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ON SOME STRONGLY NONLINEAR ELLIPTIC PROBLEMS IN L^1-DATA WITH A NONLINEARITY HAVING A CONSTANT SIGN IN ORLICZ SPACES VIA PENALIZATION METHODS

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ABSTRACT. This paper is concerned with the existence result of the unilateral problem associated to the equations of the type

$$Au + g(x, u, \nabla u) = f,$$

in Orlicz spaces, without assuming the sign condition in the nonlinearity g. The source term f belongs to $L^1(\Omega)$.

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1. INTRODUCTION

Throughout this paper Ω is a bounded open subset of $\mathbb{R}^N$, p is a real number such that $1 < p < \infty$ and p' is a conjugate, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.
Consider the following strongly nonlinear Dirichlet problem,

$$\left\{ \begin{array}{l} Au + g(x, u, \nabla u) = f \quad \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega, \end{array} \right. \tag{1.1}$$

where $Au = -\text{div}(a(x, u, \nabla u))$ is a Leray-Lions operator with a Carathéodory function $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which satisfying the classical Leray-Lions conditions.
And g is a nonlinear lower order terms having natural growth with respect to $|\nabla u|$, no growth with respect to u and satisfying a sign conditions, i.e.,

$$g(x, s, \xi)s \geq 0. \tag{1.2}$$

We begin by some remarks and well know about the solvability of the problem (1.1) in the L^p-case.
It will turn out that for In the variational case where (i.e.,$f \in W^{-1,p'}(\Omega)$) the reader is referred to [5]and [10] where the different approaches are applied.
If $f \in L^1(\Omega)$, existence result of (1.1) have been proved in [9], but under some additionally coercivity condition on the nonlinear term, that is,

$$|g(x, s, \xi)| \geq \gamma|\xi|^p \text{ for all } |s| \text{ some } \mu > 0. \tag{1.3}$$

It should be noted that hypothesis (1.3) is more technical and allows to solve (1.1) in $W_0^{1,p}(\Omega)$.
Unfortunately, where (1.3) is violated, the solvability of (1.1) with L^1-data is not possible in $W_0^{1,p}(\Omega)$., but the solution of (1.1) is proved in $W_0^{1,q}(\Omega)$ with $1 < q < \bar{q} = \frac{N(p-1)}{N-1}$.
Note that in all the last works, the coefficients of A and the nonlinearity have supposed to satisfy the growth conditions and coercivity of polynômial type.
Now, when trying to relax this restrictions on a and g, we are let to replace $W_0^{1,p}(\Omega)$ by a general setting of Orlicz-Sobolev spaces $W^1L_M(\Omega)$ built from an N-function M instead of $|t|^p$, where the N-function M which defines L_M is related to the actuel growth and coercivity of a and g.
In this L_M-case, we list firstly the work [13] of Gossez, where the second member f lies in $W^{-1}E_{\bar{M}}(\Omega)$ and the nonlinear term g depends only on x and u.
When $g \equiv g(x, u, \nabla u)$, the last work of Gossez is generalized in [6], but under some restriction on the used N-function M (that is M satisfies the so-called Δ_2-condition).
The case where $f \in L^1(\Omega)$, is studied in [7] but g have supposed satisfying in addition the following L_M-coercivity,

$$|g(x, s, \xi)| \geq \beta M(|\xi|). \tag{1.4}$$

The result of [7] is recovered by the work [8] where no coercivity condition as (1.4) is assumed on g but the result is restricted to N-function M satisfying the Δ_2-condition.
Concerning the obstacle problems associated to (1.1) in the Orlicz - Sobolev Spaces, we refer for this topics to [2] and [3].
It will be interesting to note that the hypothesis of a sign condition is assumed in the all previous works and it plays a crucial role for to obtain a priori estimates and existence of solutions.
Our principal goal in the present work is to obtain a solution of (1.1) with $f \in L^1(\Omega)$ in the general settings of Orlicz-Sobolev Spaces. This is done with a nonlinearity g, not satisfying nor sign condition and nor L_M-coercivity and without any restriction (as Δ_2-condition) on the N-function M .
More precisely, the existence of nonbounded solution to some nonlinear elliptic equations for

unilateral problems is investigated. No growth and no sign condition are imposed on the function $g(x, s, \xi)$ with respect to the variable s. Furthermore, the function g is assumed to garde a constant sign.
It's well known that the classical techniques used for to study the problem (1.1) are based on the following approximate problems,

$$(P_\epsilon) \begin{cases} -\text{div}(a(x, u_\epsilon, \nabla u_\epsilon)) + g_\epsilon(x, u_\epsilon, \nabla u_\epsilon) = f_\epsilon \text{ in } \Omega \\ u_\epsilon \equiv 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $g_\epsilon(x, s, \xi) = \frac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|}$ and where f_ϵ is a sequence of regular functions.
Nevertheless, this approximation can not allow to obtain the a priori estimates in our case, this is due to the fact that $u_\epsilon g(x, u_\epsilon, \nabla u_\epsilon)$ has no sign.
To overcome this difficulty, one has introduce a doubling approximation, that is we penalize the problem (P_ϵ) by,

$$(P_\epsilon^\sigma) \begin{cases} -\text{div}(a(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)) + g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma) - \frac{1}{\epsilon^2} m(T_{\frac{1}{\epsilon}}(u_\epsilon^{\sigma -})) = f_\epsilon \text{ in } \Omega \\ u_\epsilon^\sigma \equiv 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where $g_\epsilon^\sigma(x, s, \xi) = \delta_\sigma(s) g_\epsilon(x, s, \xi)$ and where $\delta_\sigma(t)$ is some increasing Lipschitz-function (see sections 4 and 5).
Our simplest model problem is the following:

$$\begin{cases} -\Delta_M u + |u|^r M(|\nabla u|) = f \text{ in } \Omega \\ u \equiv 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $r > 0$ and $\Delta_M u$ is the so-called M-Laplacian operator defined as,

$$\Delta_M u = -\text{div}(m(|\nabla u|) \frac{\nabla u}{|\nabla u|}),$$

where m is the derivatives function of the N-function M.
Note that, when we take in (1.5), $M(t) = |t|^p$ $(p > 1)$ we obtain the following L^p-problem,

$$\begin{cases} -\text{div}(|\nabla u|^{p-2} |\nabla u|) + |u|^r |\nabla u|^p = f \text{ in } \Omega \\ u \equiv 0 \qquad \text{on } \partial\Omega, \end{cases}$$

generated by the classical p-Laplacian operator.

2. PRELIMINARIES

2-1 Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e., M is continous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.
Equivalently, M admits the representation: $M(t) = \int_0^t m(s)\, ds$ where $m : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $m(0) = 0$, $m(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M} = \int_0^t \bar{m}(s)\, ds$, where $\bar{m} : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{m}(t) = \sup\{s : a(s) \leq t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k

$$M(2t) \leq kM(t) \quad \forall t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq$ some $t_0 > 0$ then M is said to satisfy the Δ_2-condition near infinity. We will extend these N-functions into even functions on all $\mathbb{R}$.
Moreover, we have the following Young's inequality,

$$\forall\, s, t \geq 0, \;\; st \leq M(t) + \overline{M}(s).$$

Let P and Q be two N-functions. $P << Q$ means that P grows essentially less rapidly than Q, i.e., for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$.
This is the case if and only if $\lim\limits_{t\to\infty} \dfrac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

2-2 Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$ is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))\,dx < +\infty (\text{ resp. } \int_\Omega M(\frac{u(x)}{\lambda})\,dx < +\infty \text{ for some } \lambda > 0).$$

$L_M(\Omega)$ is a Banach space under the norm,

$$\|u\|_{M,\Omega} = \inf\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})\,dx \leq 1\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.
The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.
The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\displaystyle\int_\Omega uv\,dx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|.\|_{\overline{M},\Omega}$.

2-3 We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a banach space under the norm,

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_{\overline{M}})$ and $\sigma(\prod L_M, \prod L_{\overline{M}})$.
The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\prod L_M, \prod E_{\overline{M}})$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

2-4 Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm. (For more details see [1]).
We recall some lemmas introduced in [6] which will be used later.

Lemma 2.1. *(cf.* [6]*) Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let M be an N-function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial}{\partial x_i}u \text{ a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 \quad \text{ a.e. in } \{x \in \Omega : u(x) \in D\} \end{cases}$$

Lemma 2.2. *(cf.* [6]*) Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continous with respect to the weak* topology $\sigma(\prod L_M, \prod E_{\overline{M}})$.*

We give now the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [6]).

