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ON THE DEGREE OF APPROXIMATION OF CONTINUOUS FUNCTIONS THAT PERTAINS TO THE SEQUENCE-TO-SEQUENCE TRANSFORMATION

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ABSTRACT. In this paper we prove analogous theorems like Leindler's [3] using the so-called A-transform of the B-transform of the partial sums of Fourier series. In addition, more than two such transforms are introduced and for them analogous results are showed as well.

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1. INTRODUCTION

Let $f(x)$ be a 2π- periodic continuous function. Let $S_n(f;x)$ denote the n-th partial sum of its Fourier series at x and let $\omega(\delta)=\omega(\delta,f)$ denote the modulus of continuity of f.

Let $A:=(a_{n,k})$ $(k,n=0,1,\dots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{S_n(f;x)\}$ be given by

$$T_{n,A}(f):=T_{n,A}(f;x):=\sum_{k=0}^{n}a_{n,k}S_k(f;x)\quad (n=0,1,\dots).$$

The deviation $T_{n,A}(f)-f$ was estimated by P. Chandra [1] and [2] for monotonic sequences $\{a_{n,k}\}$. Later on, these results are generalized by L. Leindler [3] who considered the sequences of Rest Bounded Variation and of Head Bounded Variation.

We point out that throughout of this paper we write $u=O(v)$ if there exists a positive constant C such that $u\le Cv$, and $\|\cdot\|$ denotes the supnorm.

Now, let us recall Chandra's theorems.

Theorem 1.1. *Let $\{a_{n,k}\}$ satisfy the following conditions:*

$$a_{n,k}\ge 0 \qquad \textit{and} \qquad \sum_{k=0}^{n}a_{n,k}=1, \tag{1.1}$$

$$a_{n,k}\le a_{n,k+1}\quad (k=0,1,\dots,n-1;n=0,1,\dots). \tag{1.2}$$

Suppose $\omega(t)$ is such that

$$\int_u^{\pi}t^{-2}\omega(t)dt=O\left(H(u)\right)\quad (u\to +0), \tag{1.3}$$

where $H(u)\ge 0$ and

$$\int_0^{t}H(u)du=O\left(tH(t)\right)\quad (t\to +0). \tag{1.4}$$

Then

$$\|T_{n,A}(f)-f\|=O\left(a_{n,n}H(a_{n,n})\right).$$

Theorem 1.2. *Let (1.1), (1.2) and (1.3) hold. Then*

$$\|T_{n,A}(f)-f\|=O\left(\omega(\pi/n)\right)+O\left(a_{n,n}H(\pi/n)\right).$$

If, in addition, $\omega(t)$ satisfies (1.4) then

$$\|T_{n,A}(f)-f\|=O\left(a_{n,n}H(\pi/n)\right).$$

Theorem 1.3. *Let us assume that (1.1) and*

$$a_{n,k}\ge a_{n,k+1}\quad (k=0,1,\dots,n-1;n=0,1,\dots) \tag{1.5}$$

hold. Then

$$\|T_{n,A}(f)-f\|=O\left(\omega(\pi/n)+\sum_{k=1}^{n}k^{-1}\omega(\pi/k)\sum_{r=0}^{k+1}a_{n,r}\right).$$

Theorem 1.4. *Let (1.1), (1.3), (1.4) and (1.5) hold. Then*

$$\|T_{n,A}(f)-f\|=O\left(a_{n,0}H(a_{n,0})\right).$$

L. Leindler [3] defined two classes of sequences, above-mentioned, as follows:

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers will be called of Head Bounded Variation, or briefly $\mathbf{c} \in HBVS$, if it has the property

$$\sum_{n=0}^{m-1} |c_n - c_{n+1}| \leq K(\mathbf{c})c_m$$

for all natural numbers m, or only for all $m \leq N$ if the sequence $\mathbf{c}$ has only finite nonzero terms, and the last nonzero term is c_N.

