The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 1, Article 9, pp. 1-8, 2010

ON THE PRODUCT OF M-MEASURES IN l-GROUPS

A. BOCCUTO, B. RIEČAN, AND A. R. SAMBUCINI

Received 4 June, 2007; accepted 18 July, 2008; published 4 March, 2010.

DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIA VANVITELLI, 1 I-06123 PERUGIA, ITALY
boccuto@dipmat.unipg.it
URL: http://www.dipmat.unipg.it/~boccuto

KATEDRA MATEMATIKY, FAKULTA PRÍRODNÝCH VIED, UNIVERZITA MATEJA BELA, TAJOVSKÉHO, 40
SK-97401 BANSKÁ BYSTRICA, SLOVAKIA
riecan@fpv.umb.sk

DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIA VANVITELLI, 1 I-06123 PERUGIA, ITALY
matears1@unipg.it
URL: http://www.unipg.it/~matears1

ABSTRACT. Some extension-type theorems and compactness properties for the product of l-group-valued M-measures are proved.

Key words and phrases: l-group, extension, weak σ-distributivity, product measure, countable compactness.

2000 *Mathematics Subject Classification.* 28B15.

ISSN (electronic): 1449-5910

This paper was supported by project SAS/CNR (Slovak Academy of Sciences/ Italian National Council of Researches) "Integration in abstract structures" 2007/09, grant VEGA 1/0539/08, grant APVV LPP-0046-06 and GNAMPA of CNR..

1. INTRODUCTION

In the study of probability theory, in many applications it is advisable to deal with set functions, which are not necessarily additive, but satisfy other properties: for example, continuity from below and from above for sequences of sets and "compatibility" with respect to the operations of finite suprema and infima. These functions are called *M-measures* (see [7, 12, 21]).

For example, in decision making, this is the case of the theory of intuitionistic fuzzy events (shortly IF-events), which are pairs $A = (\mu_A, \nu_A)$ of measurable functions $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1$. For a literature about IF-sets, see [1, 2, 5, 6, 12, 13, 17, 18, 19]. Another application is the theory of joint random variables: in this context the M-measure extension theorem plays a crucial role in the construction of joint observables. Moreover, to consider lattice-group or Riesz space-valued set functions allows to get applications in stochastic processes and in probabilities depending on the time and/or on the informations of the individual.

In this paper we continue the investigation dealt with in [4] and, starting from an extension-type existence theorem for M-measures with values in l-groups, we obtain existence results in the countably compact case for M-measures and product of M-measures.

2. PRELIMINARIES AND BASIC RESULTS

We begin with the following

Definition 2.1. An l-group (lattice ordered group) R is said to be

- **(2.1.1):** *Dedekind complete* if every nonempty subset of R, bounded from above, has supremum in R;
- **(2.1.2):** *super Dedekind complete*, if it is Dedekind complete and for any nonempty set $A \subseteq R$, bounded from above, there exists a countable subset $A^* \subseteq A$, having the same supremum as A.
- **(2.1.3):** A bounded double sequence $(a_{i,j})_{i,j}$ in R is called *regulator* or *(D)-sequence* if, for each $i \in \mathbb{N}$, $a_{i,j} \searrow 0$, that is $a_{i,j} \geq a_{i,j+1}$ for all $j \in \mathbb{N}$ and $\bigwedge_{j \in \mathbb{N}} a_{i,j} = 0$.
- **(2.1.4):** Given a sequence $(r_n)_n$ in R, we say that $(r_n)_n$ *(D)-converges* to an element $r \in R$ if there is a regulator $(a_{i,j})_{i,j}$, such that to every map $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive integer k with
$$|r_n - r| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$
for all $n \geq k$. In this case, we write $(D) \lim_n r_n = r$ or simply $\lim_n r_n = r$, since no confusion can arise.

Definition 2.2. We say that R is *weakly σ-distributive* if, for every (D)-sequence $(a_{i,j})_{i,j}$,
$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

Remark 2.3. Observe that in weakly σ-distributive l-groups (D)-convergence for sequences coincides with order convergence. An example of a super Dedekind complete weakly σ-distributive l-group is the space $L^0(Y, \Sigma, \nu)$, where (Y, Σ, ν) is a measure space with ν σ-additive and σ-finite, see [14].

