The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 2, Article 14, pp. 1-18, 2011

HYPERBOLIC MODELS ARISING IN THE THEORY OF LONGITUDINAL VIBRATION OF ELASTIC BARS

[1]I. FEDOTOV, [1]J. MARAIS, [2]M. SHATALOV AND [1]H.M. TENKAM

Received 10 September, 2010; accepted 27 October, 2010; published 19 April, 2011.

[1,2]DEPARTMENT OF MATHEMATICS AND STATISTICS, TSHWANE UNIVERSITY OF TECHNOLOGY, PRIVATE BAG X6680, PRETORIA 0001, SOUTH AFRICA.
fedotovi@tut.ac.za, julian.marais@gmail.com, djouosseutenkamhm@tut.ac.za

[2]MANUFACTURING AND MATERIALS, COUNCIL OF SCIENTIFIC AND INDUSTRIAL RESEARCH (CSIR), P.O. BOX 395, PRETORIA, 0001, SOUTH AFRICA.
mshatlov@csir.co.za

ABSTRACT. In this paper a unified approach to the derivation of families of one dimensional hyperbolic differential equations and boundary conditions describing the longitudinal vibration of elastic bars is outlined. The longitudinal and lateral displacements are expressed in the form of a power series expansion in the lateral coordinate. Equations of motion and boundary conditions are derived using Hamilton's variational principle. Most of the well known models in this field fall withing the frames of the proposed theory, including the classical model, and the more elaborated models proposed by by Rayleigh, Love, Bishop, Mindlin, Herrmann and McNiven. The exact solution is presented for the Mindlin-Herrmann case in terms of Green functions. Finally, deductions regarding the accuracy of the models are made by comparison with the exact Pochhammer-Chree solution for an isotropic cylinder.

Key words and phrases: Rayleigh-Love, Rayleigh-Bishop, Mindlin-Herrmann, hyperbolic operators, longitudinal vibration, elastic bar.

2000 *Mathematics Subject Classification.* Primary 35L20, 35L35, 35L82. Secondary 74H45, 74J05.

ISSN (electronic): 1449-5910

1. INTRODUCTION

Longitudinal vibrations of elastic bars are normally considered in mathematical physics in terms of a classical model described by the wave equation, according to the assumption that the bar is a thin and relatively long structure. More general theories have been formulated by taking into consideration the effects of the lateral motion by which the cross sections of a long and relatively thick bar are extended or contracted. A mathematical formulation of these models includes derivatives of order greater than two in the equation of motion. Rayleigh [15, pp.251–252], and later Love [13, pp.408–409], proposed the simplest generalization of the classical model by including the effects of the inertia of the lateral motion of the parts of the rod not situated on the axis. Bishop developed the next generalization of the theory [2]. In addition to the effects of the lateral motions, Bishop also took into account the contribution of shear effects to strain energy. The Rayleigh-Bishop model is described by a fourth-order partial differential equation (not containing the fourth time derivative). Both the Rayleigh-Love and Rayleigh-Bishop theories consider lateral displacement as being proportional to the longitudinal strain (via Poisson's ratio). The Rayleigh-Bishop model was generalized by Mindlin and Herrmann in 1950 [11, pp.510–521] and later by Mindlin and McNiven in 1960 [14]. They considered the lateral displacement proportional to an independent function of time and longitudinal coordinate. The Mindlin-Herrmann model is formulated as a system of two second order partial differential equations, which can be replaced by a single fourth order equation, resolved with respect to the highest order time derivate. In the Mindlin-McNiven model, the number of modes is not restricted. As an example, the authors considered three independent modes of displacement, two longitudinal modes and one lateral mode. The model is formulated as a system of three partial differential equations of second order, which can be replaced by a single equation of order six.

The present paper shows how to produce a whole set of hyperbolic equations and associated boundary conditions which naturally generalize the above-mentioned models. All the above-mentioned equations can be considered in the frames of the general theory of hyperbolic equations [10] and [4]. The Green functions for all these models can be constructed [5], [6] and [9].

All the equations considered are hyperbolic, the Mindlin-Herrmann equation being strictly hyperbolic [10], and the Rayleigh-Bishop equation strictly pseudo-hyperbolic [8]. The equations of higher order are not strictly hyperbolic since there are multiple roots among the roots of the highest order symbols. For the class of strictly hyperbolic operators, the theory of solvability of the Cauchy or mixed problem has been developed by many authors. A detailed survey of such problems can be found in the book of Gindikin and Volevich [10]. The Rayleigh-Bishop problem represents a special example of equations in which the highest time derivative is not involved in the equation. There are many non-stationary problems in mathematical physics that can be reduced to partial differential equations that are not resolved with respect to the highest derivatives. A detailed survey of such unresolved partial differential equations has been developed in the monograph of Demidenko and Uspenkii [3]. Conditions for the existence of energy estimates and a proof of the solvability of the Cauchy problem for Rayleigh-Bishop type equations can be found in a paper by Fedotov and Volevich [8]. A study of hyperbolic equations with multiple roots of the main part of symbols poses many difficulties and there are no complete results to date similar to strictly hyperbolic theory.

Nevertheless, in certain cases, the solution of the problems of hyperbolic equations can be obtained using the methods developed in [4], [6] and [9]. This method consists of the application of two orthogonality conditions of the system of eigenfunctions of the corresponding Sturm-Liouville problems to obtain a simple form of the Lagrangian that allows one to derive an exact (analytical) solution in the form of Green functions. The two orthogonality conditions arise naturally from the Lagrangian. The physical meaning of the first orthogonality consists of the orthogonality of eigen-velocities involved in the form of kinetic energy. The physical meaning of the second orthogonality consists of the orthogonality of two eigenfunctions involved in the expression of strain energy. This method is briefly exposed in this article by presenting the solution for a mixed Mindlin-Herrman problem.

2. DERIVATION OF THE SYSTEM OF EQUATIONS OF MOTION

Consider a short, cylindrical bar with radius R and length l which experiences longitudinal vibration along the x-axis and lateral shear vibrations, transverse to the x-axis in the direction of the r-axis and in the tangential direction. Consider an axisymmetric problem and suppose that the axial and lateral wave displacements can be written with respect to r in the form

$$u(x,r,t) = u_0(x,t) + r^2 u_2(x,t) + \cdots + r^{2m} u_{2m}(x,t) \tag{2.1}$$

and

$$w(x,r,t) = r u_1(x,t) + r^3 u_3(x,t) + \cdots + r^{2n+1} u_{2n+1}(x,t) \tag{2.2}$$

respectively. The displacements in the tangential direction are assumed to be negligible. That is, no torsional vibrations are present and $v(x,r,\varphi,t) = 0$. The longitudinal and lateral displacements defined in (2.1) and (2.2) are similar to those proposed by Mindlin and McNiven [14]. Mindlin and McNiven, however, represented the expansion of u and w in terms of Jacobi polynomials.

The representation (2.1)–(2.2) was first introduced in 1958 by Zachmanoglou and Volterra to represent longitudinal and lateral displacements in their four mode theory [12, pp.106–107]. According to the choice of m and n in (2.1)–(2.2), different models of longitudinal vibration of elastic bars can be obtained, including the well known models such as those of Rayleigh-Love, Rayleigh-Bishop, Mindlin-Herrmann and a three mode model analogous to the Mindlin-McNiven "second order approximation".

The term multimodal theory has been introduced to describe theories in which the longitudinal and lateral displacements in (2.1)–(2.2) are described by more than one independent function, where the number of modes is equal to the number of independent functions used. Theories in which the longitudinal and lateral displacements are described by a single function have been given the term unimode theory. The terms plane cross sectional and non-plane cross sectional theories will also be used in this article. A plane cross sectional theory is based on the assumption that each point x on the neutral longitudinal axis represents a plane cross section of the bar (orthogonal to the x-axis) and that, during deformation, the plane cross sections remain plane. This assumption was first made during the derivation of the classical wave equation, which is the simplest of the plane cross sectional models discussed. In a non-plane cross sectional theory, each point x can no longer be assumed to represent a plane cross section of the bar. For example, if the lowest axial shear mode is defined by the term $r^2 u_2(x,t)$, then a point x, located on the x-axis, no longer represents the longitudinal displacement of a plane cross section, but rather that of a circular paraboloid section. This concept was first introduced by Mindlin and McNiven [14].

