The Australian Journal of Mathematical Analysis and Applications

http://ajmaa.org

Volume 7, Issue 1, Article 17, pp. 1-14, 2010

APPROXIMATION OF DERIVATIVES IN A SINGULARLY PERTURBED SECOND ORDER ORDINARY DIFFERENTIAL EQUATION WITH DISCONTINUOUS TERMS ARISING IN CHEMICAL REACTOR THEORY

R. MYTHILI PRIYADHARSHINI AND N. RAMANUJAM

Received 12 March, 2008; accepted 29 July, 2008; published 16 November, 2010.

DEPARTMENT OF MATHEMATICS, BHARATHIDASAN UNIVERSITY, TIRUCHIRAPPALLI - 620 024, TAMILNADU, INDIA
matram2k3@yahoo.com
URL: http://www.bdu.ac.in/depa/science/ramanujam.htm

ABSTRACT. In this paper, a singularly perturbed second order ordinary differential equation with a discontinuous convection coefficient arising in chemical reactor theory is considered. A robust-layer-resolving numerical method is suggested. An ε-uniform global error estimate for the numerical solution and also to the numerical derivative are established. Numerical results are provided to illustrate the theoretical results.

Key words and phrases: Singular perturbation problem, Piecewise uniform mesh, Discrete derivative, Discontinuous convection coefficient, Chemical reactor theory.

2000 *Mathematics Subject Classification.* Primary 65L10, CR G1.7, 65L12. Secondary 65L70, 34B60.

ISSN (electronic): 1449-5910

The first author wishes to acknowledge the Council of Scientific and Industrial Research, New Delhi, India for its financial support.

1. INTRODUCTION

The theory of singular perturbation is not a settled direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving; large or small parameters, become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in Fluid Mechanics, edge layers in Solid Mechanics, skin layers in Electrical Applications, shock layers in Fluid and Solid Mechanics, transition points in Quantum Mechanics and surfaces in Mathematics.

In particular, boundary-value problems(BVPs) of the form

$$\begin{aligned} \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) &= f(x), \quad x \in \Omega \equiv (0,1), \\ -u'(0) = A, &\qquad u(1) + \varepsilon u'(1) = B, \end{aligned}$$

and

$$\begin{aligned} \varepsilon u''(x) + a(x)u'(x) &= b(x,y), \quad x \in \Omega, \\ -u'(0) = A, &\qquad u(1) + \varepsilon u'(1) = B, \end{aligned}$$

arise in the study of adiabatic tubular chemical flow reactors with axial diffusion [1]. In [1], O'Malley obtained asymptotic solutions of the BVPs arising in chemical reactor theory. It may be noted that the asymptotic solution constructed in [1] converge uniformly to the solution of the reduced problem of the given problem throughout the interval $[0,1]$ while the derivatives generally converge non-uniformly as $\varepsilon \to 0$ either at $x = 0(a(x) \geq \alpha > 0)$ or at $x = 1(a(x) \leq \alpha < 0)$. In [6]-[8], this type of problems are considered and computational methods are suggested.

Various methods for the numerical solution of problem involving singularly perturbed second order ordinary differential equations with non - smooth data (discontinuous source term / convection coefficient) using special piecewise uniform meshes (Shishkin mesh and Bakhvalov mesh) have been considered widely in the literature (see [1] - [16] and references therein). While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications. It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution.

In [2], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain where as Natalia Kopteva and Martin Stynes [4] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. It may be noted that the source term and convection coefficient are smooth for the problem considered in [2, 4]. R. Mythili Priyadharshini and N. Ramanujam [15], have determined estimate for the scaled derivative for a singularly perturbed reaction-convection-diffusion problem with two parameters.

In [13], the authors have obtained bounds on the errors in approximations to the scaled derivative in the whole domain in the case of discontinuous source term. R. Mythili Priyadharshini and N. Ramanujam [14], have determined estimate for the scaled derivative in the boundary layer region and non-scaled derivative in the outer region for the boundary value problems with Robin type boundary conditions and discontinuous convection coefficient and source term. Zhongdi Cen [9] has suggested a hybrid finite difference scheme for singularly perturbed convection - diffusion problem with discontinuous convection coefficient. As far as author's knowledge goes no work has been reported in the literature for finding approximation to scaled derivatives of the solution for problems having discontinuous convection coefficient for both upwind and hybrid finite difference schemes on Shishkin mesh.