Lemma 2.3. *(cf.* [6]*) Let Ω be an open subset of $\mathbb{R}^N$ with finite measure.*
Let M, P and Q be N-functions such that $Q << P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$:

$$|f(x, s)| \leq c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$.
Then the Nemytskii operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is strongly continuous from $\mathcal{P}(E_M(\Omega), \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

We define $\mathcal{T}_0^{1,M}(\Omega)$ to be the set of measurable function $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1 L_M(\Omega)$, where $T_k(s) = \max(-k, \min(k, s))$ for $s \in \mathbb{R}$ and $k \geq 0$. We gives the following lemma which is a generalization of Lemma 2.1 [4] in Orlicz spaces. The proof of this lemma is slightly modification of the preceding.

Lemma 2.4. *For every $u \in \mathcal{T}_0^{1,M}(\Omega)$, there exists a unique measurable function $v : \Omega \longrightarrow \mathbb{R}^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}}, \quad \textit{almost everywhere in } \Omega \textit{ for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$.

Lemma 2.5. *Let $\lambda \in \mathbb{R}$ and let u and v be two measurable functions defined on Ω which are finite almost everywhere, and which are such that $T_k(u)$, $T_k(v)$ and $T_k(u + \lambda v)$ belong to $W_0^1 L_M(\Omega)$ for every $k > 0$ then*

$$\nabla(u + \lambda v) = \nabla u + \lambda \nabla v \textit{ a.e. in } \Omega$$

where ∇u, ∇v and $\nabla(u + \lambda v)$ are the gradients of u, v and $u + \lambda v$ introduced in Lemma 2.4.

The proof of this lemma is similar to the proof of Lemma 2.12 [11] in the L^p case.

3. BASIC ASSUMPTIONS AND ONE FUNDAMENTAL LEMMA

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the segment property.
We now state our conditions on the differential operator,

$$Au = -\mathrm{div}(a(x, u, \nabla u)). \tag{3.1}$$

(A_1) $a(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function.
(A_2) There exist tow N-functions M and P with $P << M$, a function $c(x)$ in $E_{\overline{M}}(\Omega)$, constants k_1, k_2, k_3, k_4 such that, for a.e. x in Ω and for all $s, \zeta \in \mathbb{R}$,

$$|a(x, s, \zeta)| \leq c(x) + k_1 \overline{P}^{-1} M(k_2|s|) + k_3 \overline{M}^{-1} M(k_4|\zeta|).$$

(A_3) $[a(x, s, \zeta) - a(x, s, \zeta')](\zeta - \zeta') > 0$ for a.e. x in Ω, all s in $\mathbb{R}$ and all ζ' in $\mathbb{R}^N$, with $\zeta \neq \zeta'$.
(A_4) There exists a strictly positive constant α such that,

$$a(x, s, \zeta)\zeta \geq \alpha M(|\zeta|),$$

for a.e. x in Ω, all $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$.

Furthermore let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function having a constant sign such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ and all $\zeta \in \mathbb{R}^N$,
(G_1) $|g(x, s, \zeta)| \leq b(|s|)\,(h(x) + M(|\zeta|))$;

(G_2) $g(x, 0, \zeta) = 0$;
where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function, h is a given non-negative function in $L^1(\Omega)$.
Consider now the following Dirichlet problem:

$$\begin{cases} A(u) + g(x, u, \nabla u) = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.2}$$

and define $\tau_0^{1,M}(\Omega)$ as a set of measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W_0^1 L_M(\Omega)$, where $T_k(s) = \max(-k, \min(k, s))$ for $s \in \mathbb{R}$ and $k \geq 0$.

Lemma 3.1. *Let $(f_n)_n, f \in L^1(\Omega)$ such that,*

1) $f_n \geq 0$ *a.e. in* Ω*;*
2) $f_n \to f$ *a.e. in* Ω*;*
3) $\displaystyle\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$. *Then $f_n \to f$ strongly in $L^1(\Omega)$.*

Lemma 3.2. *Assume that $(A_1) - (A_4)$ are satisfied, and let (z_n) be a sequence in $W_0^1 L_M(\Omega)$ such that,*

a) $z_n \rightharpoonup z$ *in* $W_0^1 L_M(\Omega)$ *for* $\sigma(\Pi L_M(\Omega), \Pi E_{\overline{M}}(\Omega))$*;*
b) $(a(x, z_n, \nabla z_n))_n$ *is bounded in* $(L_{\overline{M}}(\Omega))^N$*;*
c) $\displaystyle\int_\Omega [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx \to 0$ *as n and $s \to +\infty$*
(where χ_s is the characteristic function of $\Omega_s = \{x \in \Omega, |\nabla z| \leq s\}$).

Then,

$$M(|\nabla z_n|) \to M(|\nabla z|) \text{ in } L^1(\Omega).$$

Proof. Fix $r > 0$ and let $s > r$ we have,

$$\begin{aligned} 0 &\leq \int_{\Omega_r} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)][\nabla z_n - \nabla z]\,dx \\ &\leq \int_{\Omega_s} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)][\nabla z_n - \nabla z]\,dx \\ &= \int_{\Omega_s} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx \\ &\leq \int_\Omega [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z\chi_s)][\nabla z_n - \nabla z\chi_s]\,dx. \end{aligned} \tag{3.3}$$

Which with the condition $c)$ imply that,

$$\lim_{n\to\infty} \int_{\Omega_r} [a(x, z_n, \nabla z_n) - a(x, z_n, \nabla z)][\nabla z_n - \nabla z]\,dx = 0. \tag{3.4}$$

So, following the same argument as in [12] we claim that,

$$\nabla z_n \to \nabla z \quad a.e. \text{ in } \Omega. \tag{3.5}$$

On the other hand, we have

$$\begin{aligned}\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n\,dx = &\int_\Omega [a(x,z_n,\nabla z_n)-a(x,z_n,\nabla z\chi_s)] \\ &\times [\nabla z_n - \nabla z\chi_s]\,dx \\ &+\int_\Omega a(x,z_n,\nabla z\chi_s)(\nabla z_n-\nabla z\chi_s)\,dx \\ &+\int_\Omega a(x,z_n,\nabla z_n)\nabla z\chi_s\,dx.\end{aligned} \tag{3.6}$$

Since $(a(x,z_n,\nabla z_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$, and by using (3.5), we obtain

$$a(x,z_n,\nabla z_n) \rightharpoonup a(x,z,\nabla z) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}},\Pi E_M), \tag{3.7}$$

which implies that,

$$\int_\Omega a(x,z_n,\nabla z_n)\nabla z\chi_s\,dx \to \int_\Omega a(x,z,\nabla z)\nabla z\chi_s\,dx \text{ as } n\to\infty. \tag{3.8}$$

Letting also $s\to\infty$, we obtain

$$\int_\Omega a(x,z,\nabla z)\nabla z\chi_s\,dx \to \int_\Omega a(x,z,\nabla z)\nabla z\,dx. \tag{3.9}$$

On the other hand, it is easy to see that the second term of the right hand side of (3.6) tends to 0 as $n\to\infty$ and $s\to\infty$.
Consequently, from c), (3.8) and (3.9) we have,

$$\lim_{n\to\infty}\int_\Omega a(x,z_n,\nabla z_n)\nabla z_n\,dx = \int_\Omega a(x,z,\nabla z)\nabla z\,dx. \tag{3.10}$$

Finally, the coersivity (A_4) and Lemma 3.1 allow to conclude that,

$$M(|\nabla z_n|) \longrightarrow M(|\nabla z|) \text{ in } L^1(\Omega). \tag{3.11}$$

In the sequel, since g is supposed having a constant sign, we start our study by a case where g is positive. ∎

4. CASE OF A POSITIVE NONLINEARITY

We consider first the convex set,

$$K_0 = \{u\in W_0^1L_M(\Omega);\ u\ge 0 \ a.e. \text{ in } \Omega\}. \tag{4.1}$$

This convex set is sequentially $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ closed in $W_0^1L_M(\Omega)$ [see [14]].

Remark 4.1. For each $u\in K_0\cap L^\infty(\Omega)$ there exists a sequence $v_j\in K_0\cap W_0^1E_M(\Omega)\cap L^\infty(\Omega)$ such that $v_j\to u$ for the modular convergence with $\|v_j\|_\infty$ bounded (see proposition 10, [14]).

