The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 2, Article 15, pp. 1-11, 2011

INCLUSION PROPERTIES OF A CERTAIN SUBCLASS OF STRONGLY CLOSE-TO-CONVEX FUNCTIONS

S. M. KHAIRNAR AND M. MORE

Received 3 November, 2009; accepted 30 July, 2010; published 19 April, 2011.

DEPARTMENT OF MATHEMATICS, MAHARASHTRA ACADEMY OF ENGINEERING, ALANDI -412 105, PUNE, MAHARASHTRA, INDIA,
smkhairnar2007@gmail.com
meenamores@gmail.com

ABSTRACT. The purpose of this paper is to derive some inclusion and argument properties of a new subclass of strongly close-to-convex functions in the open unit disc. We have considered an integral operator defined by convolution involving hypergeometric function in the subclass definition. The subclass also extends to the class of α-spirallike functions of complex order.

Key words and phrases: Strongly close-to-convex function, Strongly starlike function, α-spirallike function.

2000 *Mathematics Subject Classification.* Primary 30C45. Secondary 30C50.

ISSN (electronic): 1449-5910

The authors are thankful and acknowledge the support from the research projects funded by the Department of Science and Technology, (Ref. No. SR/S4/MS:544/08), Government of India, National Board of Higher Mathematics, Department of Atomic Energy, (Ref. No.NBHM/DAE/R.P.2/09), Government of India and BCUD, University of Pune (UOP), Pune, (Ref No.BCUD/14/Engg.109), and UGC New Delhi (47-992 / 09 / WRO)..

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and normalised as $f(0) = f'(z) - 1 = 0$. If f and g are analytic in Δ, we say that f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function w in Δ such that $f(z) = g(w(z))$. For a function $f(z)$ belonging to $\mathcal{A}$ we say that $f(z)$ is α-spirallike function of complex order d in Δ, if and only, if

$$\mathcal{R}e\left\{\frac{e^{i\alpha}}{d\cos\alpha}\left(z\frac{f'(z)}{f(z)}\right)\right\} > 0 \tag{1.2}$$

for some real $\alpha, |\alpha| < \pi/2, d \neq 0$ complex.

We denote by $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ the subclass of $\mathcal{A}$ consisting of all analytic functions which are starlike and convex, respectively of order η $(0 \leq \eta < 1)$ in Δ.

For $f \in \mathcal{A}$ if

$$\left|arg\left(\frac{zf'(z)}{f(z)} - \eta\right)\right| < \frac{\pi}{2}\beta \; (z \in \Delta) \tag{1.3}$$

where $0 \leq \eta < 1$ and $0 < \beta \leq 1$, then f is said to be strongly starlike of order β and type η in Δ, denoted by $\mathcal{S}^*(\beta, \eta)$. Similarly if $f \in \mathcal{A}$ satisfies

$$\left|arg\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right)\right| < \frac{\pi}{2}\beta \; (z \in \Delta) \tag{1.4}$$

for $0 \leq \eta < 1$ and $0 < \beta \leq 1$, then f is said to be strongly convex of order β and type η in Δ, denoted by $\mathcal{C}(\beta, \eta)$. It is very natural that $f \in \mathcal{A}$ is in $\mathcal{C}(\beta, \eta)$, if and only, if zf' is in $\mathcal{S}^*(\beta, \eta)$. Also note that $\mathcal{S}^*(1, \eta) = \mathcal{S}^*(\eta)$ and $\mathcal{C}(1, \eta) = \mathcal{C}(\eta)$. The classes $\mathcal{S}^*(\beta, 0)$ and $\mathcal{C}(\beta, 0)$ are studied extensively by Mocanu [12] and Nunokawa [16].

Denote by $D^\lambda : \mathcal{A} \to \mathcal{A}$ the operator defined by

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \; (\lambda > -1). \tag{1.5}$$

The operator $D^\lambda f$ is called the Ruscheweyh derivative of f of order λ. It is obvious that $D^0 f = f, D^1 f = zf'$ and

$$D^\alpha f(z) = \frac{z(z^{\alpha-1}f(z))^{(\alpha)}}{\alpha!}, \; (\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}).$$

Noor [15] has introduced an integral operator $I_n : \mathcal{A} \to \mathcal{A}$, anologous to $D^\lambda f$ as follows.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}, n \in \mathbb{N}_0$ and $f_n^{(-1)}(z)$ be defined such that

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{(1-z)^2}, \tag{1.6}$$

then

$$I_n f(z) = f_n^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f(z), \; (f \in \mathcal{A}). \tag{1.7}$$

We notice that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator I_n is called the Noor integral of n-th order of f (cf. [3], [8]), which is very important operator used in defining several subclasses of analytic functions.