Motivated by the works given in [7] and [10], the present paper consider the above singularly perturbed second order ordinary differential equation with a discontinuous convection coefficient and source term. In case of smooth data, the solution of the problem considered above exhibits a weak boundary layer. In case of non-smooth data, the solution and its derivatives exhibits a strong interior layers. Thus the analytical techniques developed in [10] are extended in a natural way to the problems considered in this paper. Since derivatives are related to flux or drag in physical and chemical applications, we obtain parameter-uniform approximations not only to the solution but also to its derivatives. Thus in this paper, motivated by the works of [4], bounds on the errors in approximating the first derivative of the solution in the fine mesh as well as in the coarse mesh are obtained separately.

Note: Through out this paper, C denotes a generic constant (sometimes subscripted) is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Note that C can take different values at different place, even in the same argument. Let $y : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\parallel y \parallel = \sup\limits_{x\in D} |y(x)|$, [2, 3].

2. CONTINUOUS PROBLEM

Consider the singularly perturbed second order ordinary differential equation with a discontinuous convection term on the unit interval $\Omega = (0, 1)$.

$$(2.1)\quad (P_\varepsilon) : \begin{cases} \text{Find} \quad u \in Y \equiv C^1(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+) \quad \text{such that} \\ Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in \Omega^- \cup \Omega^+ \\ B_0u(0) \equiv -u'(0) = A, \quad B_1u(1) \equiv u(1) + \varepsilon u'(1) = B, \\ a(x) \leq -\alpha_1 < 0 \quad \text{on } [0, d], \quad a(x) \geq \alpha_2 > 0 \quad \text{on } [d, 1], \\ b(x) \geq \beta > 0 \quad \text{on } [0, 1], \quad \alpha_1 < \beta, \\ |[a](d)| \leq C, \quad |[f](d)| \leq C, \end{cases}$$

where $0 < \varepsilon \ll 1$, $d \in \Omega$, $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$. For the functions $a(x)$ and $f(x)$ we assume they are sufficiently smooth on $\Omega^- \cup \Omega^+$. Further it is assumed that $f(x)$ and its derivatives have right and left limits at $x = d$. We denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$.

In the following, the maximum principle for (2.1) is established. Then using this principle, a stability result is derived.

Theorem 2.1. *Suppose that a function $u \in Y$ satisfies $B_0u(0) \geq 0$, $B_1u(1) \geq 0$, $Lu(x) \leq 0$, for $x \in \Omega^- \cup \Omega^+$ and $[u'](d) \leq 0$. Then $u(x) \geq 0$, for all $x \in \overline{\Omega}$.*

Proof. Using the method adopted in [11, 12], and the test function $s(x)$ as

$$s(x) = \begin{cases} 1/2 - x/8 + d/8, & x \in \Omega^- \cup \{0, d\} \\ 1/2 - x/4 + d/4, & x \in \Omega^+ \cup \{1\}, \end{cases}$$

the present theorem can be proved. ∎

Lemma 2.2. *If* $u \in Y$ *then*

$$\| u \| \leq C \max\{|B_0 u(0)|, |B_1 u(1)|, \| Lu \|_{\Omega^- \cup \Omega^+}\}.$$

Proof. Using appropriate barrier functions and applying Theorem 2.1, the present lemma can be proved. ∎

The sharper bounds on the derivatives of the solution are obtained by decomposing the solution as $u = v + w$, where $v = v_0 + \varepsilon v_1$. Here v_0 and v_1 are defined respectively to be the solutions of the problems

$$\begin{aligned} a(x)v_0'(x) - b(x)v_0(x) &= f(x), \quad x \in \Omega^- \cup \Omega^+, \\ -v_0'(0) = A, &\qquad v_0(1) = B \end{aligned}$$

and

$$\begin{aligned} a(x)v_1'(x) - b(x)v_1(x) &= -v_0'', \quad x \in \Omega^- \cup \Omega^+, \\ -v_1'(0) = 0, &\qquad v_1(1) + \varepsilon v_1'(1) = -v_0'(1). \end{aligned}$$