Theorem 4.1. *Assume that (A_1) - (A_4), (G_1) and (G_2) hold true and that $f\in L^1(\Omega)$. Then there exists at least one solution of the following unilateral problem,*

$$(P)\begin{cases} u\in\tau_0^{1,M}(\Omega), u\ge 0\ a.e. \text{ in } \Omega,\\ g(x,u,\nabla u)\in L^1(\Omega),\\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx+\int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \qquad\qquad\qquad \displaystyle\le \int_\Omega fT_k(u-v)\,dx,\\ \forall\ v\in K_0\cap L^\infty(\Omega),\ \forall k>0.\end{cases}$$

Remark 4.2. Note that the gradient of u in (P) is well defined in the weak sense (see Lemma 2.4 and Lemma 2.5)

Proof. Let us define,

$$g_\epsilon(x,s,\xi) = \frac{g(x,s,\xi)}{1+\epsilon|g(x,s,\xi)|} \tag{4.2}$$

and consider the following approximate problem,

$$(P_\epsilon)\begin{cases} -\text{div}a(x,u_\epsilon,\nabla u_\epsilon) + g_\epsilon(x,u_\epsilon,\nabla u_\epsilon) = f_\epsilon \ \text{ in } \ \Omega \\ u_\epsilon = 0 \qquad \text{on } \ \partial\Omega, \end{cases} \tag{4.3}$$

where f_ϵ is a regular function such that f_ϵ strongly converges to f in $L^1(\Omega)$
and $\|f_\epsilon\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$. Note that $g_\epsilon(x,s,\xi)$ satisfies the following conditions,

$$|g_\epsilon(x,s,\xi)| \leq |g(x,s,\xi)| \leq b(|s|)(h(x)+M(|\xi|))$$

and

$$|g_\epsilon(x,s,\xi)| \leq \frac{1}{\epsilon}.$$

Nevertheless, it seems different to obtain a priori estimates, due to the fact that the quantity $u_\epsilon g(x,u_\epsilon,\nabla u_\epsilon)$ has no sign.
In order to avoid this inconvenience, we approach the sign function by an increasing Lipschitz function.
Set,

$$\delta_\sigma(s) = \begin{cases} \frac{s-\sigma}{s} & \text{if } \ s \geq \sigma > 0 \\ 0 & \text{if } \ |s| \leq \sigma \\ \frac{-s-\sigma}{s} & \text{if } \ s < -\sigma < 0. \end{cases}$$

Now, we set

$$g_\epsilon^\sigma(x,s,\xi) = \delta_\sigma(s) g_\epsilon(x,s,\xi). \tag{4.4}$$

Remark that $g_\epsilon^\sigma(x,s,\xi)$ has a same sign as s.
Now, we are in opposition to approximate our initial unilateral problem by the following penalized problem,
(4.5)

$$(P_\epsilon^\sigma)\begin{cases} u_\varepsilon^\sigma \in W_0^1L_M(\Omega) \\ \displaystyle\int_\Omega \langle Au_\epsilon^\sigma, u_\epsilon^\sigma - v\rangle + \int_\Omega g_\epsilon^\sigma(x,u_\epsilon^\sigma,\nabla u_\epsilon^\sigma)(u_\epsilon^\sigma - v)\,dx - \frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma-}))(u_\varepsilon^\sigma - v)\,dx \\ \displaystyle\qquad\qquad = \int_\Omega f_\varepsilon(u_\varepsilon^\sigma - v)\,dx \\ \forall\, v \in W_0^1L_M(\Omega), \end{cases}$$

where $m(t)$ is the derivatives function of $M(t)$.

From Gossez and Mustonen ([14], Proposition 5), the problem (4.5) has at least one solution. ∎

4.1. Study of the approximate problem with respect to ϵ.

4.1.1. ***A priori estimates*** . Taking $v = u_\varepsilon^\sigma - T_k(u_\varepsilon^\sigma)$ as test in (4.5), we obtain

$$\begin{aligned}&\int_\Omega a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla T_k(u_\varepsilon^\sigma)\, dx\\ &+ \int_\Omega g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)T_k(u_\varepsilon^\sigma)\, dx\\ &- \frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -}))T_k(u_\varepsilon^\sigma)\, dx\\ &= \int_\Omega f_\varepsilon T_k(u_\varepsilon^\sigma)\, dx.\end{aligned}$$

$g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)T_k(u_\varepsilon^\sigma) \geq 0$ and $-\frac{1}{\varepsilon^2}m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -}))T_k(u_\varepsilon^\sigma) \geq 0$ then we have,

$$\int_\Omega a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla T_k(u_\varepsilon^\sigma)\, dx \leq k\|f\|_{L^1(\Omega)}. \tag{4.6}$$

So, by (A_4) we get,

$$\alpha\int_\Omega M(|\nabla T_k(u_\varepsilon^\sigma)|) \leq k\|f\|_{L^1(\Omega)}. \tag{4.7}$$

Thus $(T_k(u_\varepsilon^\sigma))_\varepsilon$ is bounded in $W_0^1L_M(\Omega)$ uniformly in ε and σ, then there exists for σ fixed some $v_k^\sigma \in W_0^1L_M(\Omega)$ such that,

$$T_k(u_\varepsilon^\sigma) \rightharpoonup v_k^\sigma \ \text{ in } \ W_0^1L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M, \Pi E_{\bar{M}})$$

and

$$T_k(u_\varepsilon^\sigma) \to v_k^\sigma \ \text{ strongly in } \ E_M(\Omega). \tag{4.8}$$

4.1.2. ***Convergence in measure of*** u_ε^σ**.** Let $k > 0$. By Lemma 5.7 of [12], there exist tow positive constants c_1 and c_2 such that,

$$\int_\Omega M(c_1T_k(u_\varepsilon^\sigma))\, dx \leq c_2\int_\Omega M(|\nabla T_k(u_\varepsilon^\sigma)|)\, dx.$$

So, in virtue of (4.7), we have

$$\int_\Omega M(c_1T_k(u_\varepsilon^\sigma))\, dx \leq kc, \tag{4.9}$$

where $c = c(\|f\|_{L^1(\Omega)}, c_1, \alpha)$.
Then, we deduce that,

$$M(c_1k)\text{meas}(\{|u_\varepsilon^\sigma| > k\}) = \int_{\{|u_\varepsilon^\sigma|>k\}} M(c_1T_k(u_\varepsilon^\sigma))\, dx \leq kc.$$

Hence,

$$\text{meas}(\{|u_\varepsilon^\sigma| > k\}) \leq \frac{kc}{M(c_1k)} \ \ \forall\, \varepsilon,\ \forall\, k.$$

This yields that,

$$\text{meas}(\{|u_\varepsilon^\sigma| > k\}) \to 0 \ \text{ as } \ k \to +\infty \tag{4.10}$$

uniformly in ε and σ.
Now, we prove that $(u_\varepsilon^\sigma)_\varepsilon$ converges to some function u^σ in measure (and therefore, we can

always assume that the convergence is a.e. after passing to a suitable subsequence).
For every $\lambda > 0$, we have

$$\begin{aligned}\text{meas}(\{|u_j^\sigma - u_i^\sigma| > \lambda\}) \le\ & \text{meas}(\{|u_j^\sigma| > k\}) \\ & + \text{meas}(\{|u_i^\sigma| > k\}) \\ & + \text{meas}(\{|T_k(u_j^\sigma) - T_k(u_i^\sigma)| > \lambda\}).\end{aligned} \tag{4.11}$$

Consequently, by (4.8) we can assume that $(T_k(u_\varepsilon^\sigma))_\varepsilon$ is a Cauchy sequence in measure in Ω. Let $\eta > 0$. By (4.11) there exists some $k(\eta) > 0$ such that,

$$\text{meas}(\{|u_j^\sigma - u_i^\sigma| > \lambda\}) \le \eta \ \text{ for all } \ i, j \ge n_0(k(\eta), \lambda).$$

This proves that $(u_\varepsilon^\sigma)_\varepsilon$ is a Cauchy sequence in measure in Ω, thus converges almost every where to some measurable function u^σ. Then

$$T_k(u_\varepsilon^\sigma) \rightharpoonup T_k(u^\sigma) \ \text{ weakly in } \ W_0^1 L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M, \Pi E_{\bar{M}})$$

$$T_k(u_\varepsilon^\sigma) \to T_k(u^\sigma) \ \text{ strongly in } \ E_M(\Omega) \ \text{ and a.e. in } \ \Omega. \tag{4.12}$$

4.1.3. ***Show that*** $u^\sigma \ge 0$. Taking $v = u_\varepsilon^\sigma - T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)$ as test in (4.5), we obtain

$$\begin{aligned}\int_\Omega a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)\, dx + \int_\Omega g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)\, dx & \\ - \frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -})) T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)\, dx & \\ = \int_\Omega f_\varepsilon T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)\, dx. &\end{aligned}$$

Since $\displaystyle\int_\Omega a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)\, dx \ge 0$ and $g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma) \ge 0$ we get,

$$-\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -})) T_{\frac{1}{\varepsilon}}(u_\varepsilon^\sigma)\, dx \le \frac{1}{\varepsilon}\|f\|_{L^1(\Omega)},$$

which implies that,

$$\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -})) T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -})\, dx \le \frac{1}{\varepsilon}\|f\|_{L^1(\Omega)}.$$

Moreover, since

$$M(\tau) \le m(\tau)\tau$$

then we have,

$$\int_\Omega M(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma -}))\, dx \le \varepsilon\|f\|_{L^1(\Omega)}.$$