For real or complex numbers a, b, c different from $0, -1, -2, \cdots$, the hypergeometric series is defined by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k \tag{1.8} $$

where $(a)_k$ is the Pochhamer symbol defined in terms of Gamma function by

$$ (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)\cdots(a+k-1) \text{ for } k = 1, 2, 3, \cdots $$

and $(a)_0 = 1$. We notice that the series (1.8) converges absolutely for all $z \in \Delta$ so that it represents an analytic function in Δ. In particular, $z\, {}_2F_1(1, a; c; z) = \phi(a, c; z)$ which is the incomplete beta function. Also $\phi(a, 1; z) = \frac{z}{(1-z)^a}$, where $\phi(2, 1; z)$ is Koebe function.

N. Shukla and P. Shukla [11] studied the mapping properties of f_μ function defined by

$$ f_\mu(a, b, c)(z) = (1-\mu)z\, {}_2F_1(a, b, c; z) + \mu z(z\, {}_2F_1(a, b, c; z)\ (\mu \geq 0). \tag{1.9} $$

We define a function $(f_\mu)^{(-1)}$ on the lines of Noor [15] by

$$ f_\mu(a, b, c)(z) * (f_\mu(a, b, c)(z))^{(-1)} = \frac{z}{(1-z)^{\lambda+1}},\ (\mu \geq 0, \lambda > -1), \tag{1.10} $$

and introduce the linear operator

$$ I_\mu^\lambda(a, b, c) f(z) = ((f_\mu(a, b, c)(z)))^{-1} * f(z). \tag{1.11} $$

For $\mu = 0$ in (1.10) we obtain the operator introduced by K. I. Noor [15].

For $\lambda > -1$ we have

$$ \frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!}\, z^{k+1},\ (z \in \Delta). \tag{1.12} $$

Using (1.8) and (1.12) in (1.10), we get

$$ \sum_{k=0}^{\infty} \frac{(\mu_{k+1})(a)_k(b)_k}{(c)_k(1)_k}\, z^{k+1} * (f_\mu(a, b, c)(z))^{-1} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1}. \tag{1.13} $$

Thus $(f_\mu)^{(-1)}$ has the form

$$ (f_\mu(a, b, c)(z))^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu_{k+1})(a)_k (b)_k}\, z^{k+1},\ (z \in \Delta). $$

Equation (1.11) implies

$$ I_0^\lambda(a, \lambda+1, a) f(z) = f(z),\ I_0^1(a, 1, a) f(z) = z f'(z). $$

It can be easily shown that

$$ z(I_\mu^\lambda(a, b, c) f(z))' = (\lambda+1) I_\mu^{\lambda+1}(a, b, c) f(z) - \lambda I_\mu^\lambda(a, b, c) f(z). \tag{1.14} $$

For $\lambda > -1$ and $\mu \geq 0$ denote $K_\mu^\lambda(a, b, c, d, \alpha, \gamma, \beta, \eta, A, B)$ the class of functions $f \in \mathcal{A}$ satisfying the condition

$$ \left| arg \left(\frac{e^{i\alpha}}{d \cos\alpha} \frac{z(I_\mu^\lambda(a, b, c) f(z))'}{I_\mu^\lambda(a, b, c) g(z)} - \gamma \right) \right| < \frac{\pi}{2}\beta $$

$(0 \le \gamma < 1; 0 < \beta \le 1; z \in \Delta)$ for some $g \in S_\mu^\lambda(a,b,c,\eta,A,B)$, which is defined by

$$S_\mu^\lambda(a,b,c,\eta,A,B) = \left\{ g \in \mathcal{A} : \frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)g(z))'}{I_\mu^\lambda(a,b,c)g(z)}\right) - \eta \prec \frac{1+Az}{1+Bz} \right\}$$

$(0 \le \eta < 1; -1 \le B < A \le 1; z \in \Delta)$.