Assuming that for all n and $0 \leq m \leq n$

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \tag{1.6}$$

and

$$\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m} \tag{1.7}$$

hold, where K is an absolute constant, L. Leindler proved the following:

Theorem 1.5. *The statements of Theorems 1.1, 1.2, 1.3 and 1.4 hold with (1.7) in place of (1.2), and with (1.6) in place of (1.5), respectively; naturally maintaining all the other assumptions.*

Let $B := (b_{n,k})$ $(k, n = 0, 1, \dots)$ be another lower triangular infinite matrix of real numbers, and let the B-transform of $\{S_n(f; x)\}$ be given by

$$T_{n,B}(f) := T_{n,B}(f; x) := \sum_{k=0}^{n} b_{n,k} S_k(f; x) \quad (n = 0, 1, \dots).$$

Wishing to generalize the Theorem 1.5, we define the A-transform of the B-transform (sequence-to-sequence transformation) of the sequence $\{S_n(f; x)\}$, as in [4], by

$$T_{n,AB}(f) := T_{n,AB}(f; x) := \sum_{p=0}^{n} a_{n,p} T_{p,B}(f; x) = \sum_{p=0}^{n} \sum_{k=0}^{p} a_{n,p} b_{p,k} S_k(f; x) \quad (n = 0, 1, \dots).$$

So, the object of this paper is to prove the analogous theorems of Leindler using $T_{n,AB}(f)$ instead $T_{n,A}(f)$. Before doing this we need some lemmas given in next section.

2. Helpful Lemmas

The following lemmas are necessary for the proof of main results.

Lemma 2.1. [1] *If (1.3) and (1.4) hold then*

$$\int_0^{\pi/n} \omega(t)dt = O\left(n^{-2} H(\pi/n)\right).$$

Lemma 2.2. [2] *If (1.3) and (1.4) hold then*

$$\int_0^r t^{-1}\omega(t)dt = O\left(rH(r)\right) \quad (r \to +0).$$

Lemma 2.3. [3] *If for a fixed n the sequence $\{a_{n,k}\} \in RBVS$, then, uniformly in $0 < t \leq \pi$,*

$$\sum_{k=0}^{n} a_{n,k} \sin\left(k+\frac{1}{2}\right)t = O\left(A_{n,\tau}\right),$$

where $A_{n,m} := \sum_{r=0}^{m} a_{n,r}$ and τ denotes the integer part of $\frac{\pi}{t}$.

If $\{a_{n,k}\} \in HBVS$ then

$$\sum_{k=0}^{n} a_{n,k} \sin\left(k+\frac{1}{2}\right)t = O\left(\frac{a_{n,n}}{t}\right).$$

Lemma 2.4. *If for a fixed p, $(p = 0, 1, \ldots, n)$, the sequence $\{b_{p,k}\} \in RBVS$, then, uniformly in $0 < t \leq \pi$,*

$$\sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k} \sin\left(k+\frac{1}{2}\right)t = O\left(\sum_{p=0}^{n} a_{n,p}A_{p,\tau}\right), \tag{2.1}$$

where $A_{p,m} := \sum_{r=0}^{m} b_{p,r}$ and τ denotes the integer part of $\frac{\pi}{t}$.

If $\{b_{p,k}\} \in HBVS$ then

$$\sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k} \sin\left(k+\frac{1}{2}\right)t = O\left(\frac{\sum_{p=0}^{n} a_{n,p}b_{p,p}}{t}\right). \tag{2.2}$$

Proof. Follows analogously as the proof of Lemma 2.3 with slight changes, so we omit details. ∎

3. MAIN RESULTS

To prove our results we shall use the same technique as L. Leindler used for his results [3].

Theorem 3.1. *Let $\{a_{n,p}\}$, $\{b_{p,k}\}$ satisfy the following conditions:*

$$a_{n,p} \geq 0, \quad b_{p,k} \geq 0 \quad \text{and} \quad \sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k} = 1, \tag{3.1}$$

$$\sum_{k=0}^{m-1} |b_{p,k} - b_{p,k+1}| \leq K b_{p,m}, \quad 0 \leq m \leq p, \ \forall p. \tag{3.2}$$

Suppose $\omega(t)$ is such that

$$\int_u^{\pi} t^{-2}\omega(t)dt = O\left(H(u)\right) \quad (u \to +0), \tag{3.3}$$

where $H(u) \geq 0$ and

$$\int_0^t H(u)du = O\left(tH(t)\right) \quad (t \to +0). \tag{3.4}$$

Then

$$\|T_{n,AB}(f)-f\| = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,p} \times H\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\right)\right).$$

