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ON A CLASS OF UNIFORMLY CONVEX FUNCTIONS DEFINED BY CONVOLUTION WITH FIXED COEFFICIENT

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ABSTRACT. We define a new subclass of uniformly convex functions with negative and fixed second coefficients defined by convolution. The main object of this paper is to obtain coefficient estimates distortion bounds, closure theorems and extreme points for functions belong to this new class . The results are generalized to families with fixed finitely many coefficients.

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1. INTRODUCTION

Let $\mathcal{A}$ denote the family of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disc $\Delta = \{z : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of of order α, $0 \le \alpha < 1$ in Δ, written as $f \in S^*(\alpha)$, if $Re[zf'(z)/f(z)] > \alpha$ in Δ. A function $f \in \mathcal{A}$ is said to be convex of order $\alpha \quad 0 \le \alpha < 1$ in Δ, written as $f \in C(\alpha)$, if and only if $zf' \in S^*(\alpha)$ in $\triangle$. The class $UST(\alpha, \beta)$ is defined as the class of functions $f \in \mathcal{A}$ satisfying the inequality

$$\text{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \ge \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \forall z \in \Delta,\ 0 \le \alpha < 1,\ \beta \ge 0. \tag{1.2}$$

We note that for $\beta > 1$, if $f \in UST(\alpha, \beta)$ then $zf'(z)/f(z)$ lies in the region $G{\equiv}G(\alpha,\beta){\equiv}\{w : Re(w) > \beta|w-1|+\alpha\}$, that is, part of the complex plane which contains $w = 1$ and is bounded by the ellipse $(u-(\beta^2-\alpha)/(\beta^2-1))^2+(\beta^2/(\beta^2-1))v^2 = \beta^2(1-\alpha)^2/(\beta^2-1)^2$ with vertices at the points $((\beta+\alpha)/(\beta+1), 0)$, $((\beta-\alpha)/(\beta-1), 0)$, $((\beta^2-\alpha)/(\beta^2-1), (\alpha-1)/\sqrt{\beta^2-1})$, and $((\beta^2-\alpha)/(\beta^2-1), (-\alpha+1)/\sqrt{\beta^2-1})$. Since $\alpha < (\beta+\alpha)/(\beta+1) < 1 < (\beta-\alpha)/(\beta-1)$, we have $G \subset \{w : Re(w) > \alpha\}$ and so $U(\alpha, \beta){\subset}S^*(\alpha)$. For $\beta = 1$ if $f \in UST(\alpha, 1)$, then zf'/f covers the region which contains $w = 2$ and is bounded by the parabola $u = (v^2 + 1 - \alpha^2)/2(1-\alpha)$. Using the relation between convex and starlike functions, we define $UCV(\alpha, \beta)$ as the class of functions $f \in A$ if and only if $zf' \in UST(\alpha, \beta)$. Ronning [8] investigated the class $UCV(\alpha, 1)$ of uniformly convex functions of order α, but the class $UCV(0, 1){\equiv}UCV$ of uniformly convex functions was first defined by Goodman [3, 4]. See also Ronning [7] and Shams et al. [11].

Let the functions $\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ and $\Psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ be analytic in Δ, where $\lambda_n \ge \mu_n \ge 0$. Murugusundaramoorthy and Jahangiri [6] defined the class $U(\Phi, \Psi, \alpha, \beta)$ to be the class of functions $f \in A$ satisfying the conditions $(f * \Psi)(z) \neq 0$ and

$$Re\left\{\frac{f * \Phi}{f * \Psi} - \alpha\right\} \ge \beta\left|\frac{f * \Phi}{f * \Psi} - 1\right|, \forall z \in \Delta,\ -1 \le \alpha < 1,\ \beta > 0. \tag{1.3}$$

We further, let $UT(\Phi, \Psi, \alpha, \beta){\equiv}U(\Phi, \Psi, \alpha, \beta) \cap T$ where T consists of functions $f \in \mathcal{A}$ which are of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \ge 0)$. The class T was introduced and studied by Silverman [12]. Here in this note the operator $(*)$ stands for the convolution of two power series $f(z) = z \pm \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z \pm \sum_{n=2}^{\infty} b_n z^n$ defined by $(f * g)(z) = f(z) * g(z) = z \pm \sum_{n=2}^{\infty} a_n b_n z^n$.

