The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 1, Article 6, pp. 1-7, 2010

NEIGHBORHOODS OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER WITH NEGATIVE COEFFICIENTS

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Received 30 July, 2007; accepted 16 July, 2008; published 4 March, 2010.

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ABSTRACT. The main object of this paper is to prove several inclusion relations associated with the (n, δ) neighborhoods of various subclasses of convex functions of complex order by making use of the known concept of neighborhoods of analytic functions.

Key words and phrases: Analytic functions, Starlike functions, Convex functions, (n, δ)- neighborhood, Inclusion relations.

2000 *Mathematics Subject Classification.* Primary 30C45.

ISSN (electronic): 1449-5910

The authors are thankful to the referee for their comments.

1. INTRODUCTION

Let $\mathcal{A}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n}^{\infty} a_{k+1} z^{k+1}, \quad (a_{k+1} \geq 0; \;\; n \in \mathbb{N} := \{1, 2, 3, \dots\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\Delta = \{z : z \in \mathbb{C}, \;\; |z| < 1\}.$$

Following the earlier investigations by Goodman [5] and Ruscheweyh [8],
for any $f(z) \in \mathcal{A}(n)$ and $\delta \geq 0$, we define the (n, δ)- neighborhood of $f(z)$ by

$$N_{n,\delta}(f) := \left\{ g \in \mathcal{A}(n) : g(z) := z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \text{ and } \sum_{k=n}^{\infty} (k+1)|a_{k+1} - b_{k+1}| \leq \delta \right\}. \tag{1.2}$$

In particular, for the identity function

$$e(z) = z,$$

we immediately have

$$N_{n,\delta}(e) := \left\{ g \in \mathcal{A}(n) : g(z) := z - \sum_{k=n}^{\infty} b_{k+1} z^{k+1} \text{ and } \sum_{k=n}^{\infty} (k+1)|b_{k+1}| \leq \delta \right\}. \tag{1.3}$$

First of all, we say that a function $f(z) \in \mathcal{A}(n)$ is said to be *starlike* of complex order γ $(\gamma \in \mathbb{C} - \{0\})$, that is, $f \in \mathcal{S}_n^*(\gamma)$, if it satisfies the inequality

$$\Re \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0 \;\; (z \in \Delta; \;\; \gamma \in \mathbb{C} - \{0\}). \tag{1.4}$$

Furthermore, a function $f(z) \in \mathcal{A}(n)$ is said to be *convex* of complex order γ $(\gamma \in \mathbb{C} - \{0\})$, that is $f(z) \in \mathcal{C}_n(\gamma)$, if it satisfies the inequality

$$\Re \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf''(z)}{f'(z)} \right) \right\} > 0 \;\; (z \in \Delta; \;\; \gamma \in \mathbb{C} - \{0\}). \tag{1.5}$$

The classes $\mathcal{S}_n^*(\gamma)$ and $\mathcal{C}_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered by Nasr and Aouf [7] and Wiatrowski [10], respectively (Refer also [4]).

Let $\mathcal{S}_n(\gamma, \lambda, \beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the following inequality

$$\left| \frac{1}{\gamma} \left[\frac{\lambda z^3 f'''(z) + (1 + 2\lambda) z^2 f''(z) + zf'(z)}{\lambda z^2 f''(z) + zf'(z)} - 1 \right] \right| < \beta$$

$$(z \in \Delta; \;\; \gamma \in \mathbb{C} - \{0\}; \;\; 0 \leq \lambda \leq 1; \;\; 0 < \beta \leq 1).$$

Let $\mathcal{R}_n(\gamma, \lambda, \beta)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the following inequality

$$\left| \frac{1}{\gamma} \left[\lambda z^2 f'''(z) + (1 + 2\lambda) z f''(z) + f'(z) - 1 \right] \right| < \beta$$

$$(z \in \Delta; \;\; \gamma \in \mathbb{C} - \{0\}; \;\; 0 \leq \lambda \leq 1; \;\; 0 < \beta \leq 1).$$

The class $\mathcal{S}_n(\gamma, \lambda, \beta)$ was studied by [6].

Let $\mathcal{A}$ be class of functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$. For $f(z)$ belong to $\mathcal{A}$, Sălăgean [9] has introduced the following operator called the Sălăgean operator:

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= z f'(z), \\ &\vdots \\ D^n f(z) &= D(D^{n-1} f(z)) \ \ (n \in \mathbb{N} := \{1, 2, 3, \dots\}). \end{aligned}$$

Note that

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \ \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

Now we can write the following equalities for the functions $f(z)$ belong to the class $\mathcal{A}(n)$.