Finally, writing $\displaystyle\int_\Omega M(u_\varepsilon^{\sigma -})\, dx$ as

$$\int_\Omega M(u_\varepsilon^{\sigma -})\, dx = \int_{\{u_\varepsilon^{\sigma -} \le \frac{1}{\varepsilon}\}} M(u_\varepsilon^{\sigma -})\, dx + \int_{\{u_\varepsilon^{\sigma -} > \frac{1}{\varepsilon}\}} M(u_\varepsilon^{\sigma -})\, dx,$$

one deduce that,

$$\int_\Omega M(u_\varepsilon^{\sigma -})\, dx \le \varepsilon\|f\|_{L^1(\Omega)} + \int_{\{u_\varepsilon^{\sigma -} > \frac{1}{\varepsilon}\}} M(u_\varepsilon^{\sigma -})\, dx.$$

Hence, due to the fact that $u_\varepsilon^\sigma \to u^\sigma$ a.e. in Ω, we conclude that

$$M(u_\varepsilon^\sigma) \to M(u^\sigma) \ \text{ a.e. in } \ \Omega.$$

Also as in (4.10) we can prove that,

$$\text{meas}\{u_\varepsilon^{\sigma -} > \frac{1}{\varepsilon}\} \to 0.$$

Then,

$$M(u_\varepsilon^{\sigma -}) \to 0 \text{ as } \varepsilon \to 0,$$

which gives,

$$u^\sigma \geq 0.$$

4.1.4. ***Boundedness of*** $(a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma)))_\varepsilon$ ***in*** $(L_{\bar{M}}(\Omega))^N$. Let $w \in (E_M(\Omega))^N$ be arbitrary. By (A_3) we have,

$$[a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) - a(x, u_\varepsilon^\sigma, w)][\nabla u_\varepsilon^\sigma - w] > 0,$$

which implies that,

$$a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)w \leq a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla u_\varepsilon^\sigma - a(x, u_\varepsilon^\sigma, w)(\nabla u_\varepsilon^\sigma - w).$$

Integrating on the subset $\{x \in \Omega,\ |u_\varepsilon^\sigma| < k\}$ we obtain,

$$\begin{aligned} \int_{\{|u_\varepsilon^\sigma|<k\}} a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)w\ dx \leq &\int_{\{|u_\varepsilon^\sigma|<k\}} a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla u_\varepsilon^\sigma\ dx \\ &- \int_{\{|u_\varepsilon^\sigma|<k\}} a(x, u_\varepsilon^\sigma, w)(\nabla u_\varepsilon^\sigma - w)\ dx. \end{aligned} \tag{4.13}$$

Thanks to (4.6), we have

$$\int_{\{|u_\varepsilon^\sigma|<k\}} a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\nabla u_\varepsilon^\sigma\ dx \leq c(k). \tag{4.14}$$

On the other hand, for λ large enough, we have by using (A_2),

$$\int_{\{|u_\varepsilon^\sigma|<k\}} \bar{M}\left(\frac{a(x, u_\varepsilon^\sigma, w)}{\lambda}\right)\ dx \leq \int_\Omega \bar{M}\left(\frac{k(x)}{\lambda}\right)\ dx + \frac{k_3}{\lambda}\int_\Omega M(k_2|w|) + c \leq c_3.$$

Hence $(|a(x, u_\varepsilon^\sigma, w)|_{\chi_{\{|u_\varepsilon^\sigma|<k\}}})_\varepsilon$ is bounded in $L_{\bar{M}}(\Omega)$, which implies that the second term of the right hand side of (4.13) is also bounded.
Consequently, we obtain,

$$\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))w \leq c_4, \tag{4.15}$$

where c_4 is a positive constant depending of k.
Hence, by the theorem of Banach-Steinhaus, the sequence $a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))_\varepsilon$ remains bounded in $(L_{\bar{M}}(\Omega))^N$.
Which implies that, for all $k > 0$, there exists a function $h_{k\sigma} \in (L_{\bar{M}}(\Omega))^N$, such that

$$a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma)) \rightharpoonup h_{k\sigma} \text{ weakly in } (L_{\bar{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\bar{M}}(\Omega), \Pi E_M(\Omega)). \tag{4.16}$$

4.1.5. ***Almost every where convergence of the gradient.*** In the sequel, we use the following notations:
$\eta(\varepsilon, j, h)$ is any quantity such that

$$\lim_{h \to +\infty} \lim_{j \to +\infty} \lim_{\varepsilon \to 0} \eta(\varepsilon, j, h) = 0.$$

If the quantity we consider does not depend on one parameter among η, j and h, we will omit the dependence on the corresponding parameter: as an example, $\eta(\varepsilon, h)$ is any quantity such that

$$\lim_{h \to +\infty} \lim_{\varepsilon \to 0} \eta(\varepsilon, h) = 0.$$

Finally, we will denote (for example) by $\eta_h(\varepsilon, j)$ a quantity that depends on ε, j, h and is such that

$$\lim_{j \to +\infty} \lim_{\varepsilon \to 0} \eta_h(\varepsilon, j) = 0$$

for any fixed value of h.

We fix $k > 0$, let $\Omega_r = \{x \in \Omega, |\nabla T_k(u^\sigma(x))| \leq r\}$ and denote by χ_r the characteristic function of Ω_r.
Clearly, $\Omega_r \subset \Omega_{r+1}$ and meas$(\Omega \backslash \Omega_r) \to 0$ as $r \to \infty$.
Fix r and let $s > r$, we have

(4.17)
$$\begin{aligned} 0 &\leq \int_{\Omega_r} [a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma_\varepsilon) - a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma)][\nabla T_k(u^\sigma_\varepsilon) - \nabla T_k(u^\sigma)]\, dx \\ &\leq \int_{\Omega_s} [a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma_\varepsilon) - a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma)][\nabla T_k(u^\sigma_\varepsilon) - \nabla T_k(u^\sigma)]\, dx \\ &= \int_{\Omega_s} [a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma_\varepsilon) - a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma)\chi_s][\nabla T_k(u^\sigma_\varepsilon) - \nabla T_k(u^\sigma)\chi_s]\, dx \\ &\leq \int_{\Omega} [a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma_\varepsilon) - a(x, T_k(u^\sigma_\varepsilon), \nabla T_k(u^\sigma)\chi_s][\nabla T_k(u^\sigma_\varepsilon) - \nabla T_k(u^\sigma)\chi_s]\, dx. \end{aligned}$$

Let $k > 0$ and let $\varphi_k(s) = s e^{\gamma s^2}$, where $\gamma = (\frac{b(k)}{\alpha})^2$.
It is well know that,

$$\varphi'_k(s) - \frac{b(k)}{\alpha} |\varphi_k(s)| \geq \frac{1}{2}, \ \ \forall\, s \in \mathbb{R}. \tag{4.18}$$

Thanks to Remark 4.1 there exists a sequence $v_j \in K_0 \cap W^1_0 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u^\sigma)$ for the modular convergence in $W^1_0 L_M(\Omega)$.
Here, we define

$$\begin{aligned} w^{h\sigma}_{\varepsilon j} &= T_{2k}(u^\sigma_\varepsilon - T_h(u^\sigma_\varepsilon) + T_k(u^\sigma_\varepsilon) - T_k(v_j)) \\ w^{h\sigma}_{j} &= T_{2k}(u^\sigma - T_h(u^\sigma) + T_k(u^\sigma) - T_k(v_j)) \\ w^{h\sigma} &= T_{2k}(u^\sigma - T_h(u^\sigma)) \end{aligned}$$

where $h > 2k > 0$.
For $\eta = \exp(-4\gamma k^2)$, we define the following function as,

$$v^{h,\sigma}_{\varepsilon,j} = u^\sigma_\varepsilon - \eta \varphi_k(w^{h,\sigma}_{\varepsilon,j}). \tag{4.19}$$

We take $v_{\varepsilon,j}^{h,\sigma}$ as test function in (4.5), we obtain,

$$\begin{aligned}&\langle A(u_\varepsilon^\sigma), \eta\varphi_k(w_{\varepsilon,j}^{h,\sigma})\rangle\\&+\int_\Omega g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\eta\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\&-\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma-}))\eta\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ \le&\int_\Omega \eta f_\varepsilon\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx,\end{aligned}$$

which implies that,

$$\begin{aligned}&\langle A(u_\varepsilon^\sigma), \varphi_k(w_{\varepsilon,j}^{h,\sigma})\rangle\\&+\int_\Omega g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\&-\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma-}))\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ \le&\int_\Omega f_\varepsilon\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx.\end{aligned}$$

It follows that,

$$\begin{aligned}&\int_\Omega a(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\nabla w_{\varepsilon,j}^{h,\sigma}\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\&+\int_\Omega g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\&-\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma-}))\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ \le&\int_\Omega f_\varepsilon\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx.\end{aligned}\tag{4.20}$$