Proof. Putting $\Psi_x(t) := \frac{1}{2}\{f(x+t)+f(x-t)-2f(x)\}$ we write

$$T_{n,AB}(f;x)-f(x) = \frac{2}{\pi}\int_0^{\pi} \Psi_x(t)\left(2\sin\frac{t}{2}\right)^{-1} \sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k}\sin\left(k+\frac{1}{2}\right)t\,dt$$

and

$$\begin{aligned} \|T_{n,AB}-f\| &= O\left(\frac{2}{\pi}\right)\int_0^{\pi}\omega(t)\left(2\sin\frac{t}{2}\right)^{-1}\left|\sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k}\sin\left(k+\frac{1}{2}\right)t\right|dt \\ &= O\left(\frac{2}{\pi}\right)\left(\int_0^{\sum_{p=0}^{n} a_{n,p}b_{p,p}} + \int_{\sum_{p=0}^{n} a_{n,p}b_{p,p}}^{\pi}\right) := F_1+F_2. \end{aligned} \tag{3.5}$$

By the well-known inequality $\sin\theta \geq \frac{2}{\pi}\theta$ for $0\leq\theta\leq\frac{\pi}{2}$, (3.1) and Lemma 2.2 we get

$$F_1 = O(1)\int_0^{\sum_{p=0}^{n} a_{n,p}b_{p,p}} t^{-1}\omega(t)dt = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\times H\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\right)\right). \tag{3.6}$$

Now using (2.2) and (3.3), under condition $\{b_{p,k}\}\in HBVS$, we have

$$F_2 = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\right)\int_{\sum_{p=0}^{n} a_{n,p}b_{p,p}}^{\pi} t^{-2}\omega(t)dt = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\times H\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\right)\right).$$

Substituting estimates for F_1 and F_2 into relation (3.5) we obtain the proof of theorem 3.1. ∎

Theorem 3.2. *Let (3.1), (3.2) and (3.3) hold. Then*

$$\|T_{n,AB}(f)-f\| = O\left(\omega(\pi/n)\right) + O\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\times H(\pi/n)\right). \tag{3.7}$$

If, in addition, $\omega(t)$ *satisfies (3.4) then*

$$\|T_{n,AB}(f)-f\| = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,p}\times H(\pi/n)\right). \tag{3.8}$$

Proof. As in the proof of Theorem 3.1 we can write

$$\begin{aligned} \|T_{n,AB}-f\| &= O\left(\frac{2}{\pi}\right)\int_0^{\pi}\omega(t)\left(2\sin\frac{t}{2}\right)^{-1}\left|\sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k}\sin\left(k+\frac{1}{2}\right)t\right|dt \\ &= O\left(\frac{2}{\pi}\right)\left(\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right) := H_1+H_2. \end{aligned} \tag{3.9}$$

Using the well-known inequalities $\sin\theta\geq\frac{2}{\pi}\theta$ for $0\leq\theta\leq\frac{\pi}{2}$, $|\sin t|\leq t$ and (3.1) we have

(3.10)
$$H_1 = O(1)\int_0^{\pi/n}\omega(t)\left|\sum_{p=0}^{n}\sum_{k=0}^{p}a_{n,p}b_{p,k}\left(k+\frac{1}{2}\right)\right|dt = O(n)\int_0^{\pi/n}\omega(t)dt = O\left(\omega(\pi/n)\right).$$

From (2.2) and (3.3) for H_2 we get

$$H_2 = O\left(\sum_{p=0}^{n}a_{n,p}b_{p,p}\right)\int_{\pi/n}^{\pi}t^{-2}\omega(t)dt = O\left(\sum_{p=0}^{n}a_{n,p}b_{p,p}\times H(\pi/n)\right). \tag{3.11}$$

Inserting estimates (3.10) and (3.11) to (3.9) we prove (3.7).
Now we pass to the proof of (3.8). Since $\{b_{p,k}\}\in HBVS$ then, for $k<p$ we get that

$$|b_{p,k}-b_{p,p}| \le \sum_{l=k}^{p-1}|b_{p,l}-b_{p,l+1}| \le K(\mathbf{c})b_{p,p} \le Kb_{p,p} \Rightarrow b_{p,k} \le (K+1)b_{p,p}.$$