The geometrical characteristics of deformation of the bar are defined by the following linear elastic strain tensor field:

$$\begin{aligned} &\varepsilon_{xx} = \partial_x u && \varepsilon_{rr} = \partial_r w && \varepsilon_{\varphi\varphi} = \frac{\partial_\varphi v}{r} + \frac{w}{r} = \frac{w}{r} \\ &\varepsilon_{xr} = \partial_r u + \partial_x w && \varepsilon_{\varphi x} = \frac{\partial_\varphi u}{r} + \partial_x v = 0 && \varepsilon_{\varphi r} = \partial_r v - \frac{v}{r} + \frac{\partial_\varphi w}{r} = 0 \end{aligned} \tag{2.3}$$

The compact notation $\partial_\alpha = \frac{\partial}{\partial \alpha}$ is used. Due to axial symmetry, it follows that $\varepsilon_{\varphi r} = \varepsilon_{\varphi x} = 0$. The stress tensor due to the isotropic properties of the bar is calculated from Hook's Law as follows:

$$\begin{aligned} &\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda(\varepsilon_{rr} + \varepsilon_{\varphi\varphi}) && \sigma_{xr} = \mu\varepsilon_{xr} \\ &\sigma_{rr} = (\lambda + 2\mu)\varepsilon_{rr} + \lambda(\varepsilon_{xx} + \varepsilon_{\varphi\varphi}) && \sigma_{\varphi x} = \mu\varepsilon_{\varphi x} = 0 \\ &\sigma_{\varphi\varphi} = (\lambda + 2\mu)\varepsilon_{\varphi\varphi} + \lambda(\varepsilon_{xx} + \varepsilon_{rr}) && \sigma_{\varphi r} = \mu\varepsilon_{\varphi r} = 0 \end{aligned} \tag{2.4}$$

λ and μ are Lame's constants, defined as

$$\mu = \frac{E}{2(1+\eta)} \qquad \text{and} \qquad \lambda = \frac{E\eta}{(1+\eta)(1-2\eta)}$$

where η and E are the Poisson ratio and Young modulus of elasticity respectively.

The equations of motion (and boundary conditions) can be derived using Hamilton's variational principle. The Lagrangian is defined as $L = T - P$, where

$$T = \frac{\rho}{2} \int_0^l \int_s \left(\dot{u}^2 + \dot{w}^2\right) ds dx \tag{2.5}$$

is the kinetic energy of the system, representing the energy yielded (supplied) by the displacement of the disturbances (vibrations),

$$P = \frac{1}{2} \int_0^l \int_s \left(\sigma_{xx}\varepsilon_{xx} + \sigma_{rr}\varepsilon_{rr} + \sigma_{\varphi\varphi}\varepsilon_{\varphi\varphi} + \sigma_{xr}\varepsilon_{xr}\right) ds dx \tag{2.6}$$

is the strain energy of the system, representing the potential energy stored in the bar by elastic straining, and $S = \int_s ds = \int_0^{2\pi} \int_0^R r dr d\varphi = \pi R^2$ and ρ are the cross sectional area and mass density of the bar respectively. Substituting (2.5) and (2.6) into the Lagrangian yields

$$L = T - P = L\left(u_j, \dot{u}_j, u_j'\right) = \int_0^l \Lambda\left(u_j, \dot{u}_j, u_j'\right) dx$$

where Λ is known as the Lagrangian density and j depends on the choice of n and m in (2.1)–(2.2). The upper dot and prime denote the derivative with respect to time t and axial coordinate x respectively. Hamilton's principle shows that the Lagrangian density Λ satisfies a system of Euler-Lagrange equations of motion (typically) of the form

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_0'}\right) = 0 \\ &\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_j}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_j'}\right) - \frac{\partial \Lambda}{\partial u_j} = 0 \qquad j = 1, 2, \ldots, q-1 \end{aligned} \tag{2.7}$$

where q is the number of independent modes (functions) chosen to represent longitudinal and lateral displacement in (2.1) and (2.2).

The Sturm-Liouville problem corresponding to the boundary value problem (2.7) (with associated boundary conditions) can be found using the method of separation of variables, by assuming that the independent functions can be expressed in the form of generalized Fourier series':

$$u_j(x,t) = \sum_{n=1}^{\infty} y_{jn}(x)\Phi(t), \quad j = 0,1,2,\ldots,q-1 \tag{2.8}$$

where the set of functions $\{y_{jn}(x)\}$ are the eigenfunctions corresponding to a particular eigenvalue ω_n and q is the number of independent functions. It is possible to prove that the eigenfunctions satisfy the two orthogonality conditions

$$(y_n, y_m)_1 = \|y_n\|_1^2 \delta_{nm} \qquad \text{and} \qquad (y_n, y_m)_2 = \|y_n\|_2^2 \delta_{nm} \tag{2.9}$$

where δ_{nm} is the Kronecker-Delta function.The two orthogonality conditions (2.9) can be used to find the exact solution of the problem, based on the methods developed in [4], [6] and [9]. This method consists of substituting the generalized Fourier series (2.8) into the Lagrangian of the system and using both orthogonality conditions to obtain an ordinary differential equation of order two in the unknown function $\Phi(t)$. Once the solution for $\Phi(t)$ is obtained, it is possible to build the Green functions, and hence the exact solution, of the problem. This method has been outlined in section 3 for the Mindlin-Herrmann case (system of two second order partial differential equations).

2.1. **Particular cases.**

2.1.1. *The classical theory.* The classical theory is the simplest of the models discussed in this article. The longitudinal displacement is represented by

$$u(x,t) = u_0(x,t)$$

and lateral displacements are assumed to be absent ($w = 0$). Since, for the classical theory, $\eta = 0$ (a fortiori $\lambda = 0$) and $E = 2\mu$, it follows that the Lagrangian density of the system is given by

$$\Lambda(\dot{u}_0, u_0') = \frac{1}{2}\left(\rho S \dot{u}_0^2 - ES{u_0'}^2\right) \tag{2.10}$$

which satisfies the Euler-Lagrange partial differential equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_0'}\right) = 0 \tag{2.11}$$

Substituting (2.10) into (2.11) leads to the familiar classical wave equation

$$S\left[\rho\partial_t^2 - E\partial_x^2\right] u_0 = 0 \tag{2.12}$$

which can be written in compact form as

$$SA_1 u_0 = 0 \tag{2.13}$$

where $A_1 = \rho\partial_t^2 - E\partial_x^2$ is a hyperbolic wave operator. The associated natural (free ends) $u_0'(x,t)|_{x=0,l}$ or essential (fixed ends) $u_0(x,t)|_{x=0,l}$ boundary conditions are derived directly from Hamilton's principle.

It is possible to prove that the set of eigenfunctions $\{y_{0n}(x)\}$ of the corresponding Sturm-Liouville problem satisfy the two orthogonality conditions given by (2.9), where

$$(y_n, y_m)_1 = \int_0^l y_{0n} y_{0m} dx \qquad \text{and} \qquad (y_n, y_m)_2 = \int_0^l y_{0n}' y_{0m}' dx \tag{2.14}$$

2.1.2. *The Rayleigh-Love model.* In the Rayleigh-Love model, the longitudinal and lateral displacements are represented by

$$u(x,t) = u_0(x,t) \tag{2.15}$$

$$w(x,r,t) = -r\eta u_0'(x,t) \tag{2.16}$$

That is, the lateral displacement of a particle distant r from the x axis is assumed to be proportional to the longitudinal strain. The Rayleigh-Love model is a unimode plane cross sectional model, since both the longitudinal and lateral displacements are defined in terms of a single mode of displacement, u_0, and the term $-r\eta u_0'(x,t)$ implies that all plane cross sections remain plane during lateral deformation (lateral deformation occurs in plane).