Thus, v is define by

$$Lv(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{2.2}$$

$$-v'(0) = A, \qquad v(d-) = v_0(d-) + \varepsilon v_1(d-), \tag{2.3}$$

$$v(d+) = v_0(d+) + \varepsilon v_1(d+), \qquad v(1) + \varepsilon v'(1) = B. \tag{2.4}$$

Now, we define the layer component of the decomposition as follows :

$$Lw(x) = 0, \qquad x \in \Omega^- \cup \Omega^+, \tag{2.5}$$

$$-w'(0) = -u'(0) + v'(0), \qquad [w](d) = -[v](d), \tag{2.6}$$

$$[w'](d) = -[v'](d), \qquad w(1) + \varepsilon w'(1) = 0. \tag{2.7}$$

Hence $w(d-) = u(d-) - v(d-)$ and $w(d+) = u(d+) - v(d+)$.

Lemma 2.3. *For each integer* k*, satisfying* $0 \leq k \leq 3$*, the solutions* v *and* w *of (2.2-2.4) and (2.5-2.7) respectively satisfy the following bounds*

$$\| v \| \leq C, \quad \| v^{(k)} \|_{\Omega^- \cup \Omega^+} \leq C(1 + \varepsilon^{2-k}),$$
$$|[v](d)|, \ |[v'](d)|, \ |[v''](d)| \leq C$$

and

$$|w^{(k)}(x)| \leq \begin{cases} C\varepsilon^{1-k} e^{-(d-x)\alpha_1/\varepsilon}, & x \in \Omega^-, \\ C\varepsilon^{1-k} e^{-(x-d)\alpha_2/\varepsilon}, & x \in \Omega^+. \end{cases}$$

Proof. Using the technique adopted in [2] and applying the argument separately on each of the subintervals Ω^- and Ω^+, the present theorem can be proved. ∎

3. DISCRETE PROBLEM

A fitted mesh method for the Problem (2.1) is now introduced. On Ω a piecewise uniform mesh of N mesh interval is constructed as follows. The domain $\overline{\Omega}$ is subdivided into the four subintervals $[0, d - \sigma_1] \cup [d - \sigma_1, d] \cup [d, d + \sigma_2] \cup [d + \sigma_2, 1]$ for some σ_1, σ_2 that satisfy $0 < \sigma_1 \leq \frac{d}{2}, \quad 0 < \sigma_2 \leq \frac{1-d}{2}$. On each subinterval a uniform mesh with $N/4$ mesh-intervals is placed. The interior points of the mesh are denoted by $\Omega^N = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \cup \{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\}$. Clearly $x_{N/2} = d$ and $\overline{\Omega}^N = \{x_i\}_0^N$. We now introduce the following notations for the four mesh widths $h_1 = \frac{4(d-\sigma_1)}{N}, \quad h_2 = \frac{4\sigma_1}{N}, \quad h_3 = \frac{4\sigma_2}{N}$ and $h_4 = \frac{4(1-d-\sigma_2)}{N}$. It is fitted to the singular perturbation problem (2.1) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1 = \min\{\frac{d}{2}, \frac{2\varepsilon}{\alpha}\ln N\} \quad \text{and} \quad \sigma_2 = \min\{\frac{1-d}{2}, \frac{2\varepsilon}{\alpha}\ln N\},$$

where $\alpha = \min\{\alpha_1, \alpha_2\}$. Then the fitted mesh method for the problem (2.1) is

$$P_\varepsilon^N \begin{cases} L^N U(x_i) \equiv \varepsilon\delta^2 U(x_i) + a(x_i)DU(x_i) - b(x_i)U(x_i) = f(x_i), \quad x_i \in \Omega^N \\ B_0 U(x_0) \equiv -D^+U(x_0) = A, \ B_N U(x_N) \equiv U(x_N) + \varepsilon D^- U(x_N) = B. \\ D^- U(x_{N/2}) = D^+ U(x_{N/2}) \end{cases} \tag{3.1}$$

where $\delta^2 Z_i = \frac{D^+Z_i - D^-Z_i}{(x_{i+1}-x_{i-1})/2}$ and $DZ_i = \begin{cases} D^-Z_i, \ i \le N/2 \\ D^+Z_i, \ i > N/2, \end{cases}$ where D^+ and D^- are the standard forward and backward finite difference operators, respectively. Analogous to the continuous results stated in Theorem 2.1 and Lemma 2.2 one can prove the following results.