Notice that $K_0^1(a,1,a,1,0,\gamma,1,\eta,1,-1)$ and $K_0^\lambda(a,\lambda+1,a,1,0,\gamma,1,\eta,1,-1)$ are respectively the classes of quasi-convex and close-to-convex functions of order γ and type η introduced by Noor and Alkhorasani [14] and studied by Silverman [21].
Further $K_0^\lambda(a,\lambda+1,a,1,0,0,\beta,0,1,-1)$ is the class of strongly close-to-convex functions of order β as studied by Pommerenke [18]. For starlike function $f(z)$, $K_0^\lambda(a,\lambda+1,a,d,\alpha,\gamma,1,0,1,-1)$ is the class of α-spirallike functions of order d.

For $q(z) = 1 + c_1 z + c_2 z^2 + \cdots$ which is analytic in Δ satisfies condition

$$q(z) \prec \frac{1+Az}{1+Bz}, \ (z \in \Delta),$$

if and only, if

$$\left| q(z) - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \ (B \ne -1, z \in \Delta),$$

and $\mathcal{R}e\, q(z) > \frac{1-A}{2}$ $(B = -1, z \in \Delta)$. This result is due to Silverman and Silvia [22].

2. MAIN RESULTS

We will require the following Lemmas in proving our main results.

Lemma 2.1. [5] *Let h be convex univalent in Δ with $h(0) = 1$ and $\mathcal{R}e(\beta h(z) + \gamma) > 0$ $(\beta, \gamma \in \mathbb{C})$. If p is analytic in Δ with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \ (z \in \Delta)$$

implies

$$p(z) \prec h(z) \ (z \in \Delta).$$

Lemma 2.2. [11] *Let h be convex univalent in Δ and w be analytic in Δ with $\mathcal{R}e\, w(z) \ge 0$. If p is analytic in Δ and $p(0) = h(0)$, then*

$$p(z) + w(z)zp'(z) \prec h(z) \ (z \in \Delta)$$

implies

$$p(z) \prec h(z) \ (z \in \Delta).$$

Lemma 2.3. [16] *Let p be analytic in Δ with $p(0) = 1$ and $p(z) \ne 0$ in Δ. Suppose that there exists a point $z_0 \in \Delta$ such that*

$$|arg\, p(z)| < \frac{\pi}{2}\theta \text{ for } |z| < |z_0| \tag{2.1}$$

and

$$|arg\, p(z_0)| = \frac{\pi}{2}\theta \ (0 < \theta \le 1). \tag{2.2}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\theta \tag{2.3}$$

where

$$k \geq \frac{1}{2}(s + \frac{1}{s}) \text{ when arg } p(z_0) = \frac{\pi}{2}\theta, \tag{2.4}$$

$$k \leq -\frac{1}{2}\left(s + \frac{1}{s}\right) \text{ when arg } p(z_0) = -\frac{\pi}{2}\theta, \tag{2.5}$$

where

$$p(z_0)^{1/\theta} = \pm is \ (s > 0). \tag{2.6}$$

Using Lemma (2.1), we obtain the following proposition.

Proposition 2.1. *Let $h(z)$ be convex univalent in Δ with $h(0) = 1$ and $\mathcal{R}e\ h(z) > 0$. If a function $f \in \mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(I_\mu^{\lambda+1}(a,b,c)f(z))'}{I_\mu^{\lambda+1}(a,b,c)f(z)} - \eta\right) \prec h(z) \ (0 \leq \eta < 1; z \in \Delta),$$

then

$$\frac{1}{1-\eta}\left(\frac{z(I_\mu^{\lambda}(a,b,c)f(z))'}{I_\mu^{\lambda}(a,b,c)f(z)} - \eta\right) \prec h(z) \ (0 \leq \eta < 1; z \in \Delta).$$

Proof. Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I_\mu^{\lambda}(a,b,c)f(z))'}{I_\mu^{\lambda}(a,b,c)f(z)} - \eta\right) \tag{2.7}$$

where p is analytic function with $p(0) = 1$. Using equation (1.14) we have

$$(1-\eta)p(z) + \eta = \frac{z(I_\mu^{\lambda}(a,b,c)f(z))'}{I_\mu^{\lambda}(a,b,c)f(z)}$$

$$((1-\eta)p(z) + \eta)I_\mu^{\lambda}(a,b,c)f(z) = z(I_\mu^{\lambda}(a,b,c)f(z))'$$

$$((1-\eta)p(z) + \eta)I_\mu^{\lambda}(a,b,c)f(z) = ((1+\lambda)I_\mu^{\lambda+1}(a,b,c)f(z) - \lambda I_\mu^{\lambda}(a,b,c)f(z)) \tag{2.8}$$