The families $U(\Phi, \Psi, \alpha, \beta)$ and $UT(\Phi, \Psi, \alpha, \beta)$ are comprehensive classes consisting of various well-know classes of analytic functions as well as many new ones. For example, for suitable choices of Φ, Ψ, α, and β we obtain

$$\begin{aligned} U\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \alpha, 0\right) &\equiv S^*(\alpha) \\ U\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, 0\right) &\equiv C(\alpha) \\ U\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \alpha, \beta\right) &\equiv UCT(\alpha, \beta) \end{aligned}$$

In the present investigation, we introduce a new class $UT_b(\Phi, \Psi, \alpha, \beta)$ $0 \leq b \leq 1$, with negative and fixed second coefficient. The main object of this paper is to obtain sufficient conditions for the functions $f \in UT_b(\Phi, \Psi, \alpha, \beta)$. Furthermore we obtain extreme points, distortion bounds and closure properties for $f \in UT_b(\Phi, \Psi, \alpha, \beta)$. We need the following theorem for our investigation established by Murugusundaramoorthy and Jahangiri [6] .

Theorem 1. *A necessary and sufficient condition for $f(z)$ of the form (1.3) to be in the class $UT(\Phi, \Psi, \alpha, \beta)$, $\quad -1 \leq \alpha < 1, \quad \beta > 0$ is that*

$$\sum_{n=2}^{\infty} [(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]\, a_n \leq 1-\alpha. \tag{1.4}$$

Corollary 1. *Let the function f defined by (1.3) be in the class $UT(\Phi, \Psi, \alpha, \beta)$. Then*

$$a_n \leq \frac{(1-\alpha)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]}, n \geq 2, -1 \leq \alpha < 1, \beta > 0.$$

Remark. In view of Theorem 1, we can see that $f(z)$ of the form (1.3) is in the class $UT(\Phi, \Psi, \alpha, \beta)$ then,

$$a_2 = \frac{(1-\alpha)}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2}. \tag{1.5}$$

By fixing the second coefficient, we introduce a new subclass $UT_b(\Phi, \Psi, \alpha, \beta)$ of $UT(\Phi, \Psi, \alpha, \beta)$ and obtain the following theorems.

Let $UT_b(\Phi, \Psi, \alpha, \beta)$ denote the class of functions f in $UT(\Phi, \Psi, \alpha, \beta)$ and of the form

$$f(z) = z - \frac{b(1-\alpha)}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} a_n z^n (a_n \geq 0, 0 \leq b \leq 1). \tag{1.6}$$

2. THE CLASS $UT_b(\Phi, \Psi, \alpha, \beta)$

Theorem 2. *Let the function $f(z)$ be defined by (1.6). Then $f \in UT_b(\Phi, \Psi, \alpha, \beta)$ if and only if*

$$\sum_{n=3}^{\infty} [(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]\, a_n \leq (1-b)(1-\alpha), \tag{2.1}$$

$-1 \leq \alpha < 1, \beta > 0.$

Proof. Substituting

$$a_2 = \frac{(1-\alpha)}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2}, 0 \leq b \leq 1$$

in (1.6) and simple computation leads to the desired result. ∎

Corollary 2. *Let the function f defined by (1.6) be in the class $UT_b(\Phi, \Psi, \alpha, \beta)$. Then*

$$a_n \leq \frac{(1-\alpha)(1-b)}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2}, n \geq 3, -1 \leq \alpha < 1, \beta > 0. \tag{2.2}$$

Theorem 3. *The class $UT_b(\Phi, \Psi, \alpha, \beta)$ is closed under convex linear combination.*

Proof. Let the function f be defined by (1.6) and $g(z)$ defined by

$$g(z) = z - \frac{(1-\alpha)b}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} d_n z^n, \tag{2.3}$$

where $d_n \geq 0$ and $0 \leq b \leq 1$.