An additional assumption made by Rayleigh and Love is that only the inertial effect of the lateral displacements are taken into account and the effect of stiffness on shear stress is negligible. That is, $\varepsilon_{xr} = \partial_x w \neq 0$ and $\sigma_{xr} \approx 0$. Under these assumptions, substituting (2.15) and (2.16) into (2.5) and (2.6) and using the relation $\lambda + 2\mu - 2\lambda\eta = E$ results in the following Lagrangian density of the system

$$\Lambda(\dot{u}_0, u_0', \dot{u}_0') = \frac{1}{2}\left[\rho S \dot{u}_0^2 + \rho\eta^2 I_2 \left(\dot{u}_0'\right)^2 - SE{u_0'}^2\right] \tag{2.17}$$

where $I_2 = \int_s r^2 ds = \frac{\pi}{2}R^4$ is the polar moment of inertia of the cross section. The Lagrangian density satisfies the Euler-Lagrange partial differential equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_0'}\right) - \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0'}\right) = 0 \tag{2.18}$$

Substituting (2.17) into (2.18) leads to the Rayleigh-Love equation of motion [**?**]

$$S\left(\rho\partial_t^2 - E\partial_x^2\right)u_0 - \eta^2 I_2 \partial_x^2\left(\rho\partial_t^2\right)u_0 = 0 \tag{2.19}$$

or, in compact mode

$$\left[SA_1 - \eta^2 I_2 \partial_x^2\left(\rho\partial_t^2\right)\right]u_0 = 0 \tag{2.20}$$

A combination of the natural

$$\left[SEu_0'(x,t) + \rho\eta^2 I_2 \ddot{u}_0'(x,t)\right]\Big|_{x=0,l} = 0 \tag{2.21}$$

or essential

$$u_0(x,t)\big|_{x=0,l} = 0 \tag{2.22}$$

boundary conditions can be used at the end points $x = 0$ and $x = l$.

It is possible to prove that the set of eigenfunctions $\{y_{0n}(x)\}$ of the corresponding Sturm-Liouville problem satisfy the two orthogonality conditions given by (2.9), where

$$(y_n, y_m)_1 = \int_0^l \left(Sy_{0n}y_{0m} + \eta^2 I_2 y_{0n}' y_{0m}'\right)dx \quad \text{and} \quad (y_n, y_m)_2 = \int_0^l y_{0n}' y_{0m}' dx \tag{2.23}$$

2.1.3. *The Rayleigh-Bishop model.* As in the case of the Rayleigh-Love model, the longitudinal and lateral displacements for the Rayleigh-Bishop model are defined by (2.15) and (2.16). It is clear that the Rayleigh-Bishop model is a unimode, plane cross sectional theory, for the same reasons as discussed above for the Rayleigh-Love case.

In contrast to Rayleigh and Love, the effect of stiffness on shear stress was not neglected by Bishop. That is, $\varepsilon_{xr} = \partial_x w = -\eta r u_0''$ and $\sigma_{xr} = -\mu\eta r u_0'' \neq 0$. In this case, an additional term enters the potential energy function (2.6) and the resulting Lagrangian density of the system

$$\Lambda(\dot{u}_0, u_0', \dot{u}_0', u_0'') = \frac{1}{2}\left[\rho S \dot{u}_0^2 + \rho\eta^2 I_2 \left(\dot{u}_0''\right)^2 - SE{u_0'}^2 - \eta^2 I_2 \mu {u_0''}^2\right] \tag{2.24}$$

which satisfies the Euler-Lagrange partial differential equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_0'}\right)-\frac{\partial^2}{\partial x \partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0'}\right)-\frac{\partial^2}{\partial x^2}\left(\frac{\partial \Lambda}{\partial u_0''}\right)=0 \tag{2.25}$$

yields the Rayleigh-Bishop equation of motion [9]

$$S\left(\rho \partial_t^2-E \partial_x^2\right) u_0-\eta^2 I_2 \partial_x^2\left(\rho \partial_t^2-\mu \partial_x^2\right) u_0=0 \tag{2.26}$$

or, in compact mode

$$\left[S A_1-\eta^2 I_2 \partial_x^2 A_{22}\right] u_0=0 \tag{2.27}$$

where A_1 and $A_{22}=\rho\partial_t^2-\mu\partial_x^2$ are wave operators. A combination of the natural

$$\begin{aligned} \left[S E u_0'(x, t)+\rho \eta^2 I_2 \ddot{u}_0'(x, t)-\eta^2 I_2 \mu u_0'''(x, t)\right]\big|_{x=0, l}&=0, \quad \text{and} \\ u_0''(x, t)\big|_{x=0, l}&=0 \end{aligned} \tag{2.28}$$

or the essential

$$u_0(x, t)\big|_{x=0, l}=0, \quad \text{and} \quad u_0'(x, t)\big|_{x=0, l}=0 \tag{2.29}$$

boundary conditions can be used at the end points $x=0$ and $x=l$.

It is possible to prove that the set of eigenfunctions $\{y_{0n}(x)\}$ of the corresponding Sturm-Liouville problem satisfy the two orthogonality conditions given by (2.9), where

$$\begin{aligned} (y_n, y_m)_1&=\int_0^l\left(S y_{0n} y_{0m}+\eta^2 I_2 y_{0n}' y_{0m}'\right) dx \\ (y_n, y_m)_2&=\int_0^l\left(S E y_{0n}' y_{0m}'+\eta^2 I_2 \mu y_{0n}'' y_{0m}''\right) dx \end{aligned} \tag{2.30}$$

2.1.4. *A two mode (Mindlin-Herrmann) model.* In the Mindlin-Herrmann model, the longitudinal and lateral wave displacements are defined as

$$u(x, t)=u_0(x, t) \tag{2.31}$$

$$w(x, r, t)=r u_1(x, t) \tag{2.32}$$

The Mindlin-Herrmann model is the first (and the simplest) of the multimode theories introduced, since two independent modes of displacement, $u_0(x,t)$ and $u_1(x,t)$, have been considered. The Mindlin-Herrmann model is also a plane cross sectional theory, since the term $ru_1(x,t)$ in (2.32) implies that all plane cross sections remain plane during deformation.

It should be noted that both the Rayleigh-Love and Rayleigh-Bishop models are special cases of the Mindlin-Herrmann model, where the number of independent displacement modes has been reduced from two to one by making the assumption that $\sigma_{rr}=0$ throughout the entire thickness of the bar. That is, the lateral displacements are assumed to be proportional to the longitudinal strain. The Mindlin-Herrmann model can be reduced to either Rayleigh-Love or Rayleigh-Bishop by introducing this assumption in the form of the constraint $u_1+\eta u_0'=0$ (constrained extremum). The Rayleigh-Love model makes the additional assumption that (implicitly) $\sigma_{xr}=0$ throughout the entire thickness of the bar. These assumptions are in agreement with the classical boundary conditions for three or two dimensional axisymmetric models on the free cylindrical outer surface of the bar, $\sigma_{rr}|_{r=R}=0$ and $\sigma_{xr}|_{r=R}=0$.

Substituting (2.31) and (2.32) into (2.5) and (2.6) yields the Lagrangian density of the system

$$\begin{aligned} \Lambda&=\Lambda(\dot{u}_0, \dot{u}_1, u_0', u_1', u_1) \\ &=\frac{1}{2}\left[S \dot{u}_0^2+I_2 \dot{u}_1^2-S(\lambda+2\mu) {u_0'}^2-4 S \lambda u_0' u_1-4 S(\lambda+\mu) u_1^2-I_2 \mu {u_1'}^2\right] \end{aligned} \tag{2.33}$$

which satisfies the system of Euler-Lagrange partial differential equations

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_0}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_0'}\right)=0 \\ &\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot{u}_1}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_1'}\right)-\frac{\partial \Lambda}{\partial u_1}=0 \end{aligned} \tag{2.34}$$

Substituting (2.33) into (2.34) leads to the system of equations of motion

$$\begin{aligned} &\rho S \partial_t^2 u_0 - S(\lambda+2\mu)\partial_x^2 u_0 - 2S\lambda \partial_x u_1 = 0 \\ &2S\lambda\partial_x u_0 + \rho I_2 \partial_t^2 - I_2 \mu \partial_x^2 u_1 + 4S(\lambda+\mu)u_1 = 0 \end{aligned} \tag{2.35}$$

Consider the following substitutions

$$\begin{aligned} \rho\partial_t^2 - (\lambda+2\mu)\partial_x^2 &\triangleq A_{11} \\ -2S\lambda\partial_x &\triangleq -a_{12} \\ 2S\lambda\partial_x &\triangleq a_{21} \\ \rho\partial_t^2 - \mu\partial_x^2 &\triangleq A_{22} \end{aligned}$$