Note that, $\nabla w_{\varepsilon,j}^{h,\sigma}=0$ on the set where $|u_\varepsilon^\sigma|>h+5k$, therefore, setting $s=5k+h$, we get by (4.20)

$$\begin{aligned}&\int_\Omega a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))\nabla w_{\varepsilon,j}^{h,\sigma}\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\&+\int_\Omega g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\&-\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma-}))\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ \le&\int_\Omega f_\varepsilon\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx.\end{aligned}$$

In view of (4.12), we have $\varphi_k(w_{\varepsilon,j}^{h,\sigma})\to\varphi_k(w_j^{h,\sigma})$ weakly* in $L^\infty(\Omega)$ as $\varepsilon\to 0$ and then

$$\int_\Omega f_\varepsilon\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\to\int_\Omega f\varphi_k(w_j^{h,\sigma})\,dx \text{ as } \varepsilon\to 0,$$

again tending j to infinity, we get

$$\int_\Omega f\varphi_k(w_j^{h,\sigma})\,dx\to\int_\Omega f\varphi_k(w^{h,\sigma})\,dx \text{ as } j\to+\infty.$$

Finally, by using the Lebesgue's theorem, we can deduce that,

$$\int_\Omega f\varphi_k(w^{h,\sigma})\,dx \to 0 \text{ as } h\to+\infty.$$

So that,

$$\int_\Omega f_\varepsilon\varphi_k(w^{h,\sigma}_{\varepsilon,j})\,dx = \eta(\varepsilon,j,h). \tag{4.21}$$

Note that the sign of $\varphi_k(w^{h,\sigma}_{\varepsilon,j})$ is the same as that of u^σ_ε in the set $\{x\in\Omega,\ |u^\sigma_\varepsilon|>k\}$, then we have

$$g^\sigma_\varepsilon(x,u^\sigma_\varepsilon,\nabla u^\sigma_\varepsilon)\varphi_k(w^{h,\sigma}_{\varepsilon,j})\ge 0,$$

and

$$-\frac{1}{\varepsilon^2}m(T_{\frac{1}{\varepsilon}}(u^{\sigma-}_\varepsilon))\varphi_k(w^{h,\sigma}_{\varepsilon,j})\ge 0$$

in the subset $\{x\in\Omega,\ |u^\sigma_\varepsilon|>k\}$, we deduce from (4.20) that,

$$\begin{aligned}&\int_\Omega a(x,T_s(u^\sigma_\varepsilon),\nabla T_s(u^\sigma_\varepsilon))\nabla(w^{h,\sigma}_{\varepsilon,j})\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\&+\int_{\{|u^\sigma_\varepsilon|<k\}} g^\sigma_\varepsilon(x,u^\sigma_\varepsilon,\nabla u^\sigma_\varepsilon)\varphi_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\&-\frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u^{\sigma-}_\varepsilon))(u^\sigma_\varepsilon-T_k(v_j))\exp(\gamma(w^{h,\sigma}_{\varepsilon,j})^2\\ \le&\,\eta(\varepsilon,j,h).\end{aligned} \tag{4.22}$$

Since by Remark 4.1, $v_j\ge 0$, then the third term of the left-hand side of the above inequality is positive, thus,

$$\begin{aligned}&\int_\Omega a(x,T_s(u^\sigma_\varepsilon),\nabla T_s(u^\sigma_\varepsilon))\nabla(w^{h,\sigma}_{\varepsilon,j})\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\&+\int_{\{|u^\sigma_\varepsilon|<k\}} g^\sigma_\varepsilon(x,u^\sigma_\varepsilon,\nabla u^\sigma_\varepsilon)\varphi_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\ \le&\,\eta(\varepsilon,j,h).\end{aligned} \tag{4.23}$$

Splitting the first integral one the left hand side of (4.23), where $|u^\sigma_\varepsilon|\le k$ and where $|u^\sigma_\varepsilon|>k$, we can write,

$$\begin{aligned}&\int_\Omega a(x,T_s(u^\sigma_\varepsilon),\nabla T_s(u^\sigma_\varepsilon))\nabla(w^{h,\sigma}_{\varepsilon,j})\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\&=\int_{\{|u^\sigma_\varepsilon|\le k\}} a(x,T_s(u^\sigma_\varepsilon),\nabla T_s(u^\sigma_\varepsilon))[\nabla T_k(u^\sigma_\varepsilon)-\nabla T_k(v_j)]\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\&+\int_{\{|u^\sigma_\varepsilon|>k\}} a(x,T_s(u^\sigma_\varepsilon),\nabla T_s(u^\sigma_\varepsilon))\nabla(w^{h,\sigma}_{\varepsilon,j})\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx.\end{aligned} \tag{4.24}$$

The first term of the right-hand side of the last inequality can write as,

$$\begin{aligned}&\int_{\{|u^\sigma_\varepsilon|\le k\}} a(x,T_s(u^\sigma_\varepsilon),\nabla T_s(u^\sigma_\varepsilon))\nabla(w^{h,\sigma}_{\varepsilon,j})\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx\\&=\int_\Omega a(x,T_k(u^\sigma_\varepsilon),\nabla T_k(u^\sigma_\varepsilon))[\nabla T_k(u^\sigma_\varepsilon)-\nabla T_k(v_j)]\varphi'_k(w^{h,\sigma}_{\varepsilon,j})\,dx.\end{aligned} \tag{4.25}$$

For the second term of the right hand side of (4.24) we can write, using (A_4),

$$\int_{\{|u_\varepsilon^\sigma|>k\}} a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))\nabla(w_{\varepsilon,j}^{h,\sigma})\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx \geq -\varphi_k'(2k)\int_{\{|u_\varepsilon^\sigma|>k\}} |a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))||\nabla v_j|\,dx. \tag{4.26}$$

Since $|a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))|$ is bounded in $L_{\bar M}(\Omega)$, we have for a subsequence

$$|a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))| \rightharpoonup l_{s,\sigma}$$

weakly in $L_{\bar M}(\Omega)$ for $\sigma(L_{\bar M},E_M)$ as ε tends to zero, and since

$$\nabla v_j\chi_{\{|u_\varepsilon^\sigma|>k\}} \to \nabla v_j\chi_{\{|u^\sigma|>k\}}$$

strongly in $E_M(\Omega)$ as $\varepsilon\to 0$, we have

$$-\varphi_k'(2k)\int_{\{|u_\varepsilon^\sigma|>k\}} |a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))||\nabla v_j|\,dx \to -\varphi'(2k)\int_{\{|u^\sigma|>k\}} l_{s,\sigma}|\nabla v_j|\,dx$$

as $\varepsilon\to 0$.
Using now, the modular convergence of (v_j), we get

$$-\varphi_k'(2k)\int_{\{|u^\sigma|>k\}} l_{s,\sigma}|\nabla v_j|\,dx \to -\varphi_k'(2k)\int_{\{|u^\sigma|>k\}} l_{s,\sigma}|\nabla T_k(u^\sigma)|\,dx = 0$$

as $j\to+\infty$.
Finally, we have

$$-\varphi_k'(2k)\int_{\{|u_\varepsilon^\sigma|>k\}} |a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))||\nabla v_j|\,dx = \eta_h(\varepsilon,j). \tag{4.27}$$

Combining (4.24) and (4.27), we deduce that,

$$\begin{aligned}&\int_\Omega a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))\nabla(w_{\varepsilon,j}^{h,\sigma})\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &\geq \int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))[\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(v_j)]\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx+\eta_h(\varepsilon,j).\end{aligned}$$

Which implies that,

$$\begin{aligned}&\int_\Omega a(x,T_s(u_\varepsilon^\sigma),\nabla T_s(u_\varepsilon^\sigma))\nabla(w_{\varepsilon,j}^{h,\sigma})\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &\geq \int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))-a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)]\\ &\times[\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &+\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &-\int_{\Omega\setminus\Omega_s^j} a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))\nabla T_k(v_j)\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &+\eta_h(\varepsilon,j),\end{aligned} \tag{4.28}$$

where χ_s^j denotes the characteristic function of the subset $\Omega_s^j=\{x\in\Omega:\ |\nabla T_k(v_j)|\leq s\}$.