Using the last estimate we have

$$\begin{aligned} 1 = \sum_{p=0}^{n}\sum_{k=0}^{p}a_{n,p}b_{p,k} &\le (K+1)\sum_{p=0}^{n}(p+1)a_{n,p}b_{p,p} \\ &\le 2(K+1)n\sum_{p=0}^{n}a_{n,p}b_{p,p} \\ &\Rightarrow \frac{1}{n} = O\left(\sum_{p=0}^{n}a_{n,p}b_{p,p}\right). \end{aligned} \tag{3.12}$$

By Lemma 2.1, (3.12) and (3.10) we obtain

$$H_1 = O(n)\int_0^{\pi/n}\omega(t)dt = O(n)O\left(n^{-2}H(\pi/n)\right) = O\left(\sum_{p=0}^{n}a_{n,p}b_{p,p}\times H(\pi/n)\right). \tag{3.13}$$

Thus (3.11) and (3.13) imply (3.8). This completes the proof of Theorem 3.2. ∎

Theorem 3.3. *Let us assume that (3.1) and*

$$\sum_{k=m}^{\infty}|b_{p,k}-b_{p,k+1}| \le Kb_{p,m}, \quad 0\le m\le p,\ \forall p.$$

hold. Then

$$\|T_{n,AB}(f)-f\| = O\left(\omega(\pi/n)+\sum_{p=0}^{n}\sum_{l=1}^{n-1}l^{-1}\omega(\pi/l)a_{n,p}\sum_{r=0}^{l+1}b_{p,r}\right).$$

Proof. We start from (3.9). For H_1 we use the estimate (3.10) :

$$H_1 = O\left(\omega(\pi/n)\right). \tag{3.14}$$

Applying Lemma 2.4 with (2.1) we have

$$\begin{aligned} H_2 &= O\left(\sum_{p=0}^{n}\int_{\pi/n}^{\pi} t^{-1}\omega(t)a_{n,p}A_{p,\tau}dt\right) \\ &= O\left(\sum_{p=0}^{n}\sum_{l=1}^{n-1}\int_{\pi/(l+1)}^{\pi/l} t^{-1}\omega(t)a_{n,p}A_{p,\tau}dt\right) \\ &= O\left(\sum_{p=0}^{n}\sum_{l=1}^{n-1} l^{-1}\omega(\pi/l)a_{n,p}\sum_{r=0}^{l+1}b_{p,r}\right). \end{aligned}$$

This with (3.14) and (3.9) complete the proof of our theorem. ∎

Theorem 3.4. *Let (3.1), (3.3), (3.4) hold, and* $\{b_{p,k}\} \in RBVS$*. Then*

$$\|T_{n,AB}(f) - f\| = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,0} \times H\left(\sum_{p=0}^{n} a_{n,p}b_{p,0}\right)\right). \tag{3.15}$$

Proof. Since $\{b_{p,k}\} \in RBVS$, then for $p \geq 0$ we have

$$b_{p,p} \leq \sum_{l=p}^{\infty} |b_{p,l} - b_{p,l+1}| \leq \sum_{l=0}^{\infty} |b_{p,l} - b_{p,l+1}| \leq K(\mathbf{c})b_{p,0} \leq Kb_{p,0},$$

therefore (2.1) takes the following form

$$\sum_{p=0}^{n}\sum_{k=0}^{p} a_{n,p}b_{p,k}\sin\left(k+\frac{1}{2}\right)t = O\left(\frac{\sum_{p=0}^{n} a_{n,p}b_{p,0}}{t}\right). \tag{3.16}$$

Now from (3.5)we can write

$$\|T_{n,AB} - f\| = O\left(\frac{2}{\pi}\right)\left(\int_0^{\sum_{p=0}^{n} a_{n,p}b_{p,0}} + \int_{\sum_{p=0}^{n} a_{n,p}b_{p,0}}^{\pi}\right) := L_1 + L_2.$$

For L_1, similar with (3.6), we have

$$L_1 = O(1)\int_0^{\sum_{p=0}^{n} a_{n,p}b_{p,0}} t^{-1}\omega(t)dt = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,0} \times H\left(\sum_{p=0}^{n} a_{n,p}b_{p,0}\right)\right). \tag{3.17}$$

For L_2, applying (3.16) and (3.3), we obtain

$$L_2 = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,0}\right)\int_{\sum_{p=0}^{n} a_{n,p}b_{p,0}}^{\pi} t^{-2}\omega(t)dt = O\left(\sum_{p=0}^{n} a_{n,p}b_{p,0} \times H\left(\sum_{p=0}^{n} a_{n,p}b_{p,0}\right)\right). \tag{3.18}$$

The estimates (3.17) and (3.18) prove (3.15). ∎

Remark 3.1. If we put $a_{n,0} = a_{n,1} = \cdots = a_{n,n-1} = 0$ and $a_{n,n} = 1$ in our theorems we obtain Leindler's results [3]. In other words theorem 1.5 holds.