The operators A_{11} and A_{22} are wave operators and a_{12} and a_{21} are first-order differential operators. Now the system (2.35) can be written in compact form:

$$\begin{aligned} &SA_{11}u_0 - a_{12}u_1 = 0 \\ &a_{21}u_0 + \left(I_2 A_{22} + b_{22}\right) u_1 = 0 \end{aligned} \tag{2.36}$$

where $b_{22} = 4S(\lambda + \mu)$. After elimination of one unknown function u_0 (u_1), it follows that the other function u_1 (u_0) satisfies the following equation of order four

$$\left[SA_{11}\left(I_2A_{22}+b_{22}\right)+a_{12}^2\right]u_j = 0, \qquad j=0,1. \tag{2.37}$$

Since, by definition,

$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$$

equation (2.37) can be written in the form

$$\left(I_2A_{11}A_{22}+b_{22}A_1\right)u_j = 0, \qquad j=0,1 \tag{2.38}$$

where A_{11}, A_{22} and A_1 are wave operators. A combination of the natural

$$\left.\left[(\lambda+2\mu)u_0' + 2\lambda u_1\right]\right|_{x=0,l} = 0, \quad \text{and} \quad \left.u_1'\right|_{x=0,l} = 0 \tag{2.39}$$

or essential

$$\left.u_0\right|_{x=0,l} = 0, \quad \text{and} \quad \left.u_1\right|_{x=0,l} = 0 \tag{2.40}$$

boundary conditions can be used at the end points $x = 0$ and $x = l$.

It is possible to prove that the set of eigenfunctions $\{y_{0n}(x)\}$ and $\{y_{1n}(x)\}$ of the corresponding Sturm-Liouville problem satisfy the two orthogonality conditions given by (2.9), where

$$\begin{aligned} (y_n, y_m)_1 &= \int_0^l \left(Sy_{0n}y_{0m} + I_2y_{1n}y_{1m}\right)dx \\ (y_n, y_m)_2 &= \int_0^l \left[4S(\lambda+\mu)y_{1n}y_{1m} + 2S\lambda\left(y_{0n}'y_{1m}+y_{0m}'y_{1n}\right) + S(\lambda+2\mu)y_{0n}'y_{0m}' + \right. \\ &\quad \left. + I_2\mu y_{1n}'y_{1m}'\right]dx \end{aligned} \tag{2.41}$$

2.1.5. *A three mode model.* Consider the case where the longitudinal and lateral displacements are defined by three modes of displacement as follows

$$u(x,r,t) = u_0(x,t) + r^2 u_2(x,t) \tag{2.42}$$

$$w(x,r,t) = r u_1(x,t) \tag{2.43}$$

The longitudinal and lateral displacements defined in (2.42) and (2.43) are similar to those proposed by Mindlin and McNiven as a "second order approximation" of their general theory. The resulting Lagrangian density of this system

$$\begin{aligned}\Lambda &= \Lambda(\dot u_0, \dot u_1, \dot u_2, u_0', u_1', u_2', u_1, u_2)\\ &= \frac{1}{2}\Big[S\rho \dot u_0^2 + 2I_2\rho\dot u_0 \dot u_2 + I_2\rho \dot u_1^2 + I_4 \rho \dot u_2^2 - S(\lambda+2\mu){u_0'}^2 - \mu I_2 {u_1'}^2 -\\ &\quad - I_4(\lambda+2\mu){u_2'}^2 - 4S\lambda u_0' u_1 - 4I_2\mu u_1' u_2 - 2I_2(\lambda+2\mu)u_0'u_2' -\\ &\quad -4I_2\lambda u_1 u_2' - 4S(\lambda+\mu)u_1^2 - 4I_2 u_2^2\Big]\end{aligned} \tag{2.44}$$

satisfies the following system of Euler-Lagrange partial differential equations

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot u_0}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_0'}\right) = 0\\ &\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot u_1}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_1'}\right) - \frac{\partial \Lambda}{\partial u_1} = 0\\ &\frac{\partial}{\partial t}\left(\frac{\partial \Lambda}{\partial \dot u_2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \Lambda}{\partial u_2'}\right) - \frac{\partial \Lambda}{\partial u_2} = 0\end{aligned} \tag{2.45}$$

Substituting (2.44) into (2.45) yields the system of equations of motion

$$\begin{aligned}&S\left[\rho\partial_t^2 - (\lambda+2\mu)\partial_x^2\right]u_0 - 2S\lambda\partial_x u_1 + I_2\left[\rho\partial_t^2 - (\lambda+2\mu)\partial_x^2\right]u_2 = 0\\ &2S\lambda\partial_x u_0\left[S\left(\rho\partial_t^2 - \mu\partial_x^2\right) + 4S(\lambda+\mu)\right]u_1 + 2I_2(\lambda-\mu)\partial_x u_2 = 0\\ &I_2\left[\rho\partial_t^2 - (\lambda+2\mu)\partial_x^2\right]u_0 - 2I_2(\lambda-\mu)\partial_x u_1 + I_4\left[\rho\partial_t^2 - (\lambda+2\mu)\partial_x^2\right]u_2 +\\ &\quad + 4I_2\mu u_2 = 0\end{aligned} \tag{2.46}$$

The system (2.46) can be written in matrix form as

$$\begin{pmatrix} SA_{11} & -a_{12} & I_2A_{11}\\ a_{12} & I_2A_{22}+b_{22} & a_{23}\\ I_2A_{11} & -a_{23} & I_4A_{11}+b_{33}\end{pmatrix}\begin{pmatrix}u_0\\ u_1\\ u_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \tag{2.47}$$

where $a_{23} = 2I_2(\lambda-\mu)\partial_x$ and $b_{33} = 4I_2\mu$. It is easy to prove that, after excluding two of the unknown functions from (2.47), the third unknown function satisfies the following equation of order six

$$\left(A_{22}A_{11}^2 + A_{11}W_1 + W_2\right)u_j = 0, \qquad j = 0,1,2 \tag{2.48}$$

where

$$\begin{aligned}W_1 &= \frac{16}{5R^2}\left[\rho(5\lambda+8\mu)\partial_t^2 - \mu(5\lambda+10\mu)\partial_x^2\right]\\ W_2 &= \frac{768\mu}{5R^4}\left[\rho(\lambda+\mu)\partial_t^2 - \mu(3\lambda+2\mu)\partial_x^2\right]\end{aligned}$$

A combination of the natural

$$\begin{aligned}[S(\lambda+2\mu)u_0'+2S\lambda u_1+I_2(\lambda+2\mu)u_2']|_{x=0,l}&=0,\quad \text{and}\\ [I_2\mu u_1'+2I_2\mu)u_2]|_{x=0,l}&=0,\quad \text{and}\\ [I_2(\lambda+2\mu)u_0'+2I_2\lambda u_1+I_4(\lambda+2\mu)u_2']|_{x=0,l}&=0\end{aligned}\tag{2.49}$$

or essential

$$u_0|_{x=0,l}=0,\quad \text{and}\quad u_1|_{x=0,l}=0,\quad \text{and}\quad u_2|_{x=0,l}=0\tag{2.50}$$

boundary conditions can be used at the end points $x=0$ and $x=l$.