By (4.16) and the fact that $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_{\varepsilon,j}^{h,\sigma})$ tends to $\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_j^{h,\sigma})$ strongly in $(E_M(\Omega))^N$, the third term of the right-hand side of (4.28) tends to the quantity

$$\int_\Omega h_{k,\sigma}\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_j^{h,\sigma})\,dx \text{ as } \varepsilon\to 0.$$

Letting now j tends to infinity, by using the modular convergence of v_j, we have

$$\int_\Omega h_{k,\sigma}\nabla T_k(v_j)\chi_{\Omega\setminus\Omega_s^j}\varphi'_k(w_j^{h,\sigma})\,dx \to \int_{\Omega\setminus\Omega_s^j} h_{k,\sigma}\nabla T_k(u^\sigma)\varphi'_k(w^{h,\sigma})\,dx \text{ as } j\to+\infty.$$

Finally, we get ,

$$\begin{aligned}\int_{\Omega\setminus\Omega_s^j} & a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))\nabla T_k(v_j)\varphi'_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &= -\int_{\Omega\setminus\Omega_s} h_{k,\sigma}\nabla T_k(u^\sigma)\varphi'_k(w^{h,\sigma})\,dx+\eta_h(\varepsilon,j).\end{aligned}\tag{4.29}$$

Concerning the second term of the right hand side of (4.28) we can write,

$$\begin{aligned}&\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(v_j)\chi_s^j]\varphi'_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &=\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)\nabla T_k(u_\varepsilon^\sigma)\varphi'_k(T_k(u_\varepsilon^\sigma)-T_k(v_j))\,dx\\ &-\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\varphi'_k(w_{\varepsilon,j}^{h,\sigma})\,dx.\end{aligned}\tag{4.30}$$

The first term of the right hand side of (4.30) tends to the quantity,

$$\int_\Omega a(x,T_k(u^\sigma),\nabla T_k(v_j)\chi_s^j)\nabla T_k(u^\sigma)\varphi'_k(T_k(u^\sigma)-T_k(v_j))\,dx \text{ as } \varepsilon\to 0.$$

Thanks to Lemma 2.3, we have

$a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)\varphi'_k(T_k(u_\varepsilon^\sigma)-T_k(v_j))\to a(x,T_k(u^\sigma),\nabla T_k(v_j)\chi_s^j)\varphi'_k(T_k(u^\sigma)-T_k(v_j))$ strongly in $(E_{\bar M}(\Omega))^N$ and

$$\nabla T_k(u_\varepsilon^\sigma)\rightharpoonup\nabla T_k(u^\sigma) \text{ weakly in } (L_M(\Omega))^N \text{ for } \sigma(\Pi L_M,\Pi E_{\bar M}).$$

For the second term of the right hand side of (4.30) it is easy to see that,

$$\begin{aligned}&\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\varphi'_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &\to\int_\Omega a(x,T_k(u^\sigma),\nabla T_k(v_j)\chi_s^j)\nabla T_k(v_j)\chi_s^j\varphi'_k(w_j^{h,\sigma})\,dx \text{ as } \varepsilon\to 0.\end{aligned}\tag{4.31}$$

Consequently, we have

$$\begin{aligned}&\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(v_j)\chi_s^j]\varphi'_k(w_{\varepsilon,j}^{h,\sigma})\,dx\\ &=\int_\Omega a(x,T_k(u^\sigma),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u^\sigma)-\nabla T_k(v_j)\chi_s^j]\varphi'_k(w_j^{h,\sigma})\,dx\\ &+\eta_{j,h}(\varepsilon).\end{aligned}\tag{4.32}$$

Since,

$$\nabla T_k(v_j)\chi_s^j\varphi'_k(w_j^{h,\sigma})\to\nabla T_k(u^\sigma)\chi_s\varphi'_k(w^{h,\sigma})$$

strongly in $E_M(\Omega))^N$ as $j \to \infty$, it is easy to see that,

$$\int_\Omega a(x, T_k(u^\sigma), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u^\sigma) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_j^{h,\sigma})\, dx \to 0 \text{ as } j \to +\infty.$$

Thus,

$$\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\, dx = \eta_h(\varepsilon, j). \tag{4.33}$$

Combining (4.28), (4.29) and (4.33) we get,

$$\begin{aligned} &\int_\Omega a(x, T_m(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))\nabla(w_{\varepsilon,j}^{h,\sigma})\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\, dx \\ &\geq \int_\Omega [a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma)) - a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(v_j)\chi_s^j)] \\ &\times [\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]\varphi_k'(w_{\varepsilon,j}^{h,\sigma})\, dx \\ &+ \int_{\Omega\setminus\Omega_s} h_{k\sigma}\nabla T_k(u^\sigma)\varphi_k'(0)\, dx + \eta(\varepsilon, j, h). \end{aligned} \tag{4.34}$$

We now turn to the second term of the left hand side of (4.23), we have

$$\begin{aligned} &\left|\int_{\{|u_\varepsilon^\sigma|<k\}} g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)\varphi_k(w_{\varepsilon,j}^{h,\sigma})\, dx\right| \\ &\leq b(k)\int_\Omega (h(x) + M(\nabla T_k(u_\varepsilon^\sigma))|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx \\ &\leq b(k)\int_\Omega h(x)|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx + \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))\nabla T_k(u_\varepsilon^\sigma)|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx \\ &\leq \eta(\varepsilon, j, h) + \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))\nabla T_k(u_\varepsilon^\sigma)|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx. \end{aligned}$$

The last term of the last side of this inequality reads as,

$$\begin{aligned} &\frac{b(k)}{\alpha}\int_\Omega [a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma)) - a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(v_j)\chi_s^j)] \\ &\times [\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx \\ &+ \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx \\ &- \frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))\nabla T_k(v_j)\chi_s^j|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx. \end{aligned}$$

And reasoning as above, it is easy to see that,

$$\frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx = \eta(\varepsilon, j)$$

and

$$\frac{b(k)}{\alpha}\int_\Omega a(x, T_k(u_\varepsilon^\sigma), \nabla T_k(u_\varepsilon^\sigma))\nabla T_k(v_j)\chi_s^j|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\, dx = \eta(\varepsilon, j, h).$$

So that,

$$\left|\int_{\{|u_\varepsilon^\sigma|<k\}} g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\varphi_k(w_{\varepsilon,j}^{h,\sigma})\,dx\right| \\ \le \frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma)) - a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)] \\ \times [\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|\,dx + \eta(\varepsilon,j,h). \tag{4.35}$$

Combining (4.23), (4.34) and (4.35), we obtain

$$\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma)) - a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)] \\ \times [\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j](\varphi_k'(w_{\varepsilon,j}^{h,\sigma}) - \frac{b(k)}{\alpha}|\varphi_k(w_{\varepsilon,j}^{h,\sigma})|)\,dx \\ \le \int_{\Omega\setminus\Omega_s} h_{k\sigma}\nabla T_k(u^\sigma)\varphi_k'(0)\,dx + \eta(\varepsilon,j,h), \tag{4.36}$$

which implies by using (4.18) that

$$\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma)) - a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)] \\ \times [\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]\,dx \\ \le 2\int_{\Omega\setminus\Omega_s} h_{k\sigma}\nabla T_k(u^\sigma)\varphi_k'(0)\,dx + \eta(\varepsilon,j,h). \tag{4.37}$$

Now, remark that,

$$\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma)) - a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u^\sigma)\chi_s)][\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(u^\sigma)\chi_s]\,dx \\ \le \int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma)) - a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)][\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]\,dx \\ + \int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]\,dx \\ - \int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u^\sigma)\chi_s)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(u^\sigma)\chi_s]\,dx \\ + \int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))[\nabla T_k(v_j)\chi_s^j - \nabla T_k(u^\sigma)\chi_s]\,dx. \tag{4.38}$$

We shall pass to the limit in ε and j in the last three terms of the right-hand side of the last inequality, we get

$$\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(v_j)\chi_s^j]\,dx = \eta(\varepsilon,j)$$

$$\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u)\chi_s)[\nabla T_k(u_\varepsilon^\sigma) - \nabla T_k(u)\chi_s]\,dx = \eta(\varepsilon)$$

and

$$\int_\Omega a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))[\nabla T_k(v_j)\chi_s^j - \nabla T_k(u^\sigma)\chi_s]\,dx = \eta(\varepsilon,j),$$

which implies that,

$$\begin{aligned}(4.39)\quad &\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))-a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u^\sigma)\chi_s)][\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(u^\sigma)\chi_s]\,dx\\ &=\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))-a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(v_j)\chi_s^j)][\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(v_j)\chi_s^j]\,dx\\ &+\eta(\varepsilon,j).\end{aligned}$$

Combining (4.17), (4.37) and (4.39), we have

(4.40)

$$\begin{aligned}&\int_{\Omega_r} [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))-a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u^\sigma))][\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(u^\sigma)]\,dx\\ &\leq\int_\Omega [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))-a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u^\sigma)\chi_s)][\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(u^\sigma)\chi_s]\,dx\\ &\leq 2\int_{\Omega\setminus\Omega_s} h_{k\sigma}\nabla T_k(u^\sigma)\varphi_k'(0)\,dx+\eta(\varepsilon,j,h).\end{aligned}$$

By passing to the $\limsup$ over n and letting j,h,s tend to infinity, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega_r} [a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u_\varepsilon^\sigma))-a(x,T_k(u_\varepsilon^\sigma),\nabla T_k(u^\sigma))][\nabla T_k(u_\varepsilon^\sigma)-\nabla T_k(u^\sigma)]\,dx=0.$$