Assuming that $f(z)$ and $g(z)$ are in the class $UT_b(\Phi, \Psi, \alpha, \beta)$, it is sufficient to prove that the function $H(z)$ defined by

$$H(z) = \mu f(z) + (1-\mu) g(z), (0 \leq \mu \leq 1) \tag{2.4}$$

is also in the class $UT_b(\Phi, \Psi, \alpha, \beta)$.
Since

$$H(z) = z - \frac{(1-\alpha)b}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} \{\mu a_n + (1-\mu) d_n\} z^n, \tag{2.5}$$

$a_n \geq 0, d_n \geq 0, 0 \leq b \leq 1$, we observe that,

$$\sum_{n=3}^{\infty} \frac{(1-\alpha)b}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2} (\mu a_n + (1-\mu) d_n) \leq (1-b)(1-\alpha) \tag{2.6}$$

which is in view of Theorem 2, again, implies that $H \in UT_b(\Phi, \Psi, \alpha, \beta)$ which completes the proof of the Theorem 3. ∎

Theorem 4. *Let the functions*

$$f_j(z) = z - \frac{(1-\alpha)b}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} a_{n,\,j} z^n, a_{n,j} \geq 0 \tag{2.7}$$

be in the class $UT_b(\Phi, \Psi, \alpha, \beta)$ for every j $(j = 1, 2, \ldots m)$. Then the function $F(z)$ defined by

$$F(z) = \sum_{j=1}^{m} \eta_j f_j(z), \tag{2.8}$$

is also in the class $UT_b(\Phi, \Psi, \alpha, \beta)$, where

$$\sum_{j=1}^{m} \eta_j = 1. \tag{2.9}$$

Proof. Combining the definitions (2.7) and (2.8), further by (2.9) we have

$$F(z) = z - \frac{(1-\alpha)b}{(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2} z^2 - \sum_{n=3}^{\infty} \left(\sum_{j=1}^{m} \eta_j a_{n,j} \right) z^n. \tag{2.10}$$

Since $f_j(z) \in UT_b(\Phi, \Psi, \alpha, \beta)$ for every $j = 1, 2, \ldots, m$, Theorem 2 yields

$$\sum_{n=3}^{\infty} [(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]\, a_{n,j} \le (1-b)(1-\alpha), \tag{2.11}$$

for $j = 1, 2, \ldots, m$. Thus we obtain

$$\begin{aligned} &\sum_{n=3}^{\infty} \; [(1+\beta)\lambda_n - (\alpha+\beta)\mu_n] \left(\sum_{j=1}^{m} \eta_j a_{n,j} \right) \\ &= \sum_{j=1}^{m} \eta_j \left(\sum_{n=3}^{\infty} [(1+\beta)\lambda_n - (\alpha+\beta)\mu_n] a_{n,j} \right) \\ &\le (1-b)(1-\alpha) \end{aligned}$$

in view of Theorem 2. So, $F \in UT_b(\Phi, \Psi, \alpha, \beta)$. ∎

Theorem 5. *Let*

$$f_2(z) = z - \frac{b(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]} z^2 \tag{2.12}$$

and

$$f_n(z) = z - \frac{b(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]} z^2 - \frac{(1-b)(1-\alpha)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]} z^n \tag{2.13}$$

for $n = 3, 4, \ldots$. Then f is in the class $UT_b(\Phi, \Psi, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \sigma_n f_n(z), \tag{2.14}$$

where $\sigma_n \ge 0$ and $\sum\limits_{n=2}^{\infty} \sigma_n = 1$.