Taking logarithmic derivatives on both sides of (2.8) and multiplying by z, we have

$$p(z) + \frac{zp'(z)}{\lambda + \eta + (1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(I_\mu^{\lambda+1}(a,b,c)f(z))'}{I_\mu^{\lambda+1}(a,b,c)f(z)} - \eta\right), \ (z \in \Delta.)$$

By applying Lemma (2.1) we have $p \prec h$, consequently

$$\frac{1}{1-\eta}\left(\frac{z(I_\mu^{\lambda}(a,b,c)f(z))'}{I_\mu^{\lambda}(a,b,c)f(z)} - \eta\right) \prec h(z), \ (z \in \Delta).$$

For $h(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Proposition (2.1), we have the following corollary. ∎

Corollary 2.1. *The inclusion relation $S_\mu^{\lambda+1}(a,b,c,\eta,A,B) \subset S_\mu^{\lambda}(a,b,c,\eta,A,B)$, for any $\lambda > -1, \mu \geq 0$ and $0 \leq \eta < 1, b = \lambda + 1, c = a, \lambda = \mu = 0$.*

Also, if we have $h(z) = \left(\frac{1+z}{1-z}\right)^\beta$ $(0 < \beta \leq 1)$ in Proposition (2.1), we get the following well known inclusion relation.

Corollary 2.2. $\mathcal{C}(\beta,\eta) \subset \mathcal{S}^*(\beta,\eta)$.

Proposition 2.2. *Let $h(z)$ be convex univalent in Δ with $h(0) = 1$ and $\mathcal{R}e\ h(z) > 0$. If $f(z) \in \mathcal{A}$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)f(z)} - \eta\right) \prec h(z)\ (0 \le \eta < 1, z \in \Delta),$$

then

$$\frac{1}{1-\eta}\left(\frac{zI_\mu^\lambda(a,b,c)F(z))'}{I_\mu^\lambda(a,b,c)F(z)} - \eta\right) \prec h(z)\ (0 \le \eta < 1, z \in \Delta),$$

where F is the integral operator defined by

$$F(z) = \frac{r+1}{z^r}\int_0^z t^{r-1}f(t)dt\ (r > -1). \tag{2.9}$$

Proof. From relation (2.9), we have

$$z(I_\mu^\lambda(a,b,c)F(z))' = (r+1)I_\mu^\lambda(a,b,c)f(z) - rI_\mu^\lambda(a,b,c)F(z). \tag{2.10}$$

Let

$$p(z) = \frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)F(z))'}{I_\mu^\lambda(a,b,c)F(z)} - \eta\right),$$

where p is analytic with $p(0) = 1$. By (2.10) we obtain

$$r + \eta + (1-\eta)p(z) = (r+1)\frac{I_\mu^\lambda(a,b,c)f(z)'}{I_\mu^\lambda(a,b,c)F(z)}. \tag{2.11}$$

Differentiating logarithmically both sides of (2.11) we get

$$p(z) + \frac{zp'(z)}{r+\eta+(1-\eta)p(z)} = \frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)f(z)} - \eta\right).$$

Finally by Lemma (2.1), we have

$$\frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)F(z))'}{I_\mu^\lambda(a,b,c)F(z)} - \eta\right) \prec h(z)\ (z \in \Delta).$$

∎

Taking $h(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Proposition (2.2), the following result can be derived.

Corollary 2.3. *If $f(z) \in S_\mu^\lambda(a,b,c,\eta,A,B)$, then $F(z) \in S_\mu^\lambda(a,b,c,\eta,A,B)$ where F is the operator defined by (2.9).*

Notice that, for $h(z) = \left(\frac{1+z}{1-z}\right)^\beta$ $(0 < \beta \le 1)$ in Proposition (2.2) and in view of Corollary (2.2), all functions belonging to the classes $\mathcal{S}^*(\beta,\eta)$ and $\mathcal{C}(\beta,\eta)$, respectively, preserve the angles under the integral operator defined in (2.9).