This implies by virtue of Lemma 3.2 that,

$$(4.41)\qquad \nabla u_\varepsilon^\sigma\to\nabla u^\sigma \text{ a.e. in } \Omega$$

and

$$(4.42)\qquad M(|\nabla T_k(u_\varepsilon^\sigma)|)\to M(|\nabla T_k(u^\sigma)|)\quad\text{in } L^1(\Omega).$$

4.1.6. ***Equi-integrability of the nonlinearity.*** We need to prove that,

$$(4.43)\qquad g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\to g^\sigma(x,u^\sigma,\nabla u^\sigma)\ \text{ strongly in }\ L^1(\Omega).$$

In particular it is enough to prove the equi-integrability of $g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)$. To this purpose, we take $u_\varepsilon^\sigma-T_1(u_\varepsilon^\sigma-T_h(u_\varepsilon^\sigma))\geq 0$ as test function in (4.5), we obtain,

$$\int_{\{|u_\varepsilon^\sigma|\geq h+1\}}|g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\,dx\leq\int_{\{|u_\varepsilon^\sigma|>h\}}|f_\varepsilon|\,dx.$$

Let $\eta>0$, then there exists $h(\eta)\geq 1$ such that,

$$(4.44)\qquad \int_{\{|u_\varepsilon^\sigma|\geq h(\eta)\}}|g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)\,dx\leq\frac{\eta}{2}.$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g_\varepsilon^\sigma(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)|\,dx\leq&\int_\Omega b(h(\eta))(c(x)+M(|\nabla T_{h(\eta)}(u_\varepsilon^\sigma)|)\,dx\\ &+\int_{\{|u_\varepsilon^\sigma|\geq h(\eta)\}}|g(x,u_\varepsilon^\sigma,\nabla u_\varepsilon^\sigma)|\,dx.\end{aligned}$$

In view of (4.42) there exists $\beta(\eta)>0$ such that,

$$(4.45)\qquad \int_E b(h(\eta))(h(x)+M(|\nabla T_{h(\eta)}(u_\varepsilon^\sigma)|)\,dx\leq\frac{\eta}{2}$$

for all E such that $|E| < \beta(\eta)$.
Finally, combining (4.44) and (4.45), one easily has $\int_E |g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma)| \, dx \leq \eta$ for all E such that meas$(E) < \beta(\eta)$.

4.1.7. *Passing to the limit in* ε. Let $v \in K_0 \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$, we take $u_\varepsilon^\sigma - T_k(u_\varepsilon^\sigma - v)$ as test function in (4.5), we can write,

$$\int_\Omega a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) \nabla T_k(u_\varepsilon^\sigma - v) \, dx + \int_\Omega g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) T_k(u_\varepsilon^\sigma - v) \, dx \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon^\sigma - v) \, dx, \tag{4.46}$$

which implies that,

$$\int_{\{|u_\varepsilon^\sigma - v| \leq k\}} a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) \nabla(u_\varepsilon^\sigma - v) \, dx + \int_\Omega g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) T_k(u_\varepsilon^\sigma - v) \, dx \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon^\sigma - v) \, dx.$$

i.e.,

$$\begin{aligned} \int_{\{|u_\varepsilon^\sigma - v| \leq k\}} a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) \nabla u_\varepsilon^\sigma \, dx - \int_{\{|u_\varepsilon^\sigma - v| \leq k\}} a(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) \nabla v \, dx \\ + \int_\Omega g_\varepsilon^\sigma(x, u_\varepsilon^\sigma, \nabla u_\varepsilon^\sigma) T_k(u_\varepsilon^\sigma - v) \, dx \\ \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon^\sigma - v) \, dx. \end{aligned}$$

By Fatou's lemma and the fact that,

$$a(x, T_{k+\|v\|_\infty}(u_\varepsilon^\sigma), \nabla T_{k+\|v\|_\infty}(u_\varepsilon^\sigma)) \rightharpoonup a(x, T_{k+\|v\|_\infty}(u^\sigma), \nabla T_{k+\|v\|_\infty}(u^\sigma))$$

weakly in $(L_{\bar{M}}(\Omega))^N$ for $\sigma(\Pi L_{\bar{M}}, \Pi E_M)$ on easily see that,

$$\begin{aligned} \int_{\{|u^\sigma - v| \leq k\}} a(x, u^\sigma, \nabla u^\sigma) \nabla u^\sigma \, dx - \int_{\{|u^\sigma - v| \leq k\}} a(x, T_{k+\|v\|_\infty}(u^\sigma), \nabla T_{k+\|v\|_\infty}(u^\sigma)) \nabla v \, dx \\ + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v) \, dx \\ \leq \int_\Omega f T_k(u^\sigma - v) \, dx. \end{aligned}$$

Hence,

$$\int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_k(u^\sigma - v) \, dx + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v) \, dx \leq \int_\Omega f T_k(u^\sigma - v) \, dx. \tag{4.47}$$

Now, let $v \in K_0 \cap L^\infty(\Omega)$, by Remark 4.1, there exist $v_j \in K_0 \cap W_0^1 E_M \cap L^\infty(\Omega)$, such that v_j converges to v in the modular sense. Let $l > \|v\|_\infty$, taking $v = T_l(v_j)$ in (4.47), we have

$$\int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_k(u^\sigma - T_l(v_j))\, dx + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - T_l(v_j))\, dx \\ \leq \int_\Omega f T_k(u^\sigma - T_l(v_j))\, dx.$$

We can easily pass to the limit as $j \to +\infty$, to get

$$\begin{aligned} &\int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_k(u^\sigma - T_l(v))\, dx \\ &+ \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - T_l(v))\, dx \\ &\leq \int_\Omega f T_k(u^\sigma - T_l(v))\, dx \qquad\qquad \forall\, v \in K_0 \cap L^\infty(\Omega). \end{aligned}$$

As $l \geq \|v\|_\infty$, we deduce,

$$\begin{aligned} &\int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_k(u^\sigma - v)\, dx \\ &+ \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_k(u^\sigma - v)\, dx \\ &\leq \int_\Omega f T_k(u^\sigma - v)\, dx \ \ \forall\, v \in K_0 \cap L^\infty(\Omega), \qquad\qquad \forall\, k > 0. \end{aligned} \tag{4.48}$$

4.2. **Study of the problem with respect to the σ.**

4.2.1. ***Estimates with respect to σ.*** We are going to give some estimates, on the sequence $(u^\sigma)_\sigma$ identical to (4.7).
For that, taking $v = T_s(u^\sigma - T_k(u^\sigma))$ in (4.48) and letting s tends to infinity then by the same argument as in section 4.1 we can prove that,

$$\alpha \int_\Omega M(|\nabla T_k(u^\sigma)|) \leq k \|f\|_{L^1(\Omega)}.$$

Thus, as in 4.1.2, there exists u such that $T_k(u) \in W_0^1 L_M(\Omega)$ and

$$T_k(u^\sigma) \rightharpoonup T_k(u) \ \text{ weakly in } \ W_0^1 L_M(\Omega) \ \text{ for } \ \sigma(\Pi L_M, \Pi E_{\bar{M}})$$

$$T_k(u^\sigma) \to T_k(u) \ \text{ strongly in } \ E_M(\Omega) \ \text{ and a.e in } \ \Omega.$$

So, $u^\sigma \geq 0$ a.e. in Ω and we have also $u \geq 0$. a.e in Ω.

4.2.2. ***Strong convergence of truncation with respect to σ.*** We fix $k > 0$, let $\Omega_r = \{x \in \Omega, |\nabla T_k(u(x))| \leq r\}$ and denote by χ_r the characteristic function of Ω_r. Clearly, $\Omega_r \subset \Omega_{r+1}$ and $\text{meas}(\Omega \backslash \Omega_r) \to 0$ as $r \to \infty$.