If $a_{n,0} = a_{n,1} = \cdots = a_{n,n-1} = 0$, $a_{n,n} = 1$ and

$$b_{n,k} = \begin{cases} p_k/P_n, & \text{if } \ 0 \leq k \leq n \\ 0, & \text{if } \ k > n, \end{cases}$$

where $\{p_k\}$ is a nonnegative sequence and $P_n := \sum_{k=0}^{n} p_k$, $p_0 > 0$, then the matrix is a Riesz matrix, and we write $R_n(f;x)$ for $T_{n,AB}(f;x)$.

If $a_{n,0} = a_{n,1} = \cdots = a_{n,n-1} = 0$, $a_{n,n} = 1$ and

$$b_{n,k} = \begin{cases} p_{n-k}/P_n, & \text{if } 0 \le k \le n \\ 0, & \text{if } k > n, \end{cases}$$

then the matrix is a Nörlund one, and we write $N_n(f;x)$ for $T_{n,AB}(f;x)$.

The following are corollaries of the main results.

Corollary 3.5. *Let $\omega(t)$ satisfy (3.3) and (3.4), and let $\{p_n\} \in HBVS$. Then*

$$\|R_n(f) - f\| = O\left((p_n/P_n)H(p_n/P_n)\right).$$

Corollary 3.6. *Let $\omega(t)$ satisfy (3.3) and (3.4), and let $\{p_n\} \in RBVS$. Then*

$$\|N_n(f) - f\| = O\left((p_n/P_n)H(p_n/P_n)\right).$$

4. FURTHER GENERALIZATIONS

Let $A_1 := \left(a^{(1)}_{n,k_1}\right)$, $A_2 := \left(a^{(2)}_{k_1,k_2}\right)$, $\dots, A_\nu := \left(a^{(\nu)}_{k_{\nu-1},k_\nu}\right)$ $(k_\nu, n = 0,1,\dots)$ be ν $(\nu \in \mathbb{N})$ lower triangular infinite matrices of real numbers and let us define the $A_1A_2\cdots A_\nu$-transform of $\{S_n(f;x)\}$ by

$$\begin{aligned} T_{n,A_1A_2\cdots A_\nu}(f) &:= T_{n,A_1A_2\cdots A_\nu}(f;x) \\ &:= \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu} S_{k_\nu}(f;x) \quad (n, k_\nu = 0,1,\dots). \end{aligned}$$

Let us assume that $\left\{a^{(\nu)}_{k_{\nu-1},k_\nu}\right\}$ satisfies conditions (1.6) and (1.7). The following analogous statement with Lemma 2.4 holds:

Lemma 4.1. *If for a fixed $k_{\nu-1}$, $(k_{\nu-1} = 0,1,\dots,n)$, the sequence $\left\{a^{(\nu)}_{k_{\nu-1},k_\nu}\right\} \in RBVS$, then, uniformly in $0 < t \le \pi$,*

$$\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu}\sin\left(k_\nu+\frac{1}{2}\right)t =$$
$$= O\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{\nu-1}=0}^{k_{\nu-2}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu-1)}_{k_{\nu-2},k_{\nu-1}} A_{k_{\nu-1},\tau}\right),$$

where $A_{k_{\nu-1},m} := \sum_{r=0}^{m} a^{(\nu)}_{k_{\nu-1},r}$ and τ denotes the integer part of $\frac{\pi}{t}$.