Theorem 2.1. *Let $f \in \mathcal{A}$ and $0 < \beta \le 1, 0 \le \gamma < 1$. If*

$$\left|arg\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{z(I_\mu^{\lambda+1}(a,b,c)f(z))'}{I_\mu^{\lambda+1}(a,b,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta$$

for some $g \in S_\mu^{\lambda+1}(a,b,c,\eta,A,B)$, then

$$\left|arg\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{z(I_\mu^{\lambda}(a,b,c)f(z))'}{I_\mu^{\lambda}(a,b,c)g(z)} - \gamma\right)\right| < \frac{\pi}{2}\theta$$

where θ $(0<\theta\le 1)$ *is the solution of the equation:*

$$\delta=\begin{cases}\theta+\frac{2}{\pi}\tan^{-1}\left(\frac{\theta\sin\frac{\pi}{2}(1-t_1)}{\frac{(1-\eta)(1+A)}{1+B}+\eta+\lambda+\theta\cos\frac{\pi}{2}(1-t_1)}\right) & \textit{for}\quad B\neq -1\\ \theta & \textit{for}\quad B=-1\end{cases}\tag{2.12}$$

and

$$t_1=\frac{2}{\pi}\sin^{-1}\left(\frac{(1-\eta)(A-B)}{(1-\eta)(1-AB)+(\eta+\lambda)(1-B^2)}\right).\tag{2.13}$$

Proof. Let

$$p(z)=\frac{1}{1-\gamma}\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)g(z)}-\gamma\right).$$

Simplifing using (1.14), we get

$$((1-\gamma)p(z)+\gamma)I_\mu^\lambda(a,b,c)g(z)=\frac{e^{i\alpha}}{d\cos\alpha}\left((1+\lambda)I_\mu^{\lambda+1}(a,b,c)f(z)-\lambda I_\mu^\lambda(a,b,c,)f(z)\right).$$

Differentiating above relation and multiplying by z, we obtain

$$\begin{aligned}&(1-\gamma)zp'(z)I_\mu^\lambda(a,b,c)g(z)+((1-\gamma)p(z)+\gamma)z(I_\mu^\lambda(a,b,c)g(z))'\\ &=\frac{e^{i\alpha}}{d\cos\alpha}\left((1+\lambda)z(I_\mu^{\lambda+1}(a,b,c)f(z))'-\lambda z(I_\mu^\lambda(a,b,c)f(z))'\right).\end{aligned}\tag{2.14}$$

Notice that from Corollary (2.1), $g\in S_\mu^{\lambda+1}(a,b,c,\eta,A,B)$ implies
$g\in S_\mu^\lambda(a,b,c,\eta,A,B)$. Let

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)g(z))'}{I_\mu^\lambda(a,b,c)g(z)}-\eta\right).$$

Using (1.14) again, we get

$$(1-\eta)q(z)+\eta+\lambda=(\lambda+1)\frac{I_\mu^{\lambda+1}(a,b,c)g(z)}{I_\mu^\lambda(a,b,c)g(z)}.\tag{2.15}$$

Relations (2.14) and (2.15) together imply

$$p(z)+\frac{zp'(z)}{(1-\eta)q(z)+\eta+\lambda}=\frac{e^{i\alpha}}{d\cos\alpha}\frac{1}{(1-\gamma)}\left(\frac{zI_\mu^{\lambda+1}(a,b,c)f(z)}{I_\mu^{\lambda+1}(a,b,c)g(z)}-\gamma\right).$$

With arguments similar to the proof of Proposition (2.1) we obtain the first part of the result.

By using the result of Silverman and Silvia [22], we have

$$\left|q(z)-\frac{1-AB}{1-B^2}\right|<\frac{A-B}{1-B^2}\ (B\neq -1, z\in\Delta)\tag{2.16}$$

and

$$\mathcal{R}e\ q(z)>\frac{1-A}{2}\ (B=-1, z\in\Delta).\tag{2.17}$$

Relation (2.16) and (2.17) together imply

$$(1-\eta)q(z)+\eta+\lambda=\rho e^{i\frac{\pi}{2}\phi}$$

where

$$\frac{(1-\eta)(1-A)}{1-B}+\eta+\lambda<\rho<\frac{(1-\eta)(1+A)}{1+B}+\eta+\lambda$$

$-t_1 < \phi < t_1$ for $B \neq -1$ when t_1 is given by (2.13) and

$$\frac{(1-\eta)(1-A)}{2} + \eta + \lambda < \rho < \infty, \ -1 < \phi < 1 \text{ for } B = -1.$$

Notice that $p(z)$ is analytic in Δ with $p(0) = 1$ and $\mathcal{R}e\, p(z) > 0$ in Δ in view of the assumption and Lemma (2.2) with $w(z) = \frac{1}{(1-\eta)q(z)+\eta+\lambda}$. Thus $p(z) \neq 0$ in Δ.