From now on, let X be a set and $\mathcal{W}$ be an algebra of subsets of X.

Definition 2.4. A family $\mathcal{A}$ of subsets of X is called *monotone class* if the following properties hold:

(a): $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$,

(b): $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for every non-increasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}$.

Definition 2.5. A family $\mathcal{K}$ of subsets of X is called *countably compact class*, if for every sequence $(C_n)_n$ of elements of $\mathcal{K}$ we get $\bigcap_{i\in\mathbb{N}} C_i \neq \emptyset$ whenever $\bigcap_{i=1}^{n} C_i \neq \emptyset$ for any $n \in \mathbb{N}$.

Definition 2.6. A set function $\lambda : \mathcal{W} \to R$ is said to be *countably compact*, if there is a countably compact class $\mathcal{K}$ with the property that for any $A \in \mathcal{W}$ there corresponds a (D)-sequence $(a_{i,j})_{i,j}$ such that to any $\varphi \in \mathbb{N}^{\mathbb{N}}$ two sets $B \in \mathcal{W}$, $C \in \mathcal{K}$ can be found, with $B \subseteq C \subseteq A$ and $\lambda(A \setminus B) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$.

In the sequel we will use the following fundamental results ([8, 9, 10, 20]).

Lemma 2.7. ([20], Theorem 3.2.3) *Let $\{(a_{i,j}^{(n)})_{i,j} : n \in \mathbb{N}\}$ be any countable family of regulators. Then for each fixed element $u \in R$, $u \geq 0$, there exists a regulator $(a_{i,j})_{i,j}$ such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$,*

$$u \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Theorem 2.8. ([11], Theorem 1.6.B) *If $\mathcal{A} \subseteq \mathcal{P}(X)$ is a monotone class of sets such that*

(c): $\mathcal{W} \subseteq \mathcal{A}$,

then $\mathcal{A}$ includes the σ-algebra of subsets of X $\sigma(\mathcal{W})$ generated by $\mathcal{W}$.

Lemma 2.9. ([10], Lemma 413R) *Let $\mathcal{K}$ be a countably compact class of sets. Then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ such that $K \cup L \in \mathcal{K}^*$ and $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}^*$ whenever $K, L \in \mathcal{K}^*$ and $(K_n)_n$ is a sequence in $\mathcal{K}^*$.*

Definition 2.10. A set function $\mu : \mathcal{W} \to R$ is called *M-measure* if it satisfies the following properties:

(2.10.i): $\mu(\emptyset) = 0$;
(2.10.ii): $\mu(A \cup B) = \mu(A) \vee \mu(B) = \sup(\mu(A), \mu(B))$ for all $A, B \in \mathcal{W}$;
(2.10.iii): $\mu(A \cap B) = \mu(A) \wedge \mu(B) = \inf(\mu(A), \mu(B))$ for any A, $B \in \mathcal{W}$;
(2.10.iv): μ is *continuous* both from below and from above, that is: if $A_n \nearrow A$, (resp. $B_n \searrow B$), A_n, A $(B_n,\ B) \in \mathcal{W}$, $n \in \mathbb{N}$, then $\mu(A_n) \nearrow \mu(A)$ $(\mu(B_n) \searrow \mu(B))$.

It is known that

Theorem 2.11. ([4], Theorem 3.1) *Let R be a super Dedekind complete weakly σ-distributive l-group. For every bounded R-valued M-measure μ, defined on a ring $\mathcal{W}$, there is a unique M-measure $\overline{\mu}$ defined on the σ-ring $\sigma(\mathcal{W})$ generated by $\mathcal{W}$, extending μ.*

The line of the proof of this theorem is the following: at the first step, the M-measure μ is extended to $\mathcal{W}^+$, which is the class of all sets A of the type

$$A := \cup_{n=1}^{\infty} A_n, \quad \text{with } A_n \subset A_{n+1}, \quad A_n \in \mathcal{W} \quad \text{for all } n \in \mathbb{N},$$

by setting $\mu^+(A) = \lim_n \mu(A_n)$ (the limit exists in R and does not depend on the choice of the sequence $(A_n)_n$).