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= z f'(z) = z - \sum_{k=n}^{\infty} (k+1) a_{k+1} z^{k+1}, \\ D^2 f(z) &= = D(Df(z)) = z - \sum_{k=n}^{\infty} (k+1)^2 a_{k+1} z^{k+1}, \\ &\vdots \\ D^{\Omega} f(z) &= D(D^{\Omega-1} f(z)) = z - \sum_{k=n}^{\infty} (k+1)^{\Omega} a_{k+1} z^{k+1} \ \ (\Omega \in \mathbb{N} \cup \{0\}). \end{aligned}$$

Finally, in the terms of the Sălăgean operator, let $\mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$ denote the subclass of $\mathcal{A}(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$\left| \frac{1}{\gamma} \left[\frac{\lambda D^{\Omega+3} f(z) + (1-\lambda) D^{\Omega+2} f(z)}{\lambda D^{\Omega+2} f(z) + (1-\lambda) D^{\Omega+1} f(z)} - 1 \right] \right| < \beta \tag{1.6}$$

$$(z \in \Delta; \ \ \gamma \in \mathbb{C} - \{0\}; \ \ 0 \le \lambda \le 1; \ \ 0 < \beta \le 1; \ \ \Omega \in \mathbb{N} \cup \{0\}).$$

Also, let $\mathcal{R}_n(\gamma, \lambda, \beta, \Omega)$ denote the subclass of $\mathcal{A}(n)$ consisting of $f(z)$ which satisfy the inequality

$$\left| \frac{1}{\gamma} \left[\lambda (D^{\Omega+2} f(z))' + (1-\lambda)(D^{\Omega+1} f(z))' - 1 \right] \right| < \beta, \tag{1.7}$$

$$(z \in \Delta; \ \ \gamma \in \mathbb{C} - \{0\}; \ \ 0 \le \lambda \le 1; \ \ 0 < \beta \le 1; \ \ \Omega \in \mathbb{N} \cup \{0\}).$$

Clearly, in these cases of the class $\mathcal{S}_n(\gamma, 0, 1, 0)$ we have the following relationship:

$$\mathcal{S}_n(\gamma, 0, 1, 0) \subset \mathcal{C}_n(\gamma), \quad (n \in \mathbb{N}, \gamma \in \mathbb{C} - \{0\}).$$

The main object of the present paper is to investigate the (n, δ)-neighborhoods of the following subclasses $\mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$ and $\mathcal{R}_n(\gamma, \lambda, \beta, \Omega)$ of $\mathcal{A}(n)$.See also the earlier works [1, 2, 3].

2. INCLUSION RELATIONS INVOLVING THE (n, δ)-NEIGHBORHOOD $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving (n, δ)- neighborhood, we shall require the following lemmas.

Lemma 2.1. *Let the function $f(z) \in \mathcal{A}(n)$ be defined by (1.1), then $f(z)$ is in the class $\mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$ if and only if*

$$\sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)(k+\beta|\gamma|)a_{k+1} \leq \beta|\gamma|. \tag{2.1}$$

Proof. We suppose that $f(z) \in \mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$. Then by appealing the condition (1.6) we get,

$$\Re\left\{\frac{\lambda D^{\Omega+3}f(z) + (1-\lambda)D^{\Omega+2}f(z)}{\lambda D^{\Omega+2}f(z) + (1-\lambda)D^{\Omega+1}f(z)} - 1\right\} > -\beta|\gamma| \tag{2.2}$$

That is,

$$\Re\left\{\frac{-\sum_{k=n}^{\infty}(k+1)^{\Omega+1}k(\lambda k+1)a_{k+1}z^{k+1}}{z - \sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)a_{k+1}z^{k+1}}\right\} > -\beta|\gamma|, \quad (z \in \Delta) \tag{2.3}$$

Now choose the values of z on the real axis and let $z \to 1^-$ through real values. Then inequality (2.3) immediately yields the desired condition (2.1).

Conversely, by applying the hypothesis (2.1) and letting $|z| = 1$, we find that

$$\begin{aligned}
&\left|\frac{\lambda D^{\Omega+3}f(z) + (1-\lambda)D^{\Omega+2}f(z)}{\lambda D^{\Omega+2}f(z) + (1-\lambda)D^{\Omega+1}f(z)} - 1\right| \\
&= \left|\frac{\sum_{k=n}^{\infty}(k+1)^{\Omega+1}k(\lambda k+1)a_{k+1}z^{k+1}}{z - \sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)a_{k+1}z^{k+1}}\right| \\
&\leq \frac{\beta|\gamma|\left\{1 - \sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)a_{k+1}\right\}}{1 - \sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)a_{k+1}} \\
&= \beta|\gamma|.
\end{aligned}$$

Hence, by maximum modulus theorem, we have $f(z) \in \mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$, which evidently completes the proof of Lemma 2.1.

Similarly, we can prove the following result.

Lemma 2.2. *Let the function $f(z) \in \mathcal{A}(n)$ be defined by (1.1), then $f(z)$ is in the class $\mathcal{R}_n(\gamma, \lambda, \beta, \Omega)$ if and only if*

$$\sum_{k=n}^{\infty}(k+1)^{\Omega+2}(\lambda k+1)a_{k+1} \leq \beta|\gamma|. \tag{2.4}$$

Theorem 2.3. *Let*

$$\delta = \frac{\beta|\gamma|}{(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)} \quad (|\gamma| < 1), \tag{2.5}$$

then $\mathcal{S}_n(\gamma, \lambda, \beta, \Omega) \subset N_{n,\delta}(e)$.

