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SOME INEQUALITIES FOR GRAMIAN NORMAL OPERATORS AND FOR GRAMIAN SELF-ADJOINT OPERATORS IN PSEUDO-HILBERT SPACES

LOREDANA CIURDARIU

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DEPARTMENT OF MATHEMATICS, "POLITEHNICA" UNIVERSITY OF TIMISOARA, PIATA VICTORIEI, NO. 2, 300006-TIMISOARA, ROMANIA.
cloredana43@yahoo.com

ABSTRACT. Several inequalities for gramian normal operators and for gramian self-adjoint operators in pseudo-Hilbert spaces are presented.

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1. INTRODUCTION

The following two results were presented in [2] and [6].

We recall that a locally convex space Z is called *admissible in the Loynes sense* if the following conditions are satisfied:

- Z is complete;
- there is a closed convex cone in Z, denoted Z_+, that defines an order relation on Z (that is $z_1 \leq z_2$ if $z_2 - z_1 \in Z_+$);
- there is an involution in Z, $Z \ni z \to z^* \in Z$ (that is $z^{**} = z$, $(\alpha z)^* = \overline{\alpha} z^*$, $(z_1 + z_2)^* = z_1^* + z_2^*$), such that $z \in Z_+$ implies $z^* = z$;
- the topology of Z is compatible with the order (that is, there exists a basis of convex solid neighborhoods of the origin);
- any monotonously decreasing sequence in Z_+ is convergent.

We say that a set $C \in Z$ is called *solid* if $0 \leq z' \leq z''$ and $z'' \in C$ implies $z' \in C$.

Let Z be an admissible space in the Loynes sense. A linear topological space $\mathcal{H}$ is called *pre-Loynes Z–space* if it satisfies the following properties: $\mathcal{H}$ is endowed with a Z–valued *inner product* (gramian), i.e. there exists an application $\mathcal{H} \times \mathcal{H} \ni (h,k) \to [h,k] \in Z$ having the properties: $[h,h] \geq 0$; $[h,h] = 0$ implies $h = 0$; $[h_1 + h_2, h] = [h_1, h] + [h_2, h]$; $[\lambda h, k] = \lambda [h,k]$; $[h,k]^* = [k,h]$; for all $h, k, h_1, h_2 \in \mathcal{H}$ and $\lambda \in \mathbb{C}$.

The topology of $\mathcal{H}$ is the weakest locally convex topology on $\mathcal{H}$ for which the application $\mathcal{H} \ni h \to [h,h] \in Z$ is continuous. Moreover, if $\mathcal{H}$ is a complete space with this topology, then $\mathcal{H}$ is called Loynes *Z–space*.

Let Z an admissible space in the Loynes sense, see [6] and $\mathcal{H}$ be a Loynes Z-space. As in Hilbert spaces, see [3], we can show also that in pseudo-Hilbert spaces for any two operators $T, U \in \mathcal{B}^*(\mathcal{H})$, see [2], we have

$$||TU - UT|| \leq 2 \min\{||T||, ||U||\} \min\{||T - U||, ||T + U||\}.$$

In addition, if N is a gramian normal operator in $\mathcal{B}^*(\mathcal{H})$ then

$$||NN^* - N^*N|| \leq 2||N|| \min\{||N - N^*||, ||N + N^*||\}.$$

If we have three operators, $T_1,\ T_2,\ T_3 \in \mathcal{B}^*(\mathcal{H})$ then

$$||T_1T_2T_3 - T_3T_2T_1|| \leq 2(||T_1|| \min\{||T_3||, ||T_2||\}||T_2 - T_3||$$

$$+||T_2|| \min\{||T_1||, ||T_3||\}||T_3 - T_1|| + ||T_3|| \min\{||T_2||, ||T_1||\}||T_2 - T_1||).$$

Indeed,

$$||T_1T_2T_3 - T_3T_2T_1|| = ||T_1(T_2T_3 - T_3T_2) + (T_1T_3 - T_3T_1)T_2 + T_3(T_1T_2 - T_2T_1)||$$

$$\leq ||T_1||||T_2T_3 - T_3T_2|| + ||T_2||||T_1T_3 - T_3T_1|| + ||T_3||||T_1T_2 - T_2T_1||$$

and now we use the first inequality above.