Successively, it is extended to $\mathcal{W}^*$, which is the ideal generated by $\mathcal{W}^+$, by setting $\mu^*(A) = \inf\{\mu^+(B) : B \in \mathcal{W}^+, B \supset A\}$ for every $A \in \mathcal{W}^*$. Then it is proved that $\overline{\mu} := \mu^*_{|\sigma(\mathcal{W})}$ is an M-measure, extending μ. Finally, in order to prove uniqueness, observe that, if $\nu : \sigma(\mathcal{W}) \to R$ is an M-measure which extends μ, then it coincides with $\overline{\mu}$ on a monotone family containing $\mathcal{W}$, and this concludes the assertion.

We denote by $\overline{\mu} : \sigma(\mathcal{W}) \to R$ this extension. Observe that:

Theorem 2.12. *If μ is countably compact, then $\overline{\mu}$ is countably compact too.*

Proof: Let $\mathcal{K}$ be the countably compact class related with countable compactness of μ and $\mathcal{K}^* \supseteq \mathcal{K}$ be a countably compact class associated with $\mathcal{K}$, closed with respect to finite unions and countable intersections, existing by virtue of Lemma 2.9. Set

$$\begin{aligned}(2.1)\quad \mathcal{L} \;=\; & \{A \in \sigma(\mathcal{W}) : \text{ there is a regulator } (a_{i,j}^{(A)})_{i,j} \text{ such that } \forall\, \varphi \in \mathbb{N}^{\mathbb{N}} \\ & \exists\, B_A \in \sigma(\mathcal{W}),\, C_A \in \mathcal{K}^* : B_A \subseteq C_A \subseteq A \text{ and } \overline{\mu}(A \setminus B_A) \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(A)}\}.\end{aligned}$$

It is enough to prove that $\mathcal{L}$ satisfies Theorem 2.8. First of all, the inclusion $\mathcal{W} \subseteq \mathcal{L}$ follows directly from the definition of countably compact measure. We now prove that $\mathcal{L}$ is a monotone class.

If $(A_n)_n$ is a non-decreasing sequence in $\mathcal{L}$, let $A = \bigcup_{n\in\mathbb{N}} A_n$. Since $\overline{\mu}$ is continuous from below, there is a regulator $(b_{i,j})_{i,j}$ with the property that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive integer k with $\overline{\mu}(A \setminus A_n) \le \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$ whenever $n \ge k$. By (2.1), in correspondence with A_k let $(a_{i,j}^{(k)})_{i,j}$ be the associated regulator, $B_k \in \sigma(\mathcal{W})$, $C_k \in \mathcal{K}^*$ be such that $B_k \subseteq C_k \subseteq A_k \subseteq A$ and

$$\overline{\mu}(A_k \setminus B_k) \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}^{(k)}.$$

Set $c_{i,j}^{(k)} = 2(a_{i,j}^{(k)} + b_{i,j})$, $i,j \in \mathbb{N}$. The double sequence $(c_{i,j})_{i,j}$ is a regulator. We obtain:

$$\overline{\mu}(A \setminus B_k) \;\le\; \overline{\mu}(A \setminus A_k) + \overline{\mu}(A_k \setminus B_k) \le \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}^{(k)};$$

moreover, by Lemma 2.7, there exists a (D)-sequence $(f_{i,j})_{i,j}$ such that

$$\overline{\mu}(A \setminus B_k) \le \overline{\mu}(\Omega) \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} c_{i,\varphi(i+n)}^{(n)} \right) \le \bigvee_{i=1}^{\infty} f_{i,\varphi(i)}.$$

Let now $(A_n)_n$ be a non-increasing sequence in $\mathcal{L}$, and set $A = \cap_{n\in\mathbb{N}} A_n$. In correspondence with A_n, let B_n, C_n, $(a_{i,j}^{(n)})_{i,j}$ be as in (2.1). Set $B = \cap_{n\in\mathbb{N}} B_n,\ C = \cap_{n\in\mathbb{N}} C_n$. Then there is a positive integer p such that

$$\overline{\mu}(A \setminus B) \;\le\; \overline{\mu}(A_p \setminus B_p) \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i+p)}^{(p)}$$

and, by Lemma 2.7, there exists a (D)-sequence $(g_{i,j})_{i,j}$ such that

$$\overline{\mu}(A_p \setminus B_p) \;\le\; \overline{\mu}(\Omega) \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \le \bigvee_{i=1}^{\infty} g_{i,\varphi(i)}.$$