If there exists a point $z_0 \in \Delta$ such that conditions (2.1) and (2.2) are satisfied, then by Lemma (2.3) we obtain (2.3) by the restrictions (2.4), (2.5) and (2.6).

Firstly suppose that $p(z_0)^{1/\theta} = is\ (s > 0)$. For the case $B \neq -1$, we obtain

$$\begin{aligned}
&arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\eta)q(z_0)+\eta+\lambda}\right)\\
&= arg\, p(z_0) + arg(1 + i\theta k(\rho e^{i\frac{\pi}{2}\theta})^{-1})\\
&\geq \frac{\pi}{2}\theta + \tan^{-1}\left(\frac{\theta k \sin\frac{\pi}{2}(1-\theta)}{\rho + \theta k\cos\frac{\pi}{2}(1-\theta)}\right)\\
&\geq \frac{\pi}{2}\theta + \tan^{-1}\left(\frac{\theta \sin\frac{\pi}{2}(1-t_1)}{(1-\eta)\frac{(1+A)}{1+B} + \eta + \lambda + \theta\cos\frac{\pi}{2}(1-t_1)}\right)\\
&\geq \frac{\pi}{2}\delta
\end{aligned}$$

where δ and t_1 are given by (2.12) and (2.13), respectively. Likewise for the case $B = -1$, we have

$$arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\eta)q(z_0)+\eta+\lambda}\right) \leq -\frac{\pi}{2}\theta.$$

This is a contradiction to the assumption of our theorem. Now, suppose that $p(z_0)^{1/\theta} = -is\ (s > 0)$. For the case $B \neq -1$, applying the same method as before, we obtain

$$\begin{aligned}
&arg\left(p(z_0) + \frac{zp'(z_0)}{(1-\eta)q(z_0)+\eta+\lambda}\right)\\
&\leq -\frac{\pi}{2}\theta - \tan^{-1}\left(\frac{\theta \sin\frac{\pi}{2}(1-t_1)}{\frac{(1-\eta)(1+A)}{1+B} + \eta + \lambda + \theta\cos\frac{\pi}{2}(1-t_1)}\right)\\
&= -\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and t_1 are given by (2.12) and (2.13), respectively. Similarly for the case $B = -1$, we have

$$arg\left(p(z_0) + \frac{z_0 p'(z_0)}{(1-\eta)q(z_0)+\eta+\lambda}\right) \leq -\frac{\pi}{2}\theta.$$

These are contradictions to the assumption.
Consequently, the proof of the theorem is complete. ■

Next we note some interesting results that can be derived from Theorem (2.1).

Corollary 2.4. *The inclusion relation*

$$K_\mu^{\lambda+1}(a,b,c,d,\alpha,\beta,\eta,A,B) \subset K_\mu^{\lambda}(a,b,c,d,\alpha,\beta,\eta,A,B),$$

holds for $\lambda > -1, \mu \geq 0, \mathcal{R}e\ a > 0, d \in \mathbb{C}\setminus\{0\}, 0 < \beta \leq 1, 0 \leq \eta < 1, 0 \leq \gamma < 1$*, and* $-1 \leq B < A \leq 1$.

Taking $\lambda = 0, \mu = 0, b = 1, c = a$ *in Theorem (2.1), we obtain the following result.*

Corollary 2.5. *Let $f \in \mathcal{A}$. If*

$$\left|arg\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{(zf'(z))'}{g'(z)}-\gamma\right)\right| < \frac{\pi}{2}\beta\;(0\le\gamma<1, 0<\beta\le 1),$$

for some $g \in S_0^1(a,1,a,\eta,A,B)$, then

$$\left|arg\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{zf'(z)}{g(z)}-\gamma\right)\right| < \frac{\pi}{2}\theta$$

where θ $(0<\theta\le 1)$ is the solution of equation (2.12).

Remark 2.1. *For $d=1, \alpha=0$ we obtain the corresponding result of Cho and Kim* [3]. *Further, if we take $\beta=1, A=1$ and $B=-1$ along with $d=1$ and $\alpha=0$ in Corollary (2.5) we notice that every quasi-convex function of order γ and type β is close-to-convex of order γ and type θ which is exactly the result obtained by Noor* [13].

Taking $\mu=\lambda=\gamma=0, b=1, c=a, B\to A$ $(A<1)$ and $g(z)=z$ in Theorem (2.1) we obtain the following result.