2. THE RESULTS

The following result is a generalization of Theorem 1 of Dragomir (see [4]), for gramian normal commuting operators on pseudo-Hilbert spaces.

Proposition 2.1. *Let $\mathcal{H}$ be a Loynes Z-space and $N_k, M_k \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1, ..., n\},\ n \geq 2$ be gramian normal operators which commute, i.e. $N_iN_j = N_jN_i,\ M_iM_j = M_jM_i,\ N_iM_j = M_jN_i,\ (\forall)\ i, j \in \{1, ..., n\}$ with the property that $\sum_{k=1}^n N_kM_k = 0$.*

Then

$$\max_{i\in\{1,...,n\}}\{|N_iM_i|\} \leq \frac{1}{2}(\sum_{k=1}^n |N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^n |M_k|^2)^{\frac{1}{2}}$$

if the maximum exists.

Moreover, the constant $\frac{1}{2}$ can not be replaced by a smaller constant in previous inequality.

Now putting in Proposition 2.1, $\sqrt{p_k}N_k$ instead of N_k and $\sqrt{p_k}M_k$ instead of M_k we have

Theorem 2.2. *Let $\mathcal{H}$ be a Loynes Z-space and $N_k, M_k \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1, ..., n\},\ n \geq 2$ be gramian normal operators which commute, and $p_k \geq 0,\ k \in \{1, ..., n\}$, $\sum_{k=1}^n p_k = 1$ with the property that $\sum_{k=1}^n p_kN_kM_k = 0$.*

Then

$$\max_{i\in\{1,...,n\}}\{p_i|N_iM_i|\} \leq \frac{1}{2}(\sum_{k=1}^n p_k|N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^n p_k|M_k|^2)^{\frac{1}{2}},$$

if the maximum exists.

Moreover, the constant $\frac{1}{2}$ can not be replaced by a smaller constant in previous inequality.

If we take in Theorem 2.2, $M_k = I,\ k \in \{1, ..., n\}$ the above inequality becomes:

$$\max_{i\in\{1,...,n\}}\{p_i|N_i|\} \leq \frac{1}{2}(\sum_{k=1}^n p_k|N_k|^2)^{\frac{1}{2}},$$

when $\sum_{k=1}^n p_kN_k = 0$. Because $\sum_{k=1}^n p_k(N_k - \sum_{j=1}^n p_jN_j) = 0$ when $\sum_{k=1}^n p_k = 1$ we can replace above N_i with $N_i - \sum_{j=1}^n p_jN_j$ obtaining the below inequality.

The equality in the next corollary follows from

$$\begin{aligned}
|N_k - \sum_{j=1}^n p_jN_j|^2 &= (N_k^* - \sum_{j=1}^n p_jN_j^*)(N_k - \sum_{j=1}^n p_jN_j) \\
&= |N_k|^2 + |\sum_{j=1}^n p_jN_j|^2 - N_k^*\sum_{j=1}^n p_jN_j - \sum_{j=1}^n p_jN_j^*N_k
\end{aligned}$$

by summing after k from 1 to n, last equality multiplied by p_k.

Corollary 2.3. *If $N_k,\ k \in \{1, ..., n\}$ are n gramian normal commutative operators in $\mathcal{B}^*(\mathcal{H})$ and $p_k \geq 0,\ k \in \{1, ..., n\}$ with $\sum_{k=1}^n p_k = 1$, then we have that*

$$\begin{aligned}
\max_{i\in\{1,..,n\}}\{p_i|N_i - \sum_{j=1}^n p_jN_j|\} &\leq \frac{1}{2}(\sum_{k=1}^n p_k|N_k - \sum_{j=1}^n p_jN_j|^2)^{\frac{1}{2}} \\
&= \frac{1}{2}(\sum_{k=1}^n p_k|N_k|^2 - |\sum_{j=1}^n p_jN_j|^2)^{\frac{1}{2}},
\end{aligned}$$

if the maximum exists.