Moreover $B \subseteq C \subseteq A$, $B \in \sigma(\mathcal{W})$ and $C \in \mathcal{K}^*$, since $\mathcal{K}^*$ is closed with respect to countable intersections. This concludes the proof. $\square$

3. EXISTENCE OF PRODUCT MEASURES

Let R be a super Dedekind complete weakly σ-distributive l-group, $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \nu)$ be two measure spaces, where $\mathcal{S}$, $\mathcal{T}$ are algebras and μ, ν are R-valued countably compact M-measures. We want to define the product measure of μ and ν. By Theorem 2.12 there exist two countably compact M-measures $\overline{\mu} : \sigma(\mathcal{S}) \to R$, $\overline{\nu} : \sigma(\mathcal{T}) \to R$, extending μ and ν respectively.

Let now $\mathcal{E}$ be the family of the *elementary sets*, of the type

$$\bigcup_{l=1}^{n} (A_l \times B_l), \tag{3.1}$$

where $n \in \mathbb{N}$, $A_l \in \sigma(\mathcal{S})$, $B_l \in \sigma(\mathcal{T})$, $l = 1, \ldots, n$, and $(A_l \times B_l) \cap (A_s \times B_s) = \emptyset$ whenever $l \neq s$. We prove the following:

Theorem 3.1. *Let μ, ν be R-valued countably compact M-measures as above. Then there is exactly one countably compact M-measure $\overline{\kappa} : \sigma(\mathcal{E}) \to R$ with*

$$\overline{\kappa}(A \times B) = \overline{\mu}(A) \wedge \overline{\nu}(B) \quad \text{for all } A \in \sigma(\mathcal{S}), B \in \sigma(\mathcal{T}). \tag{3.2}$$

Proof: Set

$$\kappa(A \times B) = \overline{\mu}(A) \wedge \overline{\nu}(B), \quad A \in \sigma(\mathcal{S}),\ B \in \sigma(\mathcal{T}).$$

Let now $E = A \times B, F = C \times D \in \sigma(\mathcal{S}) \times \sigma(\mathcal{T})$. We get:

$$\begin{aligned}
\kappa(E \cup F) &= \kappa\left((A \times B) \cup (C \times D)\right) \\
&= \kappa[((A \setminus C) \times B) \cup ((A \cap C) \times (B \cup D)) \cup ((C \setminus A) \times D)] \\
&= (\overline{\mu}(A \setminus C) \wedge \overline{\nu}(B)) \vee (\overline{\mu}(A \cap C) \wedge \overline{\nu}(B \cup D)) \vee (\overline{\mu}(C \setminus A) \wedge \overline{\nu}(D)) \\
&= (\overline{\mu}(A \setminus C) \wedge \overline{\nu}(B)) \vee (\overline{\mu}(A \cap C) \wedge \overline{\nu}(B)) \vee \\
&\vee \ (\overline{\mu}(A \cap C) \wedge \overline{\nu}(D)) \vee (\overline{\mu}(C \setminus A) \wedge \overline{\nu}(D)) \\
&= (\overline{\mu}(A) \wedge \overline{\nu}(B)) \vee (\overline{\mu}(C) \wedge \overline{\nu}(D)) = \kappa(E) \vee \kappa(F).
\end{aligned}$$

Analogously, it is possible to check that

$$\kappa(E \cap F) = \kappa(E) \wedge \kappa(F) \quad \text{for all } E, F \in \sigma(\mathcal{S}) \times \sigma(\mathcal{T}).$$

If $E, F \in \mathcal{E}$, the analogous results follow by virtue of the distributive laws. So, $\kappa : \mathcal{E} \to R$ can be defined as follows:

$$\kappa\left(\bigcup_{l=1}^{n}(A_l \times B_l)\right) = \bigvee_{l=1}^{n}(\overline{\mu}(A_l) \wedge \overline{\nu}(B_l)).$$

Now, in order to prove that κ is countably compact, we show that for each $A \in \mathcal{E}$ a (D)-sequence $(c_{i,j})_{i,j}$ can be found, with the property that:

$$\forall \varphi \in \mathbb{N}^{\mathbb{N}}, \exists B \in \mathcal{E},\, \exists C \in \mathcal{K} \quad \text{with } B \subseteq C \subseteq A \text{ and } \kappa(A \setminus B) \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{3.3}$$