Proof. We suppose that $f(z)$ can be expressed from (2.14). Then we have

$$\begin{aligned} f(z) &= z - \frac{b(1-\beta)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]} z^2 - \sum_{n=3}^{\infty} \sigma_n \frac{(1-b)(1-\alpha)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]} z^n \\ &= z - \sum_{n=2}^{\infty} A_n z^n, \end{aligned} \tag{2.15}$$

where

$$A_2 = \frac{b(1-\beta)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]} \tag{2.16}$$

and

$$A_n = \sigma_n \frac{(1-b)(1-\alpha)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]}, \; n = 3, 4, \ldots \tag{2.17}$$

Therefore,

$$\begin{aligned} \sum_{n=2}^{\infty}[(1+\beta)\lambda_n-(\alpha+\beta)\mu_n]A_n &= b(1-\beta)+\sum_{n=3}^{\infty}\sigma_n(1-b)(1-\beta) \\ &= (1-\beta)[b+(1-\sigma_2)(1-b)] \\ &\leq (1-\alpha), \end{aligned} \tag{2.18}$$

and it follows from Theorem 1 and Theorem 2, f is in the class $UT_b(\Phi,\Psi,\alpha,\beta)$. Conversely, we suppose that $f(z)$ defined by (1.6) is in the class $UT_b(\Phi,\Psi,\alpha,\beta)$. Then by using (2.2), we get

$$a_n \leq \frac{(1-b)(1-\alpha)}{[(1+\beta)\lambda_n-(\alpha+\beta)\mu_n]}, (n\geq 3). \tag{2.19}$$

Setting

$$\sigma_n = \frac{[(1+\beta)\lambda_n-(\alpha+\beta)\mu_n]}{(1-b)(1-\alpha)}a_n, (n\geq 3) \tag{2.20}$$

and

$$\sigma_2 = 1-\sum_{n=3}^{\infty}\sigma_n, \tag{2.21}$$

we have (2.14). This completes the proof of Theorem 6. ∎

Corollary 3. *The extreme points of the class $UT_b(\Phi,\Psi,\alpha,\beta)$ are functions $f_n(z)$, $n\geq 3$ given by Theorem 6.*

3. DISTORTION THEOREMS

In order to obtain distortion bounds for function $f\in UT_b(\Phi,\Psi,\alpha,\beta)$ first we prove the following lemmas.

Lemma 1. *Let the function $f_3(z)$ be defined by*

$$f_3(z) = z-\frac{b(1-\alpha)}{(1+\beta)\lambda_2-(\alpha+\beta)\mu_2}z^2-\frac{(1-b)(1-\alpha)}{(1+\beta)\lambda_3-(\alpha+\beta)\mu_3}z^3\,. \tag{3.1}$$

Then, for $0\leq r<1$ and $0\leq b\leq 1$,

$$|f_3(re^{i\theta})| \geq r-\frac{b(1-\alpha)}{(1+\beta)\lambda_2-(\alpha+\beta)\mu_2}r^2-\frac{(1-b)(1-\alpha)}{(1+\beta)\lambda_3-(\alpha+\beta)\mu_3}r^3 \tag{3.2}$$

with equality for $\theta=0$. For either $0\leq b<b_0$ and $0\leq r\leq r_0$ or $b_0\leq b\leq 1$,

$$|f_3(re^{i\theta})| \leq r+\frac{b(1-\alpha)}{(1+\beta)\lambda_2-(\alpha+\beta)\mu_2}r^2-\frac{(1-b)(1-\alpha)}{(1+\beta)\lambda_3-(\alpha+\beta)\mu_3}r^3 \tag{3.3}$$

with equality for $\theta=\pi$, where

$$\begin{aligned} b_0 &= \frac{1}{2(1-\alpha)}\{-\{[(1+\beta)\lambda_3-(\alpha+\beta)\mu_3]+4[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2]-(1-\alpha)\} \\ &\quad +\left(\{[(1+\beta)\lambda_3-(\alpha+\beta)\mu_3]+4[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2]-(1-\alpha)\}^2 - \right. \\ &\quad \left. 16(1-\alpha)[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2]\right)^{1/2} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} r_0 &= \frac{1}{b(1-b)(1-\alpha)}\{-2(1-b)\{(1+\beta)\lambda_2-(\alpha+\beta)\mu_2\} \\ &\quad +[4(1-b)^2\{(1+\beta)\lambda_2-(\alpha+\beta)\mu_2\}^2+ \\ &\quad b^2(1-b)(1-\alpha)[(1+\beta)\lambda_3-(\alpha+\beta)\mu_3]^{1/2}\}. \end{aligned} \tag{3.5}$$