Proposition 2.4. *If we consider $\mathcal{H}$ a Hilbert space and $N_k, M_k \in \mathcal{B}(\mathcal{H})$ as in Proposition 2.1, the same inequality will be satisfied.*

As an analogue of Theorem 3 from [4], we have also for gramian normal commuting operators:

Theorem 2.5. *Let $\mathcal{H}$ be a Loynes Z-space and $N_k, M_k \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1, ..., n\},\ n \geq 2$ be gramian normal operators which commute, i.e. $N_iN_j = N_jN_i,\ M_iM_j = M_jM_i,\ N_iM_j = M_jN_i,\ (\forall)\ i, j \in \{1, ..., n\}$ with the property that $\sum_{k=1}^n N_kM_k = 0$ and $p_k \geq 0,\ k \in \{1, 2, ..., n\}$ with $\sum_{k=1}^n p_k = 1$. Then we have the inequality:*

$$\sum_{j=1}^n p_j|N_j|^2 \sum_{k=1}^n |M_k|^2 + \sum_{j=1}^n p_j|M_j|^2 \sum_{k=1}^n |N_k|^2 \leq \sum_{k=1}^n |N_k|^2 \sum_{k=1}^n |M_k|^2.$$

Proof. Using the proof of the Proposition 2.1, we obtain that

$$\begin{aligned} 0 &\leq |N_iM_i|^2 \leq (\sum_{k=1}^n |N_k|^2 - |N_i|^2)(\sum_{k=1}^n |M_k|^2 - |M_i|^2) \\ &= \sum_{k=1}^n |N_k|^2 \sum_{k=1}^n |M_k|^2 + |N_i|^2|M_i|^2 - |N_i|^2 \sum_{k=1}^n |M_k|^2 - \sum_{k=1}^n |N_k|^2|M_i|^2. \end{aligned}$$

This implies:

$$|N_i|^2 \sum_{k=1}^n |M_k|^2 + |M_i|^2 \sum_{k=1}^n |N_k|^2 \leq \sum_{k=1}^n |N_k|^2 \sum_{k=1}^n |M_k|^2,$$

$(\forall)\ i \in \{1, 2, ..., n\}$. If we multiply by $p_i \geq 0$ the last inequality and then sum over $i,\ i \in \{1, 2, ..., n\}$ we obtain the first inequality. ∎

Corollary 2.6. *Taking into account the conditions from the previous theorem we will have:*

$$(\sum_{j=1}^n p_j|N_j|^2 \sum_{k=1}^n |M_k|^2)^{\frac{1}{2}} (\sum_{j=1}^n p_j|M_j|^2 \sum_{k=1}^n |N_k|^2)^{\frac{1}{2}} \leq \frac{1}{2}(\sum_{k=1}^n |N_k|^2 \sum_{k=1}^n |M_k|^2).$$

Moreover, the constant $\frac{1}{2}$ can not be replaced by a smaller constant in previous inequality.

Proof. Using the inequality from the previous theorem and the following,

$$\begin{aligned} & 2(\sum_{j=1}^n p_j|N_j|^2 \sum_{k=1}^n |M_k|^2)^{\frac{1}{2}} (\sum_{j=1}^n p_j|M_j|^2 \sum_{k=1}^n |N_k|^2)^{\frac{1}{2}} \\ \leq & \sum_{j=1}^n p_j|N_j|^2 \sum_{k=1}^n |M_k|^2 + \sum_{j=1}^n p_j|M_j|^2 \sum_{k=1}^n |N_k|^2 \end{aligned}$$

we will obtain the inequality of the above corollary. ∎

Remark 2.1. Moreover, the constant $\frac{1}{2}$ can not be replaced by a smaller constant in previous inequality.

In order to prove the sharpness of the constant we assume that there is $D > 0$ such that

$$(\sum_{j=1}^n p_j|N_j|^2 \sum_{k=1}^n |M_k|^2)^{\frac{1}{2}} (\sum_{j=1}^n p_j|M_j|^2 \sum_{k=1}^n |N_k|^2)^{\frac{1}{2}} \leq D(\sum_{k=1}^n |N_k|^2 \sum_{k=1}^n |M_k|^2).$$