As $\overline{\mu}$ and $\overline{\nu}$ are countably compact measures, there are countably compact classes $\mathcal{K}_1 \subseteq \mathcal{P}(X)$, $\mathcal{K}_2 \subseteq \mathcal{P}(Y)$, with the following property: for all $A \in \sigma(\mathcal{S})$ and $B \in \sigma(\mathcal{T})$ there is a regulator

$(\alpha_{i,j})_{i,j}$ such that $\forall\,\varphi \in \mathbb{N}^{\mathbb{N}}\ \exists\ E \in \sigma(\mathcal{S})$, $C \in \mathcal{K}_1$, $F \in \sigma(\mathcal{T})$, $D \in \mathcal{K}_2$, with $E \subseteq C \subseteq A$, $F \subseteq D \subseteq B$,

$$\sup\{\overline{\mu}(A \setminus E), \overline{\nu}(B \setminus F)\} \leq \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)}.$$

Set now

$$\mathcal{H} = \{E = C \times D :\ C \in \mathcal{K}_1,\ D \in \mathcal{K}_2\}\,.$$

It is easy to see that $\mathcal{H}$ is countably compact. By [10], Lemma 451H and Lemma 2.9 there exists a countably compact class $\mathcal{K}$ containing $\mathcal{H}$, and closed with respect to finite unions and countable intersections.

Let A be any element of $\mathcal{E}$. There exist $n \in \mathbb{N}$, $A_l \in \mathcal{S}$, $B_l \in \mathcal{T}$, $l = 1, \dots, n$ such that $A = \cup_{l=1}^{n}(A_l \times B_l)$. Since $\overline{\mu}$, $\overline{\nu}$ are countably compact M-measures, then to $l = 1, \dots, n$ there correspond two (D)-sequences $(a^{(l)}_{i,j})_{i,j}$, $(b^{(l)}_{i,j})_{i,j}$ and $H_l \in \mathcal{K}_1$, $G_l \in \mathcal{K}_2$, $E_l, F_l \in \mathcal{S}$ with $E_l \subseteq H_l \subseteq A_l$, $F_l \subseteq G_l \subseteq B_l$,

$$\overline{\mu}(A_l \setminus E_l) \leq \bigvee_{i=1}^{\infty} a^{(l)}_{i,\varphi(i+l)},$$

$$\overline{\nu}(B_l \setminus F_l) \leq \bigvee_{i=1}^{\infty} b^{(l)}_{i,\varphi(i+l)}.$$

By Lemma 2.7, there are two regulators $(a_{i,j})_{i,j}$, $(b_{i,j})_{i,j}$, with

$$[\overline{\mu}(X)] \wedge \sum_{l=1}^{\infty}\left(\bigvee_{i=1}^{\infty} a^{(l)}_{i,\varphi(i+l)}\right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)},$$

$$[\overline{\nu}(Y)] \wedge \sum_{l=1}^{\infty}\left(\bigvee_{i=1}^{\infty} b^{(l)}_{i,\varphi(i+l)}\right) \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \quad \text{for all } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Put $c_{i,j} = a_{i,j} \vee b_{i,j}$, $i, j \in \mathbb{N}$, and

$$C = \bigcup_{l=1}^{n}(H_l \times G_l), \qquad B = \bigcup_{l=1}^{n}(E_l \times F_l):$$

we get $B \subseteq C \subseteq A$, $C \in \mathcal{K}$, and

$$\begin{aligned}
\kappa(A \setminus B) &= \kappa\left(\left[\bigcup_{l=1}^{n}(A_l \times B_l)\right] \setminus \left[\bigcup_{s=1}^{n}(E_s \times F_s)\right]\right)\\
&= \kappa\left(\left[\bigcup_{l=1}^{n}((A_l \setminus E_l) \times B_l)\right] \cup \left[\bigcup_{s=1}^{n}(A_s \times (B_s \setminus F_s))\right]\right)\\
&= \left[\bigvee_{l=1}^{n}(\overline{\mu}(A_l \setminus E_l) \wedge \overline{\nu}(B_l))\right] \vee \left[\bigvee_{s=1}^{n}(\overline{\mu}(A_s) \wedge \overline{\nu}(B_s \setminus F_s))\right]\\
&\leq \left[\bigvee_{l=1}^{n}\overline{\mu}(A_l \setminus E_l)\right] \vee \left[\bigvee_{s=1}^{n}\overline{\nu}(B_s \setminus F_s)\right] \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.
\end{aligned}$$

Countable compactness of κ follows.

