dx \tag{2.1}$$

holds if and only if $p > 1$ and the constant $\left(\frac{p}{p-1}\right)^p$ is sharp.

Remark 2.1. If $p > 1$, $\alpha_k = \frac{1}{k}$, $k = 1, 2, ..., n$, $\alpha_k = 0$, $k \ge n+1$, we obtain the Bougoffa inequality (1.1) and the constant in (1.1) is in fact sharp.

Proof. Let $p > 1$. According to the more general Arithmetic-Geometric Mean inequality

$$\prod_{k=1}^\infty g_k^{\alpha_k}(x) \le \sum_{k=1}^\infty \alpha_k g_k(x),$$

we have that

$$\left(\prod_{k=1}^{\infty} F_k^{\alpha_k}(x)\right)^p \leq \left(\sum_{k=1}^{\infty} \alpha_k F_k(x)\right)^p = \left(\int_0^x \left(\sum_{k=1}^{\infty} \alpha_k f_k(t)\right) dt\right)^p . \tag{2.2}$$

By using Hardy's inequality (1.2) with the function $\sum_{k=1}^{\infty} \alpha_k f_k(t)$ and (2.2) the inequality (2.1) is proved. The constant in the inequality is sharp since by applying it with $f_k(t) = f(t)$, $k = 1, 2, ...$, it reduces to (1.2) and it is known that the constant in this inequality is sharp.

Now, let $0 < p < 1$. Then (2.2) still holds. But then (2.1) cannot hold in general since by applying it with $f_k(x) = f(x)$, $k = 1, 2, ...$, it reduces to the inequality

$$\int_0^{\infty} \left(\frac{1}{x}\int_0^x f(t)\, dt\right)^p dx \leq \left(\frac{p}{1-p}\right)^p \int_0^{\infty} f^p(x) dx$$

but it is well known that this is not true. In fact, it just holds in the reversed direction. The proof is complete. ∎

Remark 2.2. For $p < 0$ it is known that (1.2) still holds but now (2.2) holds in the reversed direction so our proof above does not work so we leave it as an open question whether (2.1) holds in this case or not.

By using the technique above and other results from the rich theory of Hardy type inequalities (see e.g. the books [3], [4], [7], [8] and [10]) we can now give the similar results in most cases. Here we only give the following multidimensional weighted version of the result above. In what follows we use bold letters to denote the n-tuples of real numbers, e.g. $\mathbf{x} = (x_{1,}..., x_n)$ or $\mathbf{t} = (t_1, ..., t_n)$. In particular, we set $\mathbf{x} = (x_{1,}..., x_n) \in \mathbb{R}^n$ and $\mathbf{t} = (t_1, ..., t_n) \in \mathbb{R}^n$.

Theorem 2.2. *Let $p > 0$, $p \neq 1$, $m \neq 1$ and $n \in \mathbb{Z}_+$. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a positive sequence such $\sum_{k=1}^{\infty} \alpha_k = 1$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence of integrable functions on $[\mathbf{0}, \mathbf{b}]$, $0 < \mathbf{b} \leq \infty$, and let*

$$F_k(\mathbf{x}) = \int_0^{x_1} ... \int_0^{x_n} f_k(\mathbf{t}) dt_1...dt_n,\ k = 1, 2, ...$$

Then the inequality

$$\int_0^{b_1} ... \int_0^{b_n} x_1^{-m}...x_n^{-m} \left(\prod_{k=1}^{\infty} F_k^{\alpha_k}(\mathbf{x})\right)^p dx_1...dx_n$$

$$\leq \left(\frac{p}{|m-1|}\right)^{np} \int_0^{b_1} ... \int_0^{b_n} \prod_{i=1}^{n} \left[1 - \frac{x_i}{b_i}\right]^{\frac{m-1}{p}} \left(\sum_{k=1}^{\infty} \alpha_k f_k(\mathbf{x})\right)^p x_1^{p-m}...x_n^{p-m} dx_1...dx_n \tag{2.3}$$

holds if and only if $p > 1$ and the constant $\left(\frac{p}{m-1}\right)^{np}$ is sharp.

Proof. We just use Theorem 3.1 in [9] instead of the classical Hardy's inequality (1.2) and the proof is similar to the proof of Theorem 2.1. We omit the details. ∎

Remark 2.3. Let $m = p > 1$, $b_1 = b_2 = ... = b_n = \infty$. Then (2.3) also follows from a result of Pachpatte [11]. If also $n = 1$, then the result in Theorem 2.2 coincides with that in Theorem 2.1.

Remark 2.4. It is well known that Hardy's inequality (1.2) implies its discrete analogue

$$\sum_{k=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{k=1}^{\infty}{\alpha_k}^p, \tag{2.4}$$

which holds for $p>1$ and also for $p<0$.

A limiting case of (2.4) is the famous Carleman inequality from 1922 (see [2]):

$$\sum_{k=1}^{\infty}\sqrt[n]{a_1a_2...a_k} \leq e\sum_{k=1}^{\infty}\alpha_k. \tag{2.5}$$

Just use the Arithmetic-Geometric inequality in the left hand side of (2.4) and replace a_k by $a_k^{\frac{1}{p}}$ in (2.4) and we find that

$$\sum_{k=1}^{\infty}\sqrt[n]{a_1a_2...a_k} \leq \left(\frac{p}{p-1}\right)^p\sum_{k=1}^{\infty}\alpha_k.$$

Let $p\to\infty$ so that $\left(\frac{p}{p-1}\right)^p \to e$ and (2.5) follows.

In a similar way we see that Polyá-Knopp's inequality

$$\int_0^{\infty}\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right)dx \leq e\int_0^{\infty}f(x)dx$$

follows by just using a continuous version of the Arithmetic-Geometric inequality, namely

$$\exp\left(\frac{1}{x}\int_0^x \ln f(t)dt\right) \leq \frac{1}{x}\int_0^x f(t)dt, \tag{2.6}$$

Hardy's inequality (1.2) and letting $p\to\infty$.

Many other proofs and historical remarks concerning the inequalities (2.5) and (2.6) can be found in the paper [5] but the proofs presented above seems to be most natural and elementary ones. We remark that the constant e is sharp in both of the inequalities (2.5) and (2.6) .

Remark 2.5. By just using (2.4) and the technique above we can also state and prove a discrete version of Theorem 1.1 and also obtain similar results by using other discrete Hardy type inequalities e.g. a discrete version of Theorem 2.1.

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