Indeed if we take $n = 2,\ N_1 = N,\ N_2 = -M,\ M_1 = M,\ M_2 = N,\ p_1 = p,\ p_2 = 1 - p$, then the inequality becomes:

$$\begin{aligned} &((p|N|^2 + (1-p)|M|^2)(|M|^2 + |N|^2))^{\frac{1}{2}} ((|N|^2 + |M|^2)(p|M|^2 + (1-p)|N|^2))^{\frac{1}{2}} \\ &\leq D(|N|^2 + |M|^2)^2. \end{aligned}$$

Now if we take $p = \frac{1}{2}$ then we have

$$\frac{1}{2}(|M|^2 + |N|^2)^2 \leq D(|N|^2 + |M|^2)^2,$$

i.e. $D \geq \frac{1}{2}$.

Corollary 2.7. *Let $M_k^{'} \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1, ..., n\},\ n \geq 2$ be gramian normal operators which commute and $p_k,\ k \in \{1, ..., n\}$ a sequence with the property $\sum_{k=1}^{n} p_k = 1$. Then:*

$$(\sum_{k=1}^{n} |M_k^{'} - \sum_{l=1}^{n} p_l M_l^{'}|^2)^{\frac{1}{2}} (\sum_{j=1}^{n} p_j |M_j^{'} - \sum_{l=1}^{n} p_l M_l^{'}|^2)^{\frac{1}{2}}$$
$$\leq \frac{1}{2} \frac{(\sum_{k=1}^{n} p_k^2)^{\frac{1}{2}}}{(\sum_{j=1}^{n} p_j^3)^{\frac{1}{2}}} \sum_{k=1}^{n} |M_k^{'} - \sum_{l=1}^{n} p_l M_l^{'}|^2.$$

Proof. As in Corollary 3.1, see [4], it is obvious that if we replace in Corollary 2.6 N_k by $p_k I$ and M_k by $M_k^{'} - \sum_{l=1}^{n} p_l M_l^{'}$ for $k \in \{1, ..., n\}$, because $\sum_{i=1}^{n} N_i M_i = \sum_{i=1}^{n} p_i (M_i^{'} - \sum_{l=1}^{n} p_l M_l^{'}) = 0$, we deduce the desired result. ∎

Theorem 2.8. *Let $\mathcal{H}$ be a Loynes Z-space and $N_k, M_k \in \mathcal{B}^*(\mathcal{H}),\ k \in \{1, ..., n\},\ n \geq 2$ be gramian normal operators which commute, i.e. $N_i N_j = N_j N_i,\ M_i M_j = M_j M_i,\ N_i M_j = M_j N_i,\ (\forall)\ i, j \in \{1, ..., n\}$ with the property that*

$$\sum_{k=1}^{n} p_k N_k M_k = 0 \quad \textit{and} \quad p_k \geq 0,\ k \in \{1, 2, ..., n\} \quad \textit{with} \quad \sum_{k=1}^{n} p_k = 1.$$

Then we have the inequalities:

$$\max_{i \in \{1,...,n\}} \{p_i |N_i||M_i|\} (\sum_{k=1}^{n} p_k |N_k|^2)^{\frac{1}{2}} (\sum_{k=1}^{n} p_k |M_k|^2)^{\frac{1}{2}}$$
$$\leq \frac{1}{2} \max_{i \in \{1,...,n\}} [p_i |N_i|^2 \sum_{k=1}^{n} p_k |M_k|^2 + p_i |M_i|^2 \sum_{k=1}^{n} p_k |N_k|^2] \tag{2.1}$$
$$\leq \frac{1}{2} \sum_{k=1}^{n} p_k |N_k|^2 \sum_{k=1}^{n} p_k |M_k|^2,$$

if the respective maximum exists in each case.