Theorem 3.1. *Suppose that a mesh function $Z(x_i)$ satisfies $B_0 Z(x_0) \ge 0$, $B_N Z(x_N) \ge 0$, $L^N Z(x_i) \le 0$, $x_i \in \Omega^N$ and $D^+Z(d) - D^-Z(d) \le 0$. Then $Z(x_i) \ge 0$ for all $x_i \in \overline{\Omega}^N$.*

Theorem 3.2. *If $U(x_i)$ is the solution of the problem (3.1), then*

$$|U(d)| \le C.$$

In order to obtain sharper error bound on the discrete derivative, we decompose the discrete solution as $U = V + W$, where $V = V_L + V_R$ and $W = W_L + W_R$.

Define the mesh functions V_L and V_R to be the solutions of the following discrete problems

$$L^N V_L(x_i) = f(x_i), \quad \text{for} \quad i = 1, ..., N/2 - 1, \tag{3.2}$$

$$-D^+V_L(x_0) = -v'(0), \quad V_L(x_{N/2}) = v(d-) \tag{3.3}$$

and

$$L^N V_R(x_i) = f(x_i), \quad \text{for} \quad i = N/2 + 1, ..., N - 1, \tag{3.4}$$

$$V_R(x_{N/2}) = v(d+), \quad V_R(x_N) + \varepsilon D^- V_R(x_N) = v(1) + \varepsilon v'(1). \tag{3.5}$$

Define the mesh functions W_L and W_R to be the solutions of the following system of finite difference equations

$$L^N W_L(x_i) = 0 \qquad \text{for} \quad i = 1, ..., N/2 - 1, \tag{3.6}$$

$$L^N W_R(x_i) = 0 \qquad \text{for} \quad i = N/2 + 1, ..., N - 1, \tag{3.7}$$

$$-D^+W_L(x_0) = 0, \qquad W_R(x_N) + \varepsilon D^- W_R(x_N) = 0, \tag{3.8}$$

$$W_R(x_{N/2}) + V_R(x_{N/2}) = W_L(x_{N/2}) + V_L(x_{N/2}), \tag{3.9}$$

$$D^+W_R(x_{N/2}) + D^+V_R(x_{N/2}) = D^-W_L(x_{N/2}) + D^-V_L(x_{N/2}). \tag{3.10}$$

Now, we can define $U(x_i)$ to be

$$U(x_i) = V(x_i) + W(x_i) = \begin{cases} V_L(x_i) + W_L(x_i), & \text{for} \quad x_i \in \{0\} \cup (\Omega^N \cap \Omega^-), \\ V_L(x_i) + W_L(x_i) & = V_R(x_i) + W_R(x_i), \ \text{for} \quad x_i = d, \\ V_R(x_i) + W_R(x_i), & \text{for} \quad x_i \in (\Omega^N \cap \Omega^+) \cup \{1\}. \end{cases} \tag{3.11}$$

Lemma 3.3. *At each mesh points $x_i \in \overline{\Omega}^N$, the smooth component of the error satisfies the estimate*

$$|(V - v)(x_i)| \le \begin{cases} C(d - x_i)N^{-1}, & \textit{for} \quad i = 0, ..., N/2 \\ C(3 - x_i)N^{-1}, & \textit{for} \quad i = N/2, ..., N. \end{cases}$$

Proof. We have the inequalities

$$\begin{aligned}|B_0(V-v)(x_0)| = |-D^+(V-v)(x_0)| &\leq C(x_{i+1}-x_i) \parallel v^{(2)} \parallel \\ &\leq CN^{-1}\end{aligned}$$

and

$$\begin{aligned}|B_N(V-v)(x_N)| = |(V-v)(x_N)+\varepsilon D^-(V-v)(x_N)| &\leq C\varepsilon(x_i-x_{i-1}) \parallel v^{(2)} \parallel \\ &\leq CN^{-1}.\end{aligned}$$