Proof. For $f(z) \in \mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$, Lemma 2.1 immediately yields

$$(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|)\sum_{k=n}^{\infty}a_{k+1} \leq \beta|\gamma|$$

so that

$$\sum_{k=n}^{\infty} a_{k+1} \leq \frac{\beta|\gamma|}{(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|)}. \tag{2.6}$$

On the other hand, we also find from (2.1) and (2.6) that

$$\begin{aligned} \sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)(k+\beta|\gamma|)a_{k+1} &\leq \beta|\gamma| \\ \Rightarrow \sum_{k=n}^{\infty}(k+1)^{\Omega+1}(\lambda k+1)(k+1-1+\beta|\gamma|)a_{k+1} &\leq \beta|\gamma| \end{aligned}$$

$$\begin{aligned} &\Rightarrow (n+1)^{\Omega+1}(\lambda n+1)\sum_{k=n}^{\infty}(k+1)a_{k+1} \\ &\leq \beta|\gamma| + (1-\beta|\gamma|)(n+1)^{\Omega+1}(\lambda n+1)\sum_{k=n}^{\infty} a_{k+1} \\ &\leq \beta|\gamma| + (1-\beta|\gamma|)(n+1)^{\Omega+1}(\lambda n+1)\frac{\beta|\gamma|}{(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|)} \\ &= \beta|\gamma| + (1-\beta|\gamma|)\frac{\beta|\gamma|}{n+\beta|\gamma|} = \frac{(n+1)\beta|\gamma|}{n+\beta|\gamma|}. \end{aligned}$$

Thus

$$\sum_{k=n}^{\infty}(k+1)a_{k+1} \leq \frac{\beta|\gamma|}{(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)} = \delta,$$

which, in view of (1.3) proves Theorem 2.3.

Similarly, by applying Lemma 2.2 instead of Lemma 2.1. We can prove the following.

Theorem 2.4. *Let*

$$\delta = \frac{\beta|\gamma|}{(n+1)^{\Omega+1}(\lambda n+1)}$$

then $\mathcal{R}_n(\gamma,\lambda,\beta,\Omega) \subset N_{n,\delta}(e)$.

3. NEIGHBORHOOD PROPERTIES FOR THE FUNCTION CLASSES $\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ AND $\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$

In this section, we determine the neighborhood for each of the classes

$$\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega) \quad \text{and} \quad \mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega),$$

which we define as follows. A function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{S}_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ if there exists a function $g(z) \in \mathcal{S}_n(\gamma,\lambda,\beta,\Omega)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\alpha, \quad (z \in \Delta, 0 \leq \alpha < 1). \tag{3.1}$$

Analogously, a function $f(z) \in \mathcal{A}(n)$ is said to be in the class $\mathcal{R}_n^{(\alpha)}(\gamma,\lambda,\beta,\Omega)$ if there exists a function $g(z) \in \mathcal{R}_n(\gamma,\lambda,\beta,\Omega)$ such that inequality (3.1) holds true.

Theorem 3.1. *If* $g(z) \in \mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$ *and*

$$\alpha = 1 - \frac{\delta(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)}{(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|) - \beta|\gamma|},$$

then

$$N_{n,\delta}(g) \subset \mathcal{S}_n^{(\alpha)}(\gamma, \lambda, \beta, \Omega).$$

Proof. Assuming that $f(z) \in N_{n,\delta}(g)$. We find from the definition (1.2) that

$$\sum_{k=n}^{\infty}(k+1)|a_{k+1} - b_{k+1}| \leq \delta,$$

which readily implies the coefficient inequality

$$\sum_{k=n}^{\infty} |a_{k+1} - b_{k+1}| \leq \frac{\delta}{n+1} \quad (n \in \mathbb{N}). \tag{3.2}$$

Next, since $g(z) \in \mathcal{S}_n(\gamma, \lambda, \beta, \Omega)$, we have from equation (2.6)

$$\sum_{k=n}^{\infty} b_{k+1} \leq \frac{\beta|\gamma|}{(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|)}, \tag{3.3}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{k=n}^{\infty} |a_{k+1} - b_{k+1}|}{1 - \sum_{k=n}^{\infty} b_{k+1}} \\ &\leq \frac{\delta}{n+1} \cdot \frac{1}{1 - \frac{\beta|\gamma|}{(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|)}} \\ &= \frac{\delta(n+1)^{\Omega}(\lambda n+1)(n+\beta|\gamma|)}{(n+1)^{\Omega+1}(\lambda n+1)(n+\beta|\gamma|) - \beta|\gamma|} = 1 - \alpha, \end{aligned}$$

which completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is similar to that of Theorem 3.1, and hence the details are omitted.

Theorem 3.2. *If* $g(z) \in \mathcal{R}_n(\gamma, \lambda, \beta, \Omega)$ *and*

$$\alpha = 1 - \frac{\delta(n+1)^{\Omega+1}(\lambda n+1)}{(n+1)^{\Omega+2}(\lambda n+1) - \beta|\gamma|},$$

then

$$N_{n,\delta}(g) \subset \mathcal{R}_n^{(\alpha)}(\gamma, \lambda, \beta, \Omega).$$

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