It is possible to prove that the set of eigenfunctions $\{y_{0n}(x)\}$, $\{y_{1n}(x)\}$ and $\{y_{2n}(x)\}$ of the corresponding Sturm-Liouville problem satisfy the two orthogonality conditions given by (2.9), where

$$\begin{aligned}(y_n,y_m)_1&=\int_0^l\left[Sy_{0n}y_{0m}+I_2\left(y_{0n}y_{2m}+y_{0m}y_{2n}\right)+I_4y_{2n}y_{2m}+I_2y_{1n}y_{1m}\right]dx\\ (y_n,y_m)_2&=\int_0^l\left[4S(\lambda+\mu)y_{1n}y_{1m}+2S\lambda\left(y_{0n}'y_{1m}+y_{0m}'y_{1n}\right)+S(\lambda+2\mu)y_{0n}'y_{0m}'+\right.\\ &\quad+I_2(\lambda+2\mu)\left(y_{on}'y_{2m}'+y_{0m}'y_{2n}'\right)+I_4(\lambda+2\mu)y_{2n}'y_{2m}'+\\ &\quad+2I_2\mu\left(y_{1n}'y_{2m}+y_{1m}'y_{2n}\right)+2I_2\lambda\left(y_{1n}y_{2m}'+y_{1m}y_{2n}'\right)+\\ &\quad\left.+I_2\mu y_{1n}'y_{1m}'+4I_2\mu y_{2n}y_{2m}\right]dx\end{aligned}\tag{2.51}$$

2.1.6. *A four mode model.* Consider a model where the longitudinal and lateral displacements are defined by four modes of displacement as

$$u(x,r,t)=u_0(x,t)+r^2u_2(x,t)\tag{2.52}$$

$$w(x,r,t)=ru_1(x,t)+r^3u_3(x,t)\tag{2.53}$$

In a similar fashion as was described for the three mode model above, the system of equations resulting from (2.52) and (2.53) may be written in matrix form as

$$\begin{pmatrix}SA_{11}&-a_{12}&I_2A_{11}&-a_{14}\\ a_{12}&I_2A_{22}+b_{22}&a_{23}&I_4A_{22}+b_{24}\\ I_2A_{11}&-a_{23}&I_4A_{11}+b_{33}&-a_{34}\\ a_{14}&I_4A_{22}+b_{24}&a_{34}&I_6A_{22}+c_{44}\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\\u_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\tag{2.54}$$

where $a_{14}=4I_2\lambda\partial_x$, $a_{23}=2I_2(\lambda-\mu)\partial_x$, $a_{34}=2I_4(2\lambda-\mu)\partial_x$, $I_6=\int_s r^6ds=\frac{pi}{4}R^8$ and b_{22}, b_{33}, b_{44} and b_{24} are numbers depending on λ, μ, I_2, I_4 and I_6. Hence, the corresponding equation of order eight has the highest term of the form $A_{11}^2A_{22}^2$. A combination of the natural

$$\begin{aligned}[S(\lambda+2\mu)u_0'+2S\lambda u_1+I_2(\lambda+2\mu)u_2'+4I_2\lambda u_3]|_{x=0,l}&=0,\quad \text{and}\\ [I_2\mu u_1'+2I_2\mu)u_2+I_4\mu u_3']|_{x=0,l}&=0,\quad \text{and}\\ [I_2(\lambda+2\mu)u_0'+2I_2\lambda u_1+I_4(\lambda+2\mu)u_2'+4I_4\lambda u_3]|_{x=0,l}&=0,\quad \text{and}\\ [I_4\mu u_1'+2I_4\mu u_2+I_6\mu u_3']|_{x=0,l}&=0\end{aligned}\tag{2.55}$$

or essential

$$u_0|_{x=0,l}=0,\quad \text{and}\quad u_1|_{x=0,l}=0,\quad \text{and}\quad u_2|_{x=0,l}=0,\quad \text{and}\quad u_3|_{x=0,l}=0\tag{2.56}$$

boundary conditions can be used at the end points $x=0$ and $x=l$.

It is possible to prove that the set of eigenfunctions $\{y_{0n}(x)\}$, $\{y_{1n}(x)\}$, $\{y_{2n}(x)\}$ and $\{y_{3n}(x)\}$ of the corresponding Sturm-Liouville problem satisfy the two orthogonality conditions given by (2.9), where

$$(y_n, y_m)_1 = \int_0^l [Sy_{0n}y_{0m} + I_2(y_{0n}y_{2m} + y_{0m}y_{2n}) + I_4y_{2n}y_{2m} + I_2y_{1n}y_{1m} + \\ + I_4(y_{1n}y_{3m} + y_{1m}y_{3n}) + I_6y_{3n}y_{3m}]\,dx$$

and

$$\begin{aligned}(y_n, y_m)_2 = \int_0^l & [4S(\lambda+\mu)y_{1n}y_{1m} + 2S\lambda(y'_{0n}y_{1m} + y'_{0m}y_{1n}) + S(\lambda+2\mu)y'_{0n}y'_{0m} + \\ & + I_2(\lambda+2\mu)(y'_{on}y'_{2m} + y'_{0m}y'_{2n}) + I_4(\lambda+2\mu)y'_{2n}y'_{2m} + \\ & + 2I_2\mu(y'_{1n}y_{2m} + y'_{1m}y_{2n}) + 2I_2\lambda(y_{1n}y'_{2m} + y_{1m}y'_{2n}) + \\ & + I_2\mu y'_{1n}y'_{1m} + 4I_2\mu y_{2n}y_{2m} + 4I_2\lambda(y'_{0n}y_{3m} + y'_{0m}y_{3n}) + \\ & + 8I_2(\lambda+\mu)(y_{1n}y_{3m} + y_{1m}y_{3n}) + 4I_4(4\lambda+5\mu)y_{3n}y_{3m} + \\ & + I_4\mu(y'_{1n}y'_{3m} + y'_{1m}y'_{3n}) + 4I_4\lambda(y'_{2n}y_{3m} + y'_{2m}y_{3n}) + \\ & + 2I_4\mu(y_{2n}y'_{3m} + y_{2m}y'_{3n}) + I_6\mu y'_{3n}y'_{3m}]\,dx\end{aligned}$$

3. EXACT SOLUTION OF THE MINDLIN-HERRMANN PROBLEM

In what follows, the solution of one of the models considered in this article, namely that of the mixed two mode (Mindlin-Herrmann) problem (2.36), (2.39), with initial conditions given by

$$\begin{aligned} u_0(x,t)|_{t=0} &= g(x), \qquad & \dot{u}_0(x,t)|_{t=0} &= h(x) \\ u_1(x,t)|_{t=0} &= \phi(x), \qquad & \dot{u}_1(x,t)|_{t=0} &= \psi(x) \end{aligned} \tag{3.1}$$

is presented. Note that the boundary conditions (2.39) represent a bar with both ends free (natural boundary conditions). Applying the method of eigenfunction orthogonalities for vibration problems [4] to problem (2.36), (2.39), (3.1), two types of orthogonality conditions are proven for the eigenfunctions. Assume the solution of the system (2.36) is of the form

$$u_0(x,t) = y_0(x)e^{i\omega t} \qquad u_1(x,t) = y_1(x)e^{i\omega t} \tag{3.2}$$

where $i^2 = -1$. After substituting (3.2) into (2.36) and the boundary conditions (2.39) the following Surm-Liouville problem is obtained

$$\begin{aligned} & S\tilde{A}_{11}y_0 - \tilde{a}_{12}y_1 = 0 \\ & \tilde{a}_{21}y_0 + \left(I_2\tilde{A}_{22} + b_{22}\right)y_1 = 0 \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \tilde{A}_{11} &= -\omega^2\rho - (\lambda+2\mu)\frac{d^2}{dx^2} \\ \tilde{A}_{22} &= -\omega^2\rho - \mu\frac{d^2}{dx^2} \\ \tilde{a}_{12} &= 2S\lambda\frac{d}{dx} \end{aligned}$$

with the corresponding boundary conditions. Let $\{y_{0n}\}$ and $\{y_{1n}\}$ be the eigenfunctions of the Sturm-Liouville problem (3.3), which satisfy the two orthogonality conditions given by (2.41).

The solution of the problem (2.36), (2.39), (3.1) can therefore be written as

$$u_0(x,t) = S\int_0^l \left[g(\xi)\frac{\partial G_1(x,\xi,t)}{\partial t} + h(\xi)G_1(x,\xi,t)\right]d\xi + \\ + I_2\int_0^l \left[\phi(\xi)\frac{\partial G_2(x,\xi,t)}{\partial t} + \psi(\xi)G_2(x,\xi,t)\right]d\xi \tag{3.4}$$

and

$$u_0(x,t) = S\int_0^l \left[g(\xi)\frac{\partial G_3(x,\xi,t)}{\partial t} + h(\xi)G_3(x,\xi,t)\right]d\xi + \\ + I_2\int_0^l \left[\phi(\xi)\frac{\partial G_3(x,\xi,t)}{\partial t} + \psi(\xi)G_3(x,\xi,t)\right]d\xi \tag{3.5}$$

where

$$G_1(x,\xi,t) = \sum_{n=1}^{\infty}\left[\frac{y_{0n}(x)y_{0n}(\xi)\sin\omega_n t}{\omega_n\|y_n\|_1^2}\right] \quad G_2(x,\xi,t) = \sum_{n=1}^{\infty}\left[\frac{y_{0n}(x)y_{1n}(\xi)\sin\omega_n t}{\omega_n\|y_n\|_1^2}\right] \\ G_3(x,\xi,t) = \sum_{n=1}^{\infty}\left[\frac{y_{1n}(x)y_{0n}(\xi)\sin\omega_n t}{\omega_n\|y_n\|_1^2}\right] \quad G_4(x,\xi,t) = \sum_{n=1}^{\infty}\left[\frac{y_{1n}(x)y_{1n}(\xi)\sin\omega_n t}{\omega_n\|y_n\|_1^2}\right] \tag{3.6}$$

are the Green functions, and

$$\omega_n = \frac{\|y_n\|_2}{\sqrt{\rho}\|y_n\|_1}, \qquad n = 1,2,\ldots$$

are the eigenvalues (eigenfrequencies) of the problem. The solution of all other problems presented in this article can be obtained in a similar manner. The solution of the three mode problem, for example, can be obtained with six Green functions.