Proof. Using the inequality

$$|N_i M_i|^2 \leq (\sum_{k=1}^{n} |N_k|^2 - |N_i|^2)(\sum_{k=1}^{n} |M_k|^2 - |M_i|^2),\ i \in \{1, ..., n\},$$

from the proof of Proposition 2.1, with $\sqrt{p_i} N_i$ instead of N_i and $\sqrt{p_i} M_i$ instead of M_i we obtain

$$p_i^2 |N_i M_i|^2 \leq (\sum_{k=1}^{n} p_k |N_k|^2 - p_i |N_i|^2)(\sum_{k=1}^{n} p_k |M_k|^2 - p_i |M_i|^2)$$

$$= \sum_{k=1}^{n} p_k |N_k|^2 \sum_{k=1}^{n} p_k |M_k|^2 + p_i^2 |N_i|^2 |M_i|^2 - p_i |N_i|^2 \sum_{k=1}^{n} p_k |M_k|^2 - p_i |M_i|^2 \sum_{k=1}^{n} p_k |N_k|^2.$$

This means that

$$p_i |N_i|^2 \sum_{k=1}^{n} p_k |M_k|^2 + p_i |M_i|^2 \sum_{k=1}^{n} p_k |N_k|^2 \leq \sum_{k=1}^{n} p_k |N_k|^2 \sum_{k=1}^{n} p_k |M_k|^2,$$

for each $i \in \{1, ..., n\}$.

Using the hypothesis that the maximum exists and taking in the above inequality the maximum over $i \in \{1, ..., n\}$ we obtain the second part of the inequality (2.1).

For the first part, we will use the inequality,

$$p_i|N_i|^2 \sum_{k=1}^{n} p_k|N_k|^2 + p_i|M_i|^2 \sum_{k=1}^{n} p_k|N_k|^2 \geq 2p_i|N_i||M_i|(\sum_{k=1}^{n} p_k|N_k|^2)^{\frac{1}{2}}(\sum_{k=1}^{n} p_k|M_k|^2)^{\frac{1}{2}}$$

for each $i \in \{1, ..., n\}$.

This concludes the proof. ∎

3. OTHER RESULTS

Corollary 3.1. *If $n \in N$, $n \geq 2$, $A_1, A_2, ..., A_n \in \mathcal{B}^*(\mathcal{H})$, where $\mathcal{H}$ is a Loynes Z- space, are n gramian self-adjoint commutative operators and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\begin{aligned} &\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} \\ &= \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j} \end{aligned} \tag{3.1}$$

.

Corollary 3.2. *(i) If $n \in N$, $n \geq 2$, $A_1, A_2, ..., A_n \in \mathcal{B}^*(\mathcal{H})$, where $\mathcal{H}$ is a Loynes Z- space, are n gramian self-adjoint commutative operators and $a_1, a_2, ..., a_n \in R - \{0\}$ with $a_1 + a_2 + ... + a_n \neq 0$ then,*

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} \geq \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}$$

(ii) Under the above conditions, we have,

$$\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} \geq \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n}$$

.

Using Corollary 3.1 we can deduce an analogue of Theorem 5 from [7] for operators on Loynes Z- spaces.

Theorem 3.3. *If $n \in N$ $n \geq 2$, $A_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1, n}$ are n gramian self-adjoint commutative and $a_1, a_2, ..., a_n \in (0, \infty)$, then*

$$\begin{aligned} &\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} \\ &\geq A_{k,l} + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\, i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}, \end{aligned}$$

where

$$A_{k,l} = \max_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j (a_i + a_j)} = \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n,$$

if they exist.

Proof. The proof is as in [7]. Using the inequality

$$\frac{A^2}{a} + \frac{B^2}{b} \geq \frac{(A+B)^2}{a+b},$$

and Corollary 3.1, we have

$$\begin{aligned}
&\frac{A_1^2}{a_1} + \frac{A_2^2}{a_2} + ... + \frac{A_n^2}{a_n} - \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} \\
= \ & \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}
\end{aligned}$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \left(\frac{(a_k A_l - a_l A_k)^2}{a_k a_l} + \sum_{m=1,\, m \neq k,l}^{n} \left(\frac{(a_m A_l - a_l A_m)^2}{a_m a_l} \right.\right.$$

$$\left.\left. + \frac{(a_m A_k - a_k A_m)^2}{a_m a_k} \right)\right) + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\, i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}$$

$$= \frac{1}{a_1 + a_2 + ... + a_n} \left(\frac{(a_k A_l - a_l A_k)^2}{a_k a_l} + \sum_{m=1,\, m \neq k,l}^{n} \left(\frac{(a_k A_l - \frac{a_l a_k}{a_m} A_m)^2}{\frac{a_l a_k^2}{a_m}} \right.\right.$$

$$\left.\left. + \frac{(\frac{a_l a_k}{a_m} A_m - a_l A_k)^2}{\frac{a_k a_l^2}{a_m}} \right)\right) + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\, i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}$$

$$\geq \frac{1}{a_1 + a_2 + ... + a_n} \left(\frac{(a_k A_l - a_l A_k)^2}{a_k a_l} + \sum_{m=1,\, m \neq k,l}^{n} \frac{a_m (a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)} \right)$$

$$+ \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\, i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}$$

$$= \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)}) + \frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\, i,j \neq k,l} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j}.$$