Proof. We employ the technique as used by Silverman and Silvia [13]. Since

$$\begin{aligned} \frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta} &= 2(1-\alpha)r^3\sin\theta\left\{\frac{b}{(1+\beta)\lambda_2-(\alpha+\beta)\mu_2}+\frac{4(1-b)}{(1+\beta)\lambda_3-(\alpha+\beta)\mu_3}r\cos\theta\right. \\ &\quad \left. -\frac{b(1-b)(1-\beta)}{[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2](1+\beta)[\lambda_3-(\alpha+\beta)\mu_3]}\right\} \end{aligned} \tag{3.6}$$

we can see that

$$\frac{\partial|f_3(re^{i\theta})|^2}{\partial\theta}=0 \tag{3.7}$$

for $\theta_1=0$, $\theta_2=\pi$, and

$$\theta_3=\cos^{-1}\left(\frac{b[(1-b)(1-\alpha)r^2-b[(1+\beta)\lambda_3-(\alpha+\beta)\mu_3]}{4r(1-b)[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2]}\right) \tag{3.8}$$

since θ_3 is valid root only when $-1\le\cos\theta_3\le 1$. Hence we have a third root if and only if $r_0\le r<1$ and $0\le b\le b_0$. Thus the results of the theorem follow from comparing the extremal values $|f_3(re^{i\theta_k})|$, $k=1,2,3$ on the appropriate intervals. ∎

Lemma 2. *Let the functions $f_n(z)$ be defined by (2.13) and $n\ge 4$. Then*

$$|f_n(re^{i\theta})|\le|f_4(-r)|. \tag{3.9}$$

Proof. Since

$$f_n(z)=z-\frac{b(1-\beta)}{(2+k-\beta)(1+\lambda)}z^2-\frac{(1-b)(1-\beta)}{[n(1+k)-(k+\beta)](1+n\lambda-\lambda)}z^n$$

and $\dfrac{r^n}{n}$ is a decreasing function of n, we have

$$\begin{aligned} |f_n(re^{i\theta}| &\le r+\frac{b(1-\alpha)}{[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2]}r^2-\frac{(1-b)(1-\alpha)}{[(1+\beta)\lambda_4-(\alpha+\beta)\mu_4]}r^4 \\ &= -f_4(-r) \end{aligned}$$

which shows (3.9). ∎

Theorem 6. *Let the function $f(z)$ defined by (1.6) belongs to the class $UT_b(\Phi,\Psi,\alpha,\beta)$, then for $0\le r<1$,*

$$|f(re^{i\theta})|\ge r-\frac{b(1-\alpha)}{[(1+\beta)\lambda_2-(\alpha+\beta)\mu_2]}r^2-\frac{(1-b)(1-\alpha)}{[(1+\beta)\lambda_3-(\alpha+\beta)\mu_3]}r^3 \tag{3.10}$$

with equality for $f_3(z)$ at $z=r$, and

$$|f(re^{i\theta})|\le\max\{\max_\theta|f_3(re^{i\theta})|,-f_4(-r)\}, \tag{3.11}$$

where $\max_\theta|f_3(re^{i\theta})|$ is given by Lemma 1.

Proof. The proof of Theorem 6, is obtained by comparing the bounds of Lemma 1, and Lemma 2. ∎

Remark: Taking $b = 1$ in Theorem 6, we obtain the following result.

