The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 2, Article 12, pp. 1-4, 2011

THE SQUARE NUMBER BY THE APPROXIMATION

MASAKI HISASUE

Received 15 August, 2010; accepted 19 November, 2010; published 19 April, 2011.

ASAHIKAWA FUJI GIRLS' HIGH SCHOOL, ASAHIKAWA HANASAKI-CHO 6-3899, HOKKAIDO, JAPAN.
masaki@fuji.ed.jp

ABSTRACT. In this paper, we give square numbers by using the solutions of Pell's equation.

Key words and phrases: Diophantine equations, Pell equations.

2000 *Mathematics Subject Classification.* 11D75.

ISSN (electronic): 1449-5910

1. INTRODUCTION

A square number, also called a perfect square, is a figurate number of the form n^2, where n is an non-negative integer. The square numbers are 0, 1, 4, 9, 16, 25, 36, 49, $\cdots$. The difference between any perfect square and its predecessor is given by the following identity, $n^2 - (n-1)^2 = 2n - 1$. Also, it is possible to count up square numbers by adding together the last square, the last square's root, and the current root.

Squares of even numbers are even, since $(2n)^2 = 4n^2$ and squares of odd numbers are odd, since $(2n-1)^2 = 4(n^2 - n) + 1$. It follows that square roots of even square numbers are even, and square roots of odd square numbers are odd.

Let D be a positive integer which is not a perfect square. It is well known that there exist an infinite number of integer solutions of the equation $x^2 - Dy^2 = 1$, known as Pell's equation. The first non-trivial solution of this Diophantine equation, from which all others are easily computed, can be found using, e.g., the cyclic method, known in India in the 12th century, or using the slightly less efficient but more regular English method (17th century). There are other methods to compute this so-called fundamental solution, some of which are based on a continued fraction expansion of the square root of D. For example, for $D = 5$ one can take $x = 9, y = 4$. We shall always assume that D is positive but not a square, since otherwise there are no solutions.

Pell's equation has an extraordinarily rich history, to which Weil's book [10] is the best guide; see also [3, Chap. XII].

2. MAIN RESULT

A special case of the quadratic Diophantine equation having the form $x^2 - Dy^2 = 1$ where $D > 0$ is a square free number is called a Pell equation.

Given smallest solution $(x, y) = (x_1, y_1)$, a whole family of solutions can be found by taking each side to the nth power,

$$x_n^2 - Dy_n^2 = (x_1^2 - Dy_1^2)^n = 1.$$

It is closely related quadratic field $\mathbb{Q}(\sqrt{D})$. The explicit solutions of the family of the above equation is

$$x_n = \frac{\alpha^n + \beta^n}{2}, y_n = \frac{\alpha^n - \beta^n}{2\sqrt{D}},$$

where $\alpha = x_1 + \sqrt{D}y_1, \beta = x_1 - \sqrt{D}y_1$. Since $\alpha\beta = (x_1 + \sqrt{D}y_1)(x_1 - \sqrt{D}y_1) = x_1^2 - Dy_1^2 = 1$, then we have

$$\frac{x_{2n} - 1}{2D} = \frac{1}{2D}\left(\frac{\alpha^{2n} + \beta^{2n}}{2} - 1\right) = \left(\frac{\alpha^n - \beta^n}{2\sqrt{D}}\right)^2.$$

Put $f_n = \frac{\alpha^n - \beta^n}{\sqrt{D}}$, then

$$f_1 = \frac{\alpha - \beta}{\sqrt{D}} = \frac{x_1 + \sqrt{D}y_1 - (x_1 - \sqrt{D}y_1)}{\sqrt{D}} = 2y_1 \in 2\mathbb{Z},$$

$$f_2 = \frac{\alpha^2 - \beta^2}{\sqrt{D}} = 4x_1y_1 \in 2\mathbb{Z},$$

$$f_{n+2} = \frac{\alpha^{n+2} - \beta^{n+2}}{\sqrt{D}} = \frac{(\alpha^{n+1} - \beta^{n+1})(\alpha + \beta) - \alpha\beta(\alpha^n - \beta^n)}{\sqrt{D}} = 2x_1 f_{n+1} - f_n.$$

Thus $f_n \in 2\mathbb{Z}$ for any positive integer n and $\frac{x_{2n}-1}{2D}$ is a square number.