Corollary 2.6. *Let $f \in \mathcal{A}$ and $0<\delta\le 1$. If*

$$\left|arg\left(\frac{e^{i\alpha}}{d\cos\alpha}(f'(z)+zf''(z))\right)\right| < \frac{\pi}{2}\delta,$$

then

$$\left|arg\left(\frac{e^{i\alpha}}{d\cos\alpha}f'(z)\right)\right| < \frac{\pi}{2}\theta,$$

where θ $(0<\theta\le 1)$ is the solution of the equation

$$\delta=\theta+\frac{2}{\pi}\tan^{-1}\theta.$$

Note that taking $\alpha=0, d=1$ in Corollary (2.6) we obtain the result by Cho and Kim [3].

Theorem 2.2. *Let $f \in \mathcal{A}$ and $0<\delta\le 1, 0\le\gamma<1$. If*

$$\left|arg\left(\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)g(z)}-\gamma\right)\right| < \frac{\pi}{2}\delta$$

for some $g \in S_\mu^\lambda(a,b,c,\eta,A,B)$, then

$$\left|arg\left(\frac{z(I_\mu^\lambda(a,b,c)F(f(z)))'}{I_\mu^\lambda(a,b,c)F(g(z))}-\gamma\right)\right| < \frac{\pi}{2}\theta$$

where F is as defined by (2.9) and θ $(0<\theta\le 1)$ is the solution of the equation given by (2.12).

Proof. Let

$$p(z)=\frac{1}{1-\gamma}\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{z(I_\mu^\lambda(a,b,c)F(f(z)))'}{I_\mu^\lambda(a,b,c)F(g(z))}-\gamma\right).$$

Since $g \in S_\mu^\lambda(a,b,c,\eta,A,B)$, Proposition (2.2) implies that
$F(g(z)) \in S_\mu^\lambda(a,b,c,\eta,A,B)$. Proposition (2.2) implies that $F(g(z)) \in S_\mu^\lambda(a,b,c,\eta,A,B)$. In view of (2.10) we have

$$((1-\gamma)p(z)+\gamma)I_\mu^\lambda(a,b,c)F(g(z)) =$$
$$\frac{e^{i\alpha}}{d\cos\alpha}\left[(r+1)z(I_\mu^\lambda(a,b,c)f(z))'-rz(I_\mu^\lambda(a,b,c)F(f(z)))'\right]$$

Simplifying further we obtain

$$((1-\gamma)zp'(z)+((1-\gamma)p(z)+\gamma)((1-\eta)q(z)+r+\eta)$$
$$=\frac{e^{i\alpha}}{d\cos\alpha}(r+1)\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{(I_\mu^\lambda(a,b,c)F(g(z)))}$$

where

$$q(z)=\frac{1}{1-\eta}\left(\frac{z(I_\mu^\lambda(a,b,c)F(g(z)))'}{I_\mu^\lambda(a,b,c)F(g(z))}-\eta\right).$$

Consequently, we have

$$\frac{1}{1-\gamma}\left(\frac{e^{i\alpha}}{d\cos\alpha}\frac{z(I_\mu^\lambda(a,b,c)f(z))'}{I_\mu^\lambda(a,b,c)g(z)}-\gamma\right)=p(z)+\frac{zp'(z)}{(1-\eta)q(z)+\eta+r}.$$

Now following arguments similar to the proof of Theorem (2.1) the required result follows. Theorem (2.2) yields immediately the following result. ■

Corollary 2.7. *If $f(z)\in K_\mu^\lambda(a,b,c,d,\alpha,\gamma,\beta,\eta,A,B)$, then $F(f(z))\in K_\mu^\lambda(a,b,c,d,\alpha,\beta,\eta,A,B)$, where F is the integral operator defined by (2.9).*

Remark 2.2. *If we choose $\mu=0,\lambda=1,b=1,c=a$ and $\mu=0,b=\lambda+1,c=a$ with $d=1,\alpha=0,\beta=1,A=1$ and $B=-1$ in Corollary (2.7), respectively, then we obtain the corresponding results of Noor and Alkhorasani* [14]*. Moreover, taking $\mu=0,b=\lambda+1,c=a,\gamma=0,A=1,B=-1$ and $\delta=1$ in Corollary (2.7), we get the classical result by Bernardi* [1]*, which inturn implies the result by Libera* [7]*.*

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