We now prove **(2.10.iv)**, namely that $\kappa(A_n) \searrow 0$ (resp. $\kappa(B_n) \nearrow \kappa(B)$) whenever $(A_n)_n$ is a non-increasing sequence in $\mathcal{E}$, with $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$ (resp. $B_n \in \mathcal{E}$, $n \in \mathbb{N}$, $B_n \nearrow B$, $B \in \mathcal{E}$).

Pick now arbitrarily any sequence $(A_n)_n$ in $\mathcal{E}$, with $A_n \searrow \emptyset$. Since κ is countably compact, then in correspondence with each positive integer n a (D)-sequence $(d^{(n)}_{i,j})_{i,j}$ and two elements $B_n \in \mathcal{E}$, $C_n \in \mathcal{K}$ can be found, with $B_n \subseteq C_n \subseteq A_n$ and

$$\kappa(A_n \setminus B_n) \leq \bigvee_{i=1}^{\infty} d^{(n)}_{i,\varphi(i+n)}.$$

Again by Lemma 2.7, there exists a (D)-sequence $(d_{i,j})_{i,j}$ with the property that

$$[\kappa(X \times Y)] \wedge \sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} d^{(n)}_{i,\varphi(i+n)} \right) \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)}.$$

Set now $D_n = \bigcap_{l=1}^{n} C_l$. We get

$$\bigcap_{n\in\mathbb{N}} D_n \subseteq \bigcap_{n\in\mathbb{N}} A_n = \emptyset.$$

Since $\mathcal{K}$ is a countably compact class, a positive integer m can be found, with

$$\bigcap_{l=1}^{m} B_l \subseteq D_m = \bigcap_{l=1}^{m} C_l = \emptyset.$$

For each $n \geq m$ we obtain

$$\begin{aligned} \kappa(A_n) &\leq \kappa(A_m) = \kappa\left(A_m \setminus \left[\bigcap_{l=1}^{m} B_l \right] \right) = \kappa\left(\bigcup_{l=1}^{m} (A_m \setminus B_l) \right) \\ &\leq \kappa\left(\bigcup_{l=1}^{m} (A_l \setminus B_l) \right) = \bigvee_{l=1}^{m} \kappa(A_l \setminus B_l) \leq \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \end{aligned}$$

and then $\lim_n \kappa(A_n) = 0$.

Let now $B_n \in \mathcal{E}$ ($n \in \mathbb{N}$), $B_n \nearrow B$, $B \in \mathcal{E}$. Then $B \setminus B_n \searrow \emptyset$, and hence

$$\begin{aligned} \kappa(B) &= \kappa((B \setminus B_n) \cup B_n) = \kappa(B \setminus B_n) \vee \kappa(B_n) \\ &\leq \kappa(B \setminus B_n) \vee \left(\bigvee_{i=1}^{\infty} \kappa(B_i) \right). \end{aligned}$$

Thus we get:

$$\kappa(B) \leq \lim_n \kappa(B \setminus B_n) \vee \left(\bigvee_{i=1}^{\infty} \kappa(B_i) \right) = \bigvee_{i=1}^{\infty} \kappa(B_i) \leq \kappa(B),$$

and hence $\kappa(B) = \lim_i \kappa(B_i)$. Furthermore, if $C_n \searrow C$, then $C_n \setminus C \searrow \emptyset$,

$$\kappa(C_n) = \kappa((C_n \setminus C) \cup C) = \kappa(C_n \setminus C) \vee \kappa(C),$$

and

$$\bigwedge_{n=1}^{\infty} \kappa(C_n) = \left(\bigwedge_{n=1}^{\infty} \kappa(C_n \setminus C) \right) \vee \kappa(C) = 0 \vee \kappa(C) = \kappa(C).$$

Thus we proved that κ is an R-valued countably compact M-measure, defined on $\mathcal{E}$. By Theorem 2.12 there is a (unique) countably compact M-measure $\overline{\kappa}$, defined on $\sigma(\mathcal{E})$ and extending κ. $\square$

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