4. PREDICTING THE ACCURACY OF THE APPROXIMATE THEORIES

Two forms of graphical display are typically used to analyze the factors governing wave propagation for mathematical models describing the vibration of continuous systems. These are the frequency spectrum and phase velocity dispersion curves, and are obtained from the so-called frequency equation [11, pp. 54], [1, pp. 217–218], which shows the relationship between frequency ω, wave number k and phase velocity c for any particular model. In the "$k-\omega$" plane the frequency equation for each model yields a number of continuous curves, called branches. The number of branches corresponds to the number of independent functions chosen to represent u and w in (2.1) (2.2). Each branch shows the relationship between frequency ω and wave number k for a particular mode of propagation. The collection of branches plotted in the "$k-\omega$" plane is called the frequency spectrum of the system. Dispersion curves represent phase velocity c versus wave number k and can be obtained from the frequency equation by using the relation $\omega = ck$.

The different approximate models of longitudinal vibrations of rods can be analyzed and deductions can be made regarding their accuracy by plotting their frequency spectra (or dispersion curves) and comparing them with the frequency spectrum (or dispersion curves) of the exact Pochhammer-Chree frequency equation for (the longitudinal modes of vibration of) an axisymmetric problem of a cylindrical rod with free outer surface [7].

In order to find the frequency equation, it is assumed that each independent function can be represented as $u_j(x,t) = U_j e^{i(kx-\omega t)}$, where $j = 0,1,2,\ldots,q-1$ and q is the number of independent functions chosen in (2.1)–(2.2). These representations for $u_j(x,t)$ are substituted into the equation(s) of motion, yielding the frequency equation. The frequency equation thus obtained for the classical model is given by

$$-\omega^2 + c_0^2 k^2 = 0$$

which gives a single straight line with gradient equal to $c_0 = \sqrt{\frac{E}{\rho}}$, the speed of propagation of waves in a rod described by the classical wave equation. The frequency equations for the Rayleigh-Love and Rayleigh-Bishop models are given by

$$-S\omega^2 + Sc_0^2 k^2 - \eta^2 I_2 \omega^2 k^2 = 0$$

and

$$-S\omega^2 + Sc_0^2 k^2 - \eta^2 I_2 \omega^2 k^2 + \eta^2 I_2 k^4 c_2^2 = 0$$

respectively, where $c_2 = \sqrt{\frac{\mu}{\rho}}$ is the speed of propagation of shear waves in an infinite rod. Since the classical, Rayleigh-Love and Rayleigh-Bishop models are unimodal theories, their frequency equations yield a single branch in the "$k - \omega$" domain. The Rayleigh-Love and Rayleigh-Bishop models, however, do not yield straight lines as in the classical model. That is, the Rayleigh-Love and Rayleigh-Bishop models represent dispersive systems (the phase velocity c depends on the wave number k). For multimodal theories, the substitution results in a system of equations with unknowns U_j. The frequency equation can be found by equating the determinant of the coefficient matrix to zero. The frequency equation for the two mode (Mindlin-Herrmann) model, for example, can be thus obtained as

$$I_2\omega^4 - I_2\omega^2 k^2 \left(c_1^2 + c_2^2\right) - 4S\left(\omega^2 - c_0^2 k^2\right)\left(c_1^2 - c_2^2\right) + I_2 k^4 c_1^2 c_2^2 = 0$$

where $c_1 = \sqrt{\frac{\lambda+2\mu}{\rho}}$ is the speed of propagation of pressure (dilatational) waves in an infinite rod. The well known Pochhammer-Chree frequency equation is given by [11, pp. 464-473], [1, pp. 242–246]

$$\frac{2\alpha}{R}\left(\beta^2 + k^2\right) J_1(\alpha R) J_1(\beta R) - \left(\beta^2 - k^2\right) J_0(\alpha R) J_1(\beta R) - 4k^2\alpha\beta J_1(\alpha R) J_1(\beta R) = 0$$

where $J_n(x)$ is the Bessel function of the first kind of order n,

$$\alpha^2 = \frac{\omega^2}{c_1^2} - k^2, \qquad \beta^2 = \frac{\omega^2}{c_2^2} - k^2$$

and R is the outer radius of the cylinder. The Pochhammer-Chree frequency equation yields infinitely many branches in the "$k - \omega$" and "$k - c$" planes.

The figures that follow in this section have been generated for a cylindrical rod made from an Aluminum alloy with modulus of elasticity $E = 70$ GPa , mass density $\rho = 2700$ kg.m $^{-3}$ and Poisson ratio $\eta = 0.33$. It is not necessary to define the radius R of the cylinder, since all frequency spectra and dispersion curves have been generated using the normalized, dimensionless parameters

$$\Omega = \frac{\omega R}{\pi c_2}, \qquad \xi = \frac{kR}{\pi}, \qquad \bar{c} = \frac{c}{c_0}$$

that are independent of the choice of R.

Figure 1 shows the frequency spectra for the classical, Rayleigh-Love and Rayleigh-Bishop models, as well as the first branch of the two mode model and the first branch of the exact Pochhammer-Chree equation. All the models, including the classical model, approximately describe the first branch of the exact solution in a restricted "$k - \omega$" domain. The Rayleigh-Love approximation is initially more accurate than the Rayleigh-Bishop and two mode (Mindlin-Herrmann) approximations, but the values fall away rapidly for values of ξ greater that 2 (approximately), due to the limit point in the frequency spectrum. The Rayleigh-Bishop and

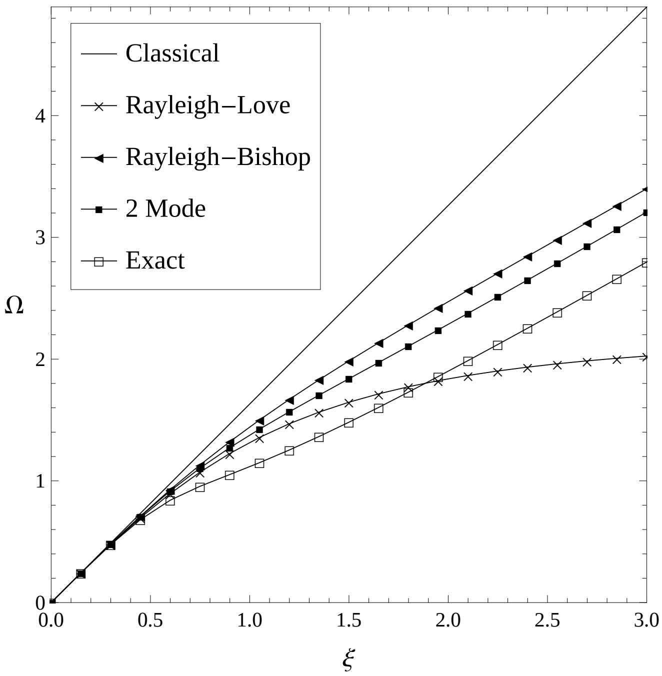

Figure 1: Frequency spectrum of the classical, Rayleigh-Love, Rayleigh-Bishop, two mode (first branch) and exact (first branch) theories.