∎

Theorem 3.4. *If* $n \in N$ $n \geq 2$, $A_i \in \mathcal{B}^*(\mathcal{H})$, $i = \overline{1,n}$, *gramian self-adjoint operators which commute as pairs and* $a_1, a_2, ..., a_n \in (0, \infty)$, *then*

$$\begin{aligned}
&\frac{1}{a_1 + a_2 + ... + a_n} \sum_{1 \leq i < j \leq n,\, i,j \neq k,l} \frac{(a_i A_j + a_j A_i)^2}{a_i a_j} \\
&\geq \frac{(A_1 + A_2 + ... + A_n)^2}{a_1 + a_2 + ... + a_n} + \sum_{k=1}^{n} \frac{A_k^2}{a_k}
\end{aligned}$$

$$-2\left[\frac{A_k^2}{a_k} + \frac{A_l^2}{a_l} + \frac{a_k + a_l}{\sum_{i=1}^n a_i} \sum_{r=1, r \neq k,l}^{n} \frac{A_r^2}{a_r} + \frac{1}{\sum_{i=1}^n a_i} \sum_{t=2, t \neq k,l}^{n-1} A_t^2\right] + A_{k,l},$$

where

$$A_{k,l} = \max_{1 \leq i < j \leq n} \frac{(a_i A_j - a_j A_i)^2}{a_i a_j (a_i + a_j)} = \frac{(a_k A_l - a_l A_k)^2}{a_k a_l (a_k + a_l)},\ 1 \leq k < l \leq n,$$

if they exist.

Proof. We use the inequality from Theorem 3.3,

$$\sum_{k=1}^{n}\frac{A_k^2}{a_k}-\frac{(\sum_{k=1}^{n}A_k)^2}{\sum_{k=1}^{n}a_k}\geq A_{k,l}+\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j-a_jA_i)^2}{a_ia_j}$$

$$=A_{k,l}+\frac{2}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j)^2+(a_jA_i)^2}{a_ia_j}$$

$$-\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}$$

$$=A_{k,l}+\frac{2}{\sum_{k=1}^{n}a_k}(\sum_{1\leq i<j\leq n}\frac{(a_iA_j)^2+(a_jA_i)^2}{a_ia_j}-\frac{(a_kA_l)^2+(a_lA_k)^2}{a_ka_l}$$

$$-\sum_{m=1,\ m\neq k,l}^{n}(\frac{(a_mA_l)^2+(a_lA_m)^2}{a_ma_l}+\frac{(a_mA_k)^2+(a_kA_m)^2}{a_ma_k}))$$

$$-\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}$$

$$=A_{k,l}+\frac{2}{\sum_{k=1}^{n}a_k}(\sum_{j=2}^{n}\frac{A_j^2}{a_j}\sum_{i=1}^{j-1}a_i+\sum_{j=2}^{n}a_j\sum_{i=1}^{j-1}\frac{A_i^2}{a_i})-\frac{2}{\sum_{i=1}^{n}a_i}\frac{(a_kA_l)^2+(a_lA_k)^2}{a_ka_l}$$

$$-\frac{2}{\sum_{i=1}^{n}a_i}((\frac{A_l^2}{a_l}+\frac{A_k^2}{a_k})\cdot\sum_{m=1,\ m\neq k,l}^{n}a_m+(a_l+a_k)\cdot\sum_{m=1,\ m\neq k,l}\frac{A_m^2}{a_m})$$

$$-\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}.$$

By elementary calculus, the last term of previous inequality becomes

$$A_{k,l}-\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}+\frac{2}{\sum_{i=1}^{n}a_i}(a_1\sum_{i=2,i\neq k,l}^{n}\frac{A_i^2}{a_i}$$