By standard local truncation error estimate and Lemma 2.3, we have

$$|L^N(V-v)(x_i)| \leq CN^{-1}.$$

Using the two mesh functions $\Psi^{\pm}(x_i) = \phi(x_i) \pm (V-v)(x_i)$, where

$$\phi(x_i) = \begin{cases} C(d-x_i)N^{-1}, & \text{for} \quad i = 0, ..., N/2 \\ C(3-x_i)N^{-1}, & \text{for} \quad i = N/2, ..., N, \end{cases}$$

we have,

$$B_0\Psi^{\pm}(x_0) = -D^+\phi(x_0) \mp D^+(V-v)(x_0) \geq 0,$$

$$\begin{aligned}L^N\Psi^{\pm}(x_i) &= a(x_i)D^-\phi(x_i) - b(x_i)C(d-x_i)N^{-1} \pm L^N(V-v)(x_i), \text{ for } i = 1, ..., \frac{N}{2}-1 \\ &\leq \alpha_1 CN^{-1} - \beta C(d-x_i)N^{-1} \pm CN^{-1}, \\ &< 0.\end{aligned}$$

Similarly, $L^N\Psi^{\pm}(x_i) \leq 0, \quad \text{for} \quad i = N/2+1, ..., N-1,$

$$B_N\Psi^{\pm}(x_N) = \phi(x_N) + \varepsilon D^-\phi(x_N) \pm B_N(V-v)(x_N) > 0$$

and

$$D^+\Psi^{\pm}(x_{N/2}) - D^-\Psi^{\pm}(x_{N/2}) = D^+\phi(x_{N/2}) - D^-\phi(x_{N/2}) = 0.$$

Applying Theorem 3.1, we get $\Psi^{\pm}(x_i) \geq 0, \forall x_i \in \overline{\Omega}^N$, which completes the prove. ∎

Theorem 3.4. *Let w be the solution of (2.5-2.7) and W the corresponding numerical solution of (3.6-3.10). Then at each mesh point $x_i \in \overline{\Omega}^N$, we have*

$$|(W-w)(x_i)| \leq CN^{-1}\ln N.$$

Proof. First we consider the case $\sigma_1 = \frac{2\varepsilon}{\alpha}\ln N$ and $\sigma_2 = \frac{2\varepsilon}{\alpha}\ln N$. Since $|U(x_{N/2})| \leq C$, we have $|W_L(x_{N/2})| \leq C$ and $|W_R(x_{N/2})| \leq C$. Using the arguments in [5], with the above transition parameters for $i \leq N/4$, we have

$$|W_L(x_i)| \leq CN^{-2}$$

and

$$|(W_L-w)(x_i)| \leq |W_L(x_i)| + |w(x_i)| \leq CN^{-2}. \tag{3.12}$$

Similarly for $i \geq 3N/4$, we have

$$|W_R(x_i)| \leq CN^{-2}$$

and

$$|(W_R-w)(x_i)| \leq |W_R(x_i)| + |w(x_i)| \leq CN^{-2}. \tag{3.13}$$

Therefore, it follows that

$$|(W_L-w)(x_{N/4})| \leq CN^{-2} \quad \text{and} \quad |(W_R-w)(x_{3N/4})| \leq CN^{-2}.$$

We have for $i = N/4+1, ..., N/2-1$,

$$|(L^N W_L - Lw)(x_i)| \leq \varepsilon h_2 |w^{(3)}(x_i)| + h_2 |w^{(2)}(x_i)| \leq \frac{Ch_2}{\varepsilon}.$$

Similarly for $i = N/2+1, ...3N/4-1$, we obtain

$$|(L^N W_R - Lw)(x_i)| \leq \varepsilon h_3 |w^{(3)}(x_i)| + h_3 |w^{(2)}(x_i)| \leq \frac{Ch_3}{\varepsilon}.$$

At the mesh point $x_{N/2} = d$, let h_2 and h_3 be the mesh interval sizes on either side of $x_{N/2}$ and $h = \max\{h_2, h_3\}$. Thus

$$\begin{aligned}
|(D^+ - D^-)(W-w)(x_{N/2})| &= |(D^+ - D^-)w(x_{N/2})| \\
&\leq |(D^+ - \frac{d}{dx})w(x_{N/2})| + |(D^- - \frac{d}{dx})w(x_{N/2})| \\
&\leq \frac{1}{2}h_3|w^{(2)}(x_i)| + \frac{1}{2}h_2|w^{(2)}(x_i)| \\
&\leq \frac{Ch}{2\varepsilon}.
\end{aligned}$$