Mindlin-Herrmann approximations are reasonably accurate over a larger "$k - \omega$ domain, but the branches asymptotically tend towards the shear wave solution, while the exact solution tends to the (Rayleigh) surface waves mode. The shear wave mode is given by the straight line $\omega(k) = c_2 k$ and the surface waves mode is given by the straight line $\omega(k) = c_R k$, where $c_R \approx 0.9320 c_2$ is the speed of propagation of surfaces waves in the rod. The factor 0.9320 is dependent on the Poisson ration $\eta = 0.33$. The phenomenon described above is illustrated in figure 2, which shows the phase velocity dispersion curves for the first branches of the exact solution and the Mindlin-Herrmann model, as well as the Rayleigh-Love and Rayleigh-Bishop models.

Frequency spectra for a selection of multimode theories are presented in figures 3 through 6. The exact Pochhammer-Chree frequency spectrum is shown by dashed lines on all the frequency spectrum plots shown for the multimode models. Figures 3 and 4 show the frequency spectrum of the two mode model described in section 2.1.4 and the three mode model described in section 2.1.5 respectively.

Figure 5 shows the frequency spectrum of the four mode model described in section 2.1.6. The frequency spectrum of a five mode model with three longitudinal and two lateral displacement modes is shown in figure 6.

It is apparent that the branches of the multimode theories approach those of the exact solution with increasing number modes. That is, the greater the amount of independent functions chosen to represent longitudinal and lateral displacements, the broader is the "$k - \omega$" domain in which the effect of longitudinal vibrations of rods could be analyzed. However, it should be noted that, regardless of the number of independent functions chosen, the first few branches of the approximate theories will tend towards the shear wave solution $\omega(k) = c_2 k$, while the remaining branches will tend towards the pressure wave solution $\omega(k) = c_1 k$ as $k \to \infty$. The first branches will not tend towards the surface wave mode, as is the case with the exact Pochhammer-Chree frequency equation, regardless of the number of independent function used. This is because

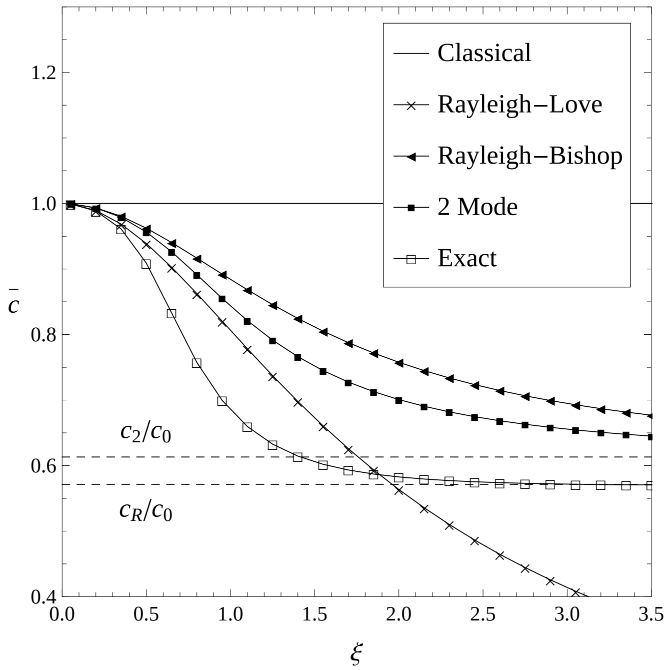

Figure 2: Dispersion curves for the classical, Rayleigh-Love, Rayleigh-Bishop, two mode (first branch) and exact (first branch) theories.

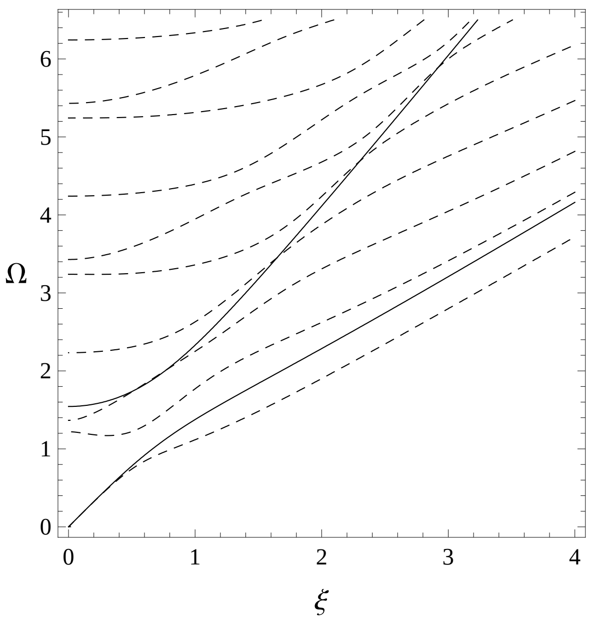

Figure 3: Frequency spectrum of a two mode model (solid line), exact solution (dashed line)

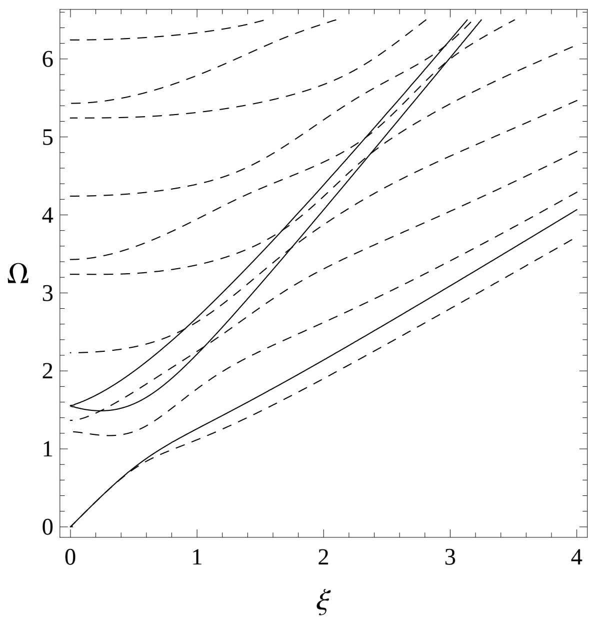

Figure 4: Frequency spectrum of a three mode model (solid line), exact solution (dashed line)

all the approximate theories discussed in this article are one dimensional theories, and can therefore not predict the effect of vibrations on the outer cylindrical surface of the rod. The approximate theories are therefore limited in their application to analysis of waves with long wavelength (small k). Furthermore, each model is limited to analysis of frequencies below that of the lowest frequency of the lowest mode omitted from the model. If frequencies higher than this were to be introduced, then the effects of the omitted modes could not be taken into account

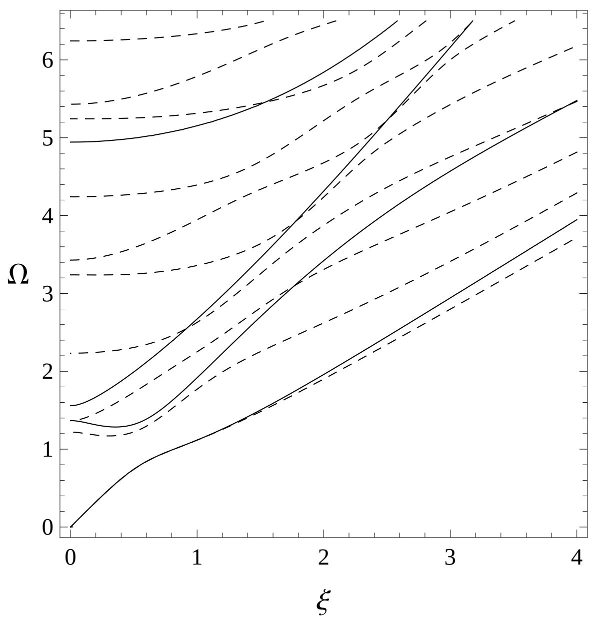

Figure 5: Frequency spectrum of a four mode model (solid line), exact solution (dashed line)

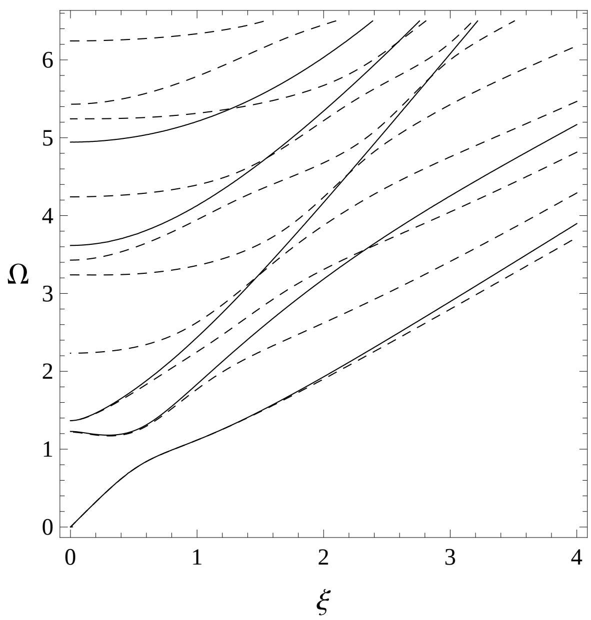

Figure 6: Frequency spectrum of a five mode model (solid line), exact solution (dashed line)

and the dynamics predicted for the system would be inaccurate. This "coupling" effect of the displacement modes is evident from coupled systems of equations of motion for the multimode models.