$$+a_2\sum_{i=3,i\neq k,l}^{n}\frac{A_i^2}{a_i}+...+a_{k-1}\sum_{i=k+1,i\neq l}^{n}\frac{A_i^2}{a_i}-a_k\sum_{i=1}^{k-1}\frac{A_i^2}{a_i}+a_{k+1}\sum_{i=k+2,i\neq l}^{n}\frac{A_i^2}{a_i}+...$$

$$+a_{l-1}\sum_{i=l+1}^{n}\frac{A_i^2}{a_i}-a_l\sum_{i=1}^{l-1}\frac{A_i^2}{a_i}+...+a_{n-1}\frac{A_n^2}{a_n}+a_2\frac{A_1^2}{a_1}+a_3(\frac{A_1^2}{a_1}+\frac{A_2^2}{a_2})+...+a_n\sum_{i=1,i\neq k,l}^{n-1}\frac{A_i^2}{a_i})$$

$$=A_{k,l}-\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}+\frac{2}{\sum_{i=1}^{n}a_i}(\sum_{t=1,t\neq k,l}^{n-1}a_t\sum_{r=t+1,r\neq k,l}^{n}\frac{A_r^2}{a_r}$$

$$-a_k\sum_{i=1}^{k-1}\frac{A_i^2}{a_i}-a_l\sum_{i=1}^{l-1}\frac{A_i^2}{a_i}+\sum_{p=2}^{n}a_p\sum_{r=1,r\neq k,l}^{p-1}\frac{A_r^2}{a_r})$$

$$=A_{k,l}-\frac{1}{\sum_{k=1}^{n}a_k}\sum_{1\leq i<j\leq n,\ i,j\neq k,l}\frac{(a_iA_j+a_jA_i)^2}{a_ia_j}+\frac{2}{\sum_{i=1}^{n}a_i}(\sum_{t=1,t\neq k,l}^{n-1}a_t\sum_{r=t+1,r\neq k,l}^{n}\frac{A_r^2}{a_r}$$

$$+\sum_{t=2,t\neq k,l}^{n} a_t \sum_{r=1,r\neq k,l}^{t-1} \frac{A_r^2}{a_r}).$$

Because,

$$\sum_{t=1,t\neq k,l}^{n-1} a_t \sum_{r=t+1,r\neq k,l}^{n} \frac{A_r^2}{a_r} + \sum_{t=2,t\neq k,l}^{n} a_t \sum_{r=1,r\neq k,l}^{t-1} \frac{A_r^2}{a_r} = \sum_{t=2,t\neq k,l}^{n-1} a_t \sum_{r=t+1,r\neq k,l}^{n} \frac{A_r^2}{a_r}$$

$$+\sum_{t=2,t\neq k,l}^{n-1} a_t \sum_{r=1,r\neq k,l}^{t-1} \frac{A_r^2}{a_r} + a_1 \sum_{r=2,r\neq k,l}^{n} \frac{A_r^2}{a_r} + a_n \sum_{r=2,r\neq k,l}^{n-1} \frac{A_r^2}{a_r} = \sum_{t=1,t\neq k,l}^{n} a_t \sum_{r=1,r\neq k,l}^{n} \frac{A_r^2}{a_r}$$

$$-\sum_{t=2,t\neq k,l}^{n-1} A_t^2 = \sum_{t=1}^{n} a_t \sum_{r=1,r\neq k,l}^{n} \frac{A_r^2}{a_r} - (a_k + a_l) \sum_{r=1,r\neq k,l}^{n} \frac{A_r^2}{a_r} - \sum_{t=2,t\neq k,l}^{n-1} A_t^2,$$

we obtain the desired inequality. ∎

Consequence 1. *If $n \in N$ $n \geq 2$, $\mathcal{H}$ being now a Hilbert space, $A_i \in \mathcal{B}(\mathcal{H})$, $i = \overline{1,n}$ gramian self-adjoint operators which commute as pairs and $a_1, a_2, ..., a_n \in (0, \infty)$, then the same inequality as in Theorem 3.3 is satisfied.*

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