Corollary 4. *Let the function $f(z)$ defined by (1.6) be in the class $UT_b(\Phi, \Psi, \alpha, \beta)$. Then for $|z| = r < 1$, we have*

$$r - \frac{(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r^2 \le |f(z)| \le r + \frac{(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r^2 \tag{3.12}$$

Lemma 3. *Let the function $f_3(z)$ be defined by (3.1). Then, for $0 \le r < 1$, and $0 \le b \le 1$,*

$$|f_3'(re^{i\theta})| \ge 1 - \frac{2b(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r - \frac{3(1-b)(1-\alpha)}{[(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3]}r^2 \tag{3.13}$$

with equality for $\theta = 0$. For either $0 \le b < b_1$ and $0 \le r \le r_1$ or $b_1 \le b \le 1$,

$$|f_3'(re^{i\theta})| \le 1 + \frac{2b(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r - \frac{3(1-b)(1-\alpha)}{[(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3]}r^2 \tag{3.14}$$

with equality for $\theta = \pi$, where

$$\begin{aligned} b_1 &= \frac{1}{6(1-\alpha)}\{-\{[(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3] - 6[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2] - 3(1-\alpha)\}\} \\ &\quad +\{([(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3] - 6[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2] - 3(1-\alpha))^2 + \\ &\quad 72(1-\alpha)[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]\}^{1/2}\} \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} r_1 &= \frac{1}{3b(1-b)(1-\alpha)}\{-3(1-b)[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2] \\ &\quad +[9(1-b)^2[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]^2 + \\ &\quad 3b^2(1-b)(1-\alpha)[(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3]]^{1/2}\}. \end{aligned} \tag{3.16}$$

Proof. The proof of Lemma 3, is much akin to the proof of Lemma 3. ∎

Theorem 7. *Let the function $f(z)$ defined by (1.6) belongs to the class $UT_b(\Phi, \Psi, \alpha, \beta)$, then for $0 \le r < 1$,*

$$|f'(re^{i\theta})| \ge 1 - \frac{2b(1-\alpha)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r - \frac{3(1-b)(1-\alpha)}{[(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3]}r^2 \tag{3.17}$$

with equality for $f_3'(z)$ at $z = r$, and

$$|f'(re^{i\theta})| \le \max\{\max_{\theta}|f_3'(re^{i\theta})|, -f_4'(-r)\}, \tag{3.18}$$

where $\max_{\theta}|f_3'(re^{i\theta})|$ is given by Lemma 3.

Remark: Putting $b = 1$ in Theorem 7, we obtain the following result.

Corollary 5. *Let the function $f(z)$ defined by (1.6) be in the class $UT_b(\Phi, \Psi, \alpha, \beta)$. Then for $|z| = r < 1$, we have*

$$1 - \frac{2(1-\beta)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r \le |f'(z)| \le 1 + \frac{2(1-\beta)}{[(1+\beta)\lambda_2 - (\alpha+\beta)\mu_2]}r \tag{3.19}$$

4. THE CLASS $UT_{b_n,m}(\Phi, \Psi, \alpha, \beta)$

Instead of fixing just the second coefficient, we can fix finitely many coefficients. Let $UT_{b_n,m}(\Phi, \Psi, \alpha, \beta)$ denote the class of functions in $UT_b(\Phi, \Psi, \alpha, \beta)$ of the form

$$f(z) = z - \sum_{n=2}^{k} \frac{b_n(1-\alpha)}{[(1+\beta)\lambda_3 - (\alpha+\beta)\mu_3]} z^n - \sum_{n=m+1}^{\infty} a_n z^n, \tag{4.1}$$

where $0 \leq \sum_{n=2}^{m} b_n = b \leq 1$. Note that $UT_{b_2,2}(\Phi, \Psi, \alpha, \beta) = UT_b(\Phi, \Psi, \alpha, \beta)$.

Theorem 8. *The extreme points of the class $U_{b_n,m}(k, \lambda, \beta)$ are*

$$f_k(z) = z - \sum_{n=2}^{m} \frac{b_n(1-\alpha)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]} z^n$$

and

$$f_n(z) = z - \sum_{n=2}^{m} \frac{b_n(1-\alpha)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]} z^n - \sum_{n=m+1}^{\infty} \frac{(1-b)(1-\beta)}{[(1+\beta)\lambda_n - (\alpha+\beta)\mu_n]} z^n$$

The details of the proof are omitted. Since the characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for $UT_b(\Phi, \Psi, \alpha, \beta)$.

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