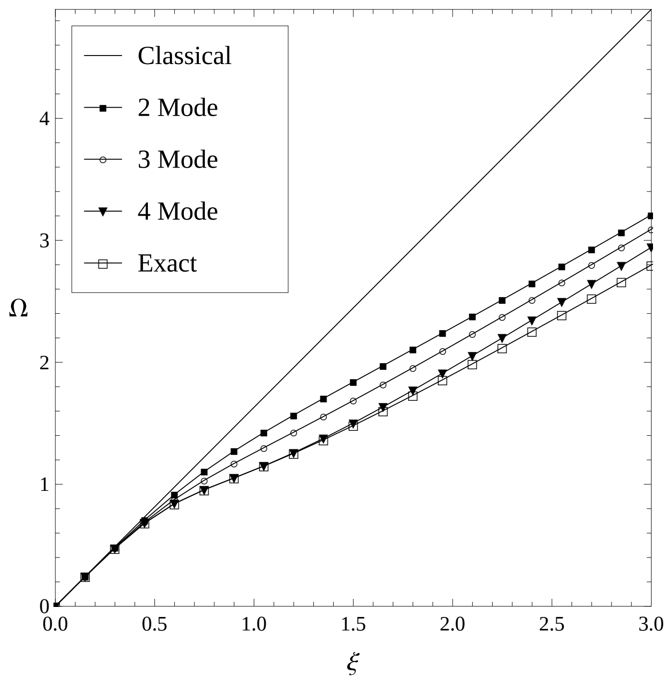

Figure 7: First branches of the two, three, four mode models, and the exact solution.

The nature of the first (lowest) branches of the multimode models with increasing number of modes is of particular interest in the design of low frequency ultrasonic transducers and waveguides, where low frequency, long wavelength waves are typically considered. Figure 7 shows the first branches of the spectral curves for the two mode, three mode and four mode models, together with the first branch of the exact Pochhammer-Chree solution. It is clear from this figure that the first branch of the approximate multimode theories approach that of the exact solution with increasing number of modes. The first branch of the five mode model has been omitted from figure 7, since it is too close to the first branch of the four mode model for them to be easily distinguished from one another in the selected region for ξ.

5. CONCLUSION

In this article, a generalized theory for the derivation of approximate models describing the longitudinal vibration of elastic bars has been presented. The models outlined in this article represent a family of one dimensional hyperbolic differential equations, since all $u_n = u_n(x,t)$. An infinite number of these approximate models have been introduced and the general procedure for derivation of the equations of motion and boundary conditions for all the models has been exposed.

The approximate models have been categorized as unimode or multimode, and plane cross sectional or non-plane cross sectional models, based on the representation for longitudinal and lateral displacements (2.1)–(2.2). The classical, Rayleigh-Love and Rayleigh-Bishop models are all unimode, plane cross sectional models. Both the Mindlin-Herrmann and the three mode models are multimode models. The Mindlin-Herrmann model is a plane cross sectional model, whereas the three mode model is a non-plane cross sectional model. All models subsequent to the three mode model (four mode, five mode, etc.) are also multimode, non-plane cross sectional models.

The orthogonality conditions have been given for all models discussed, which substantially simplify construction of the solution in terms of Green functions. The solution procedure was presented for the Mindlin-Herrmann model, using the Fourier method. The method of two eigenfunction orthogonalities (presented here) can be used to obtain the solution for all models considered in this article.

Finally, it has been shown that the accuracy of the approximate models discussed in this article approach that of the exact theory with an increasing number of modes, based on a comparison of the frequency spectra with that of the exact Pochhammer-Chree frequency equation for an infinite isotropic cylinder with free outer surface.

ACKNOWLEDGEMENT

This material is based upon work supported financially by the Tshwane University of Technology (TUT), the Council for Scientific and Industrial Research (CSIR) of South Africa and the National Research Foundation (NRF) of South Africa (NRF grant reference number EV2008111100011). Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and therefore the TUT, the CSIR and the NRF do no accept any liability in regard thereto.

REFERENCES

[1] J.D. ACHENBACH, *Wave Propagation in Elastic Solids*, Elsevier, Amsterdam, 2005.

[2] R.E.D BISHOP, Longitudinal Waves in Beams, *The Aeronautical Quarterly,* **3** (1952), pp. 280–293.

[3] G.V. DEMIDENKO and S.V USPENKII, *Equations and Systems Not Resolved With Respect to the Highest Order Derivative,* (Nauchnaya kniga, Novosibirsk, 1998. [in Russian]; Partial Differential Equations and Systems Not Solvable With Respect to the Highest Order Derivate, *Pure and Applied Mathematics 256,* Marcel Dekker, New York, 1998.

[4] I. FEDOTOV, T. FEDOTOVA, M.A. SHATALOV and H.M. TENKAM, Method of Multiple Orthogonalities for Vibration Problems, *Current Themes in Eingineering Science 2009: Selected Presentations at the World Congress on Engineering 2009, AIP conference Proceedings,* **1220** (2010), pp. 43–58.

[5] I. FEDOTOV, Y. GAI, A. POLYANIN and M. SHATALOV, Analysis of an N-Stepped Rayleigh Bar With Sections of Complex Geometry, *Appl. Math. Model.,* **32** (2008), pp. 1–11.

[6] I. FEDOTOV, A.D. POLYANIN and M. SHATALOV, Theory of Free Vibration of a Rigid Rod Based on the Rayleigh Model, *Dokl. Phys.,* **417** (2007), pp. 56–61.

[7] I. FEDOTOV, M. SHATALOV, H.M. TENKAM and J. MARAIS. Comparison of Classical and Modern Theories of Longitudinal Wave Propagation in Elastic Rods, *Proceedings of the 16th International Conference of Sound and Vibration,,* Krakow, Poland, pp. 5 –9 July, 2009.

[8] I. FEDOTOV and L.R. VOLEVICH, The Cauchy Problem for Hyperbolic Equations Not Resolved With Respect to the Highest Time Derivate, *Russ. J. Math. Phys.,* **13** (2006), pp. 279–292.

[9] Y. GAI, I. FEDOTOV and M. SHATALOV, Analysis of a Rayleigh-Bishop Model For a Thick Bar, *Proceedings of the 2006 IEEE International Ultrasonics Symposium,* Vancouver, Canada, 2–6 October, 2006, pp. 1915–1917.

[10] S.G. GINDIKIN and L.R. VOLEVICH, Mixed Problem for Partial Differential equations With Quasihomogeneous Principle Part, *Translation of Mathematical Monographs 147,* American Mathematical Society, Providence, 1996.

[11] K.F. GRAFF, *Wave Motion in Elastic Solids,* Dover Publications, New York, 1991.

[12] E.I. GRIGOLJUK & I.T. Selezov, *Mechanics of Solid Deformed Bodies, Vol. 5, Non-classical Theories of Rods, Plates and Shells,* Nauka, Moscow, 1973. [in Russian]

[13] A.E.H. LOVE, *A Treatise on the Mathematical Theory of Elasticity,* 2nd (1906) Ed., Bibliolife, 2009.

[14] R.D MINDLIN and H.D MCNIVEN, Axially Symmetric Waves in Elastic Rods, *Trans. ASME J. Appl. Mech.,* **27** (1960), pp. 145–151.

[15] J.W.S RAYLEIGH, *Theory of Sound, Vol. I,* Dover Publications, New York, 1945.