If $\left\{a^{(\nu)}_{k_{\nu-1},k_\nu}\right\} \in HBVS$ then

$$\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu}\sin\left(k_\nu+\frac{1}{2}\right)t =$$
$$= O\left(\frac{\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{\nu-1}=0}^{k_{\nu-2}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_{\nu-1}}}{t}\right).$$

Proof. The proof of this lemma is a direct consequence of Lemma 2.3. ∎

Now we shall present some theorems, that generalize the results of section 3, in which is used $T_{n,A_1A_2\cdots A_\nu}(f)$ instead $T_{n,AB}(f)$.

Theorem 4.2. *Let* $\left(a^{(1)}_{n,k_1}\right), \left(a^{(2)}_{k_1,k_2}\right), \ldots, \left(a^{(\nu)}_{k_{\nu-1},k_\nu}\right)$ *satisfy the following conditions:*

(4.1)
$$a^{(1)}_{n,k_1} \geq 0, a^{(2)}_{k_1,k_2} \geq 0, \ldots, a^{(\nu)}_{k_{\nu-1},k_\nu} \geq 0 \text{ and } \sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu} = 1,$$

$$\sum_{k_\nu=0}^{m-1} |a^{(\nu)}_{k_{\nu-1},k_\nu} - a^{(\nu)}_{k_{\nu-1},k_\nu+1}| \leq K a^{(\nu)}_{k_{\nu-1},m}, \quad 0 \leq m \leq k_{\nu-1}, \forall k_{\nu-1}. \tag{4.2}$$

Suppose $\omega(t)$ *is such that*

$$\int_u^\pi t^{-2}\omega(t)dt = O\left(H(u)\right) \quad (u \to +0), \tag{4.3}$$

where $H(u) \geq 0$ *and*

$$\int_0^t H(u)du = O\left(tH(t)\right) \quad (t \to +0). \tag{4.4}$$

Then

$$\|T_{n,A_1A_2\cdots A_\nu}(f) - f\| = O\Bigg(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_{\nu-1}} \times H\Bigg(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu-1}\Bigg)\Bigg).$$

Theorem 4.3. *Let (4.1), (4.2) and (4.3) hold. Then*

$$\|T_{n,A_1A_2\cdots A_\nu}(f) - f\| = O\left(\omega(\pi/n)\right) + O\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu-1} \times H(\pi/n)\right).$$

If, in addition, $\omega(t)$ *satisfies (4.4) then*

$$\|T_{n,A_1A_2\cdots A_\nu}(f) - f\| = O\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu)}_{k_{\nu-1},k_\nu-1} \times H(\pi/n)\right).$$

Theorem 4.4. *Let us assume that (4.1) and*

$$\sum_{k_\nu=m}^{\infty} |a^{(\nu)}_{k_{\nu-1},k_\nu} - a^{(\nu)}_{k_{\nu-1},k_\nu+1}| \leq K a^{(\nu)}_{k_{\nu-1},m}, \quad 0 \leq m \leq k_{\nu-1}, \forall k_{\nu-1}. \tag{4.5}$$

hold. Then

$$\|T_{n,A_1A_2\cdots A_\nu}(f) - f\| = = O\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_{\nu-1}=0}^{k_{\nu-2}}\sum_{l=0}^{k_{\nu-2}-1} a^{(1)}_{n,k_1}a^{(2)}_{k_1,k_2}\cdots a^{(\nu-1)}_{k_{\nu-2},k_{\nu-1}} l^{-1}\omega(\pi/l)\sum_{r=0}^{l+1} a^{(\nu)}_{k_{\nu-1},r}\right).$$

Theorem 4.5. *Let (4.1), (4.3), (4.4) and (4.5) hold. Then*

$$\|T_{n,A_1A_2\cdots A_\nu}(f)-f\| = O\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a_{n,k_1}^{(1)}a_{k_1,k_2}^{(2)}\cdots a_{k_{\nu-1},0}^{(\nu)} \times H\left(\sum_{k_1=0}^{n}\sum_{k_2=0}^{k_1}\cdots\sum_{k_\nu=0}^{k_{\nu-1}} a_{n,k_1}^{(1)}a_{k_1,k_2}^{(2)}\cdots a_{k_{\nu-1},0}^{(\nu)}\right)\right).$$

Remark 4.1. To prove the Theorems 4.2-4.5, we follow the line of the proofs of Theorems 3.1-3.4, that is why we shall omit them.

Remark 4.2. We notice that putting $\nu = 2$ in the Theorems 4.2-4.5 we obtain the results of the section 3.

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