Since

$$x_{2n}^2 - Dy_{2n}^2 = 1, y_{2n} = \sqrt{\frac{x_{2n}^2 - 1}{D}}.$$

By

$$\frac{x_{2n} - 1}{2D} = \left(\frac{f_n}{2}\right)^2$$

we have

$$x_{2n} + \sqrt{D}y_{2n} = x_{2n} + \sqrt{x_{2n}^2 - 1} = 1 + \frac{Df_n^2}{2} + \frac{1}{2}\sqrt{Df_n^2(Df_n^2 + 4)}.$$

Therefore we have

$$x_{2n} + \sqrt{D}y_{2n} > 1 + \frac{Df_n^2}{2} + \frac{1}{2}\sqrt{(Df_n^2)^2} = 1 + Df_n^2$$

and

$$x_{2n} + \sqrt{D}y_{2n} < 1 + \frac{Df_n^2}{2} + \frac{1}{2}\sqrt{(Df_n^2 + 2)^2} = 2 + Df_n^2.$$

Utilizing that $(x_2 + \sqrt{D}y_2)^n = x_{2n} + \sqrt{D}y_{2n}$, we have the following:

Theorem 2.1. *Let D be a square free positive integer, $(x, y) = (x_1, y_1), ($resp. $(x_2, y_2))$ be the smallest (resp. second smallest) solution of the equation $x^2 - Dy^2 = 1$. Let*

$$f_n = \frac{(x_1 + \sqrt{D}y_1)^n - (x_1 - \sqrt{D}y_1)^n}{\sqrt{D}},$$

Then we have

$$\frac{1}{D} + f_n^2 < \frac{(x_2 + \sqrt{D}y_2)^n}{D} < \frac{2}{D} + f_n^2.$$

By this theorem, we have the following:

Theorem 2.2. *For any positive integer k, there exist a real number g_k depending only on k such that $\lfloor \frac{g_k^n}{k} \rfloor$ are square numbers for arbitrary positive integer n, where $\lfloor \cdot \rfloor$ is a floor function.*

Proof. If k is square free, then put $g_k = x_2 + \sqrt{k}y_2$, where x_2, y_2 are the same as Theorem 2.1, if k is a square number such that $k = l^2$, where l is a non-negative number, we put $g_k = k^2$, then

$$\lfloor \frac{g_k^n}{k} \rfloor = \lfloor \frac{(l^4)^n}{l^2} \rfloor = (l^{2l-1})^2,$$

which is a square number, thus we have the theorem. ∎

References

[1] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer-Verlag, 1976

[2] E. J. BARBEAU, *Pell's Equation*, New York: Springer-Verlag, 2003.

[3] H. COHN, *Pell's Equation*, New York: Dover, 1980.

[4] J. H. CONWAY and R. K. GUY, *The Book of Numbers*, New York: Springer-Verlag, pp. 30-32, 1996

[5] G. H. HARDY and E. M. WRIGHT, *An Introduction to the Theory of Numbers*, Oxford Science Publications, 1979.

[6] R. K. GUY, *Unsolved Problems in Number Theory, 2nd ed*, New York: Springer-Verlag, 1994.

[7] M. J. JR. JACOBSON, *Solving the Pell Equation*, Cms Books in Mathematics, 2008.

[8] H. W. JR. LENSTRA, *Solving the Pell Equation, Not. Amer. Math. Soc.*, **49** (2002), pp. 182-192.

[9] O. TAUSSKY-TODD, *Sums of Squares, Amer. Math. Monthly*, **77** (1970), pp. 805-830.

[10] A. WEIL, *Number Theory, an Approach through History*, Birkhauser, Boston, 1984