Fix r and let $s > r$, we have

$$
\begin{aligned}
(4.49)\quad 0 &\leq \int_{\Omega_r} [a(x, T_k(u^\sigma), \nabla T_k(u^\sigma) - a(x, T_k(u^\sigma), \nabla T_k(u)][\nabla T_k(u^\sigma) - \nabla T_k(u)]\, dx \\
&\leq \int_{\Omega_s} [a(x, T_k(u^\sigma), \nabla T_k(u^\sigma) - a(x, T_k(u^\sigma), \nabla T_k(u)][\nabla T_k(u^\sigma) - \nabla T_k(u)]\, dx \\
&= \int_{\Omega_s} [a(x, T_k(u^\sigma), \nabla T_k(u^\sigma) - a(x, T_k(u^\sigma), \nabla T_k(u)\chi_s][\nabla T_k(u^\sigma) - \nabla T_k(u)\chi_s]\, dx \\
&\leq \int_{\Omega} [a(x, T_k(u^\sigma), \nabla T_k(u^\sigma) - a(x, T_k(u^\sigma), \nabla T_k(u)\chi_s][\nabla T_k(u^\sigma) - \nabla T_k(u)\chi_s]\, dx.
\end{aligned}
$$

Thanks to Remark 4.1, there exists a sequence $v_j \in K_0 \cap W_0^1 E_M(\Omega) \cap L^\infty(\Omega)$ which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_M(\Omega)$.
Here, we define

$$
\begin{aligned}
w_j^{h\sigma} &= T_{2k}(u^\sigma - T_h(u^\sigma) + T_k(u^\sigma) - T_k(v_j)) \\
w_j^{h} &= T_{2k}(u - T_h(u) + T_k(u) - T_k(v_j)) \\
w^{h} &= T_{2k}(u - T_h(u))
\end{aligned}
$$

where $h > 2k > 0$.
The choice of $v = T_s(u^\sigma - \varphi_k(w_j^{h\sigma}))$ as test function in (4.48), allows to have, for all $l > 0$,

$$
\begin{aligned}
&\int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla T_l(u^\sigma - T_s(u^\sigma - \varphi_k(w_j^{h\sigma}))\, dx \\
&+ \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_l(u^\sigma - T_s(u^\sigma - \varphi_k(w_j^{h\sigma}))\, dx \\
&\leq \int_\Omega f T_l(u^\sigma - T_s(u^\sigma - \varphi_k(w_j^{h\sigma}))\, dx,
\end{aligned}
$$

which implies that,

$$
\begin{aligned}
&\int_{\{|u^\sigma - \varphi(w_j^{h\sigma})| \leq s\}} a(x, u^\sigma, \nabla u^\sigma) \nabla T_l(\varphi_k(w_j^{h\sigma}))\, dx \\
&+ \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) T_l(u^\sigma - T_s(u^\sigma - \varphi_k(w_j^{h\sigma}))\, dx \\
&\leq \int_\Omega f T_l(u^\sigma - T_s(u^\sigma - \varphi_k(w_j^{h\sigma}))\, dx.
\end{aligned}
$$

Letting s tends to infinity and choosing l large enough $(l \geq |\varphi_k(2k)|)$, we deduce

$$
(4.50)\quad \int_\Omega a(x, u^\sigma, \nabla u^\sigma) \nabla \varphi_k(w_j^{h\sigma})\, dx + \int_\Omega g^\sigma(x, u^\sigma, \nabla u^\sigma) \varphi_k(w_j^{h\sigma})\, dx \leq \int_\Omega f \varphi_k(w_j^{h\sigma})\, dx.
$$

Then by using the same techniques as in 4.1.5 we can deduce that,

$$
(4.51)\qquad M(\nabla T_k(u^\sigma)) \to M(\nabla T_k(u)) \text{ strongly in } L^1(\Omega)
$$

and

$$
\nabla u^\sigma \to \nabla u \quad \text{a.e. in } \Omega.
$$

4.2.3. ***Equi-integrability of*** $g^\sigma(x,u^\sigma,\nabla u^\sigma)$ ***with respect to*** σ **.** Moreover, since g is a Carathéodory function, it is easy to see that,

$$g(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)\ \text{ a.e. in }\ \Omega\ \text{ as }\ \sigma\to 0.$$

Then, by assumption (G_2) (note that this hypothesis is only used here), it is clear that,

$$g^\sigma(x,u^\sigma,\nabla u^\sigma)=\delta_\sigma(u^\sigma)g(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)\ \text{ a.e. in }\ \{x\in\Omega,u(x)\geq 0\}.$$

Similarly, claim that,

$$g^\sigma(x,u^\sigma,\nabla u^\sigma)\to g(x,u,\nabla u)\ \text{ in }\ L^1(\Omega).$$

Indeed, taking $u^\sigma-T_1(u_\sigma-T_l(u^\sigma))$ as test function in (4.48), we obtain

$$\int_{\{|u^\sigma|>l+1\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\leq\int_{\{|u^\sigma|>l\}}|f|\,dx.$$

Let $\beta>0$, then there exists $l(\beta)\geq 1$ such that,

$$\int_{\{|u^\sigma|\geq l(\beta)\}}g^\sigma(x,u^\sigma,\nabla u^\sigma)\,dx<\frac{\beta}{2}. \tag{4.52}$$

For any measurable subset $E\subset\Omega$, we have

$$\begin{aligned}\int_E|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx&\leq\int_\Omega b(l(\beta))(c(x)+M((\nabla T_{l(\beta)}(u^\sigma)))\,dx\\&+\int_{\{|u^\sigma|\geq l(\beta)\}}|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx.\end{aligned}$$

In view of (4.51) there exist $\alpha(\beta)>$ such that

$$\int_E b(l(\beta))(c(x)+M(|(\nabla T_{l(\beta)}(u^\sigma)|)\,dx\leq\frac{\eta}{2}. \tag{4.53}$$

Finally, combining (4.52) and (4.53), one easily has $\displaystyle\int_E|g^\sigma(x,u^\sigma,\nabla u^\sigma)|\,dx\leq\eta$ for all E such that meas$(E)\leq\alpha(\beta)$.

So, as in 4.1.7, we can pass to the limit in σ and conclude. This achieves the proof of Theorem 4.1.

Remark 4.3. If we suppose that the source term f is no positive, then the unique positive solution of the problem (1.1) is the vanished function.

Indeed: If we take $v=0$ in (P), we have

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u)\,dx+\int_\Omega g(x,u,\nabla u)T_k(u)\,dx\leq\int_\Omega fT_k(u)\,dx.$$

Since $g(x,u,\nabla u)\geq 0$ and $T_k(u)\geq 0$ we deduce,

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u)\,dx\leq\int_\Omega fT_k(u)\,dx.$$

On the other hand, thanks to (A_4) and the fact that $f\leq 0$ and $u\geq 0$, we conclude

$$\alpha\int_\Omega M(|\nabla T_k(u)|)\,dx\leq\int_\Omega fT_k(u)\,dx\leq 0.$$

We can easily deduce that $T_k(u)=0$, $\forall\ k\geq 0$ by letting k tends to infinity, we have

$$u=0.$$

5. CASE WHERE THE NONLINEARITY g IS NEGATIVE

We consider,

$$\overline{K}_0 = \{u \in W_0^1 L_M(\Omega);\ u \le 0\ \ a.e. \text{ in } \Omega\}.$$

This convex set is sequentially $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closed in $W_o^1 L_M(\Omega)$ (see [14]). The nonlinearity term g is supposed a non-positive function.

Theorem 5.1. *Assume that* (A_1) - (A_4), (G_1) *and* (G_2) *hold true and that* $f \in L^1(\Omega)$. *Then there exists at least one solution of the following unilateral problem,*

$$(P)\begin{cases} u \in \tau_0^{1,M}(\Omega), u \le 0\ a.e. \text{ in } \Omega, \\ g(x,u,\nabla u) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u-v)\,dx \\ \displaystyle\qquad\qquad\qquad \le \int_\Omega f T_k(u-v)\,dx, \\ \forall\ v \in \bar{K}_0 \cap L^\infty(\Omega),\ \forall k > 0. \end{cases}$$

Proof. The same proof as in Theorem 4.1 can be applied with the following changements:

i) The Lipschitz function $\delta_\sigma(s)$ is replaced by.

$$\overline{\delta}_\sigma(s) = \begin{cases} \frac{-s-\sigma}{s} & \text{if } s \ge \sigma > 0 \\ 0 & \text{if } |s| \le \sigma \\ \frac{s+\sigma}{s} & \text{if } s < -\sigma < 0. \end{cases}$$

ii) The approximate problem becomes :

$$(\bar{P}_\epsilon^\sigma)\begin{cases} u_\varepsilon^\sigma \in W_0^1 L_M(\Omega) \\ \displaystyle\int_\Omega \langle A u_\epsilon^\sigma, u_\epsilon^\sigma - v\rangle + \int g_\epsilon^\sigma(x, u_\epsilon^\sigma, \nabla u_\epsilon^\sigma)(u_\epsilon^\sigma - v)\,dx + \frac{1}{\varepsilon^2}\int_\Omega m(T_{\frac{1}{\varepsilon}}(u_\varepsilon^{\sigma +}))(u_\varepsilon^\sigma - v)\,dx \\ \displaystyle\qquad\qquad\qquad = \int_\Omega f_\varepsilon(u_\varepsilon^\sigma - v)\,dx, \\ \forall\, v \in W_0^1 L_M(\Omega). \end{cases}$$

iii) The set K_0 considered in Remark 4.1, will be replaced by,

$$\overline{K}_0 = \{u \in W_0^1 L_M(\Omega);\ u \le 0\ \ a.e. \text{ in } \Omega\}.$$

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