Consider the mesh functions

$$\Psi^{\pm}(x_i) = \frac{Ch}{\varepsilon}\begin{cases} d - x_i, & x_i \in \Omega^N \cap (d-\sigma_1, d) \\ 3(d+\sigma_2) - x_i, & x_i \in \Omega^N \cap (d, d+\sigma_2). \end{cases}$$

Applying the discrete maximum principle to $\Psi^{\pm}(x_i) \pm (W-w)(x_i)$ over the interval $[d - \sigma_1, d+\sigma_2]$, we get the required result. Thus,

$$\begin{aligned}
|(W-w)(x_i)| &\leq \frac{Ch}{\varepsilon}, \quad \text{for} \quad i = 0, ..., N, \\
&\leq CN^{-1}\ln N.
\end{aligned}$$

Now we consider the case $\sigma_1 = \frac{d}{2}$ and $\sigma_2 = \frac{1-d}{2}$. In this case $\varepsilon^{-1} \leq C \ln N$. We have the inequalities

$$\begin{aligned}
B_0(W-w)(x_0) &= |-D^+(W-w)(x_0)| \\
&\leq C(x_{i+1} - x_i)|w''(x_i)| \\
&\leq CN^{-1}\ln N, \\
B_N(W-w)(x_N) &= |(W-w)(x_N) + \varepsilon D^-(W-w)(x_N)| \\
&\leq C\varepsilon(x_i - x_{i-1})|w''(x_i)| \\
&\leq CN^{-1}, \\
|L^N(W-w)(x_i)| &\leq CN^{-1}(|w^{(3)}(x_i)| + |w^{(2)}(x_i)|), \quad x_i \in \Omega^N, \\
&\leq C\varepsilon^{-1}N^{-1} \\
&\leq CN^{-1}\ln N
\end{aligned}$$

and

$$\begin{aligned}
|(D^+ - D^-)(W-w)(x_{N/2})| &\leq C\varepsilon^{-1}N^{-1} \\
&\leq CN^{-1}\ln N.
\end{aligned}$$

Consider the mesh functions

$$\Psi^{\pm}(x_i) = CN^{-1}\ln N \begin{cases} d - x_i, & x_i \in \overline{\Omega}^N \cap [0, d) \\ 3 - x_i, & x_i \in \overline{\Omega}^N \cap (d, 1]. \end{cases}$$

Applying Theorem 3.1 to $\Psi^{\pm}(x_i) \pm (W-w)(x_i)$, over the entire domain, we get

$$|(W-w)(x_i)| \leq CN^{-1}\ln N$$

which is the required result. ∎

Theorem 3.5. *Let u be the solution of Problem (2.1) and U be the solution of the corresponding discrete Problem (3.1). Then we have*

$$\sup_{0<\varepsilon\leq 1} \| U-u \| \leq CN^{-1}\ln N.$$

Proof. Proof follows immediately, if one applies the above Lemmas 3.3 and 3.4 to $U - u = (V-v) + (W-w)$. ∎

Remark 3.1. Following the procedure adopted in [2] and applying it separately on the intervals $[0,d]$ and $[d,1]$, one can extend the above result to obtain the global error bound

$$\sup_{0<\varepsilon\leq 1} \| \overline{U}-u \| \leq CN^{-1}\ln N,$$

where $\overline{U}$ is the piecewise linear interpolant of U on $\overline{\Omega}$.

4. ANALYSIS ON DERIVATIVE ESTIMATE

In this section, we give the $\varepsilon-$uniform error estimate between the scaled derivative of the continuous solution and the corresponding numerical solution in the fine mesh region. Further, in the coarse mesh, an estimate is obtained without scaling the derivative.

We note that the errors

$$e(x_i) \equiv U(x_i) - u(x_i),$$

satisfy the equations

$$[\varepsilon\delta^2 + a(x_i)D^+]e(x_i) = [b(x_i)e(x_i)] + \text{ truncation error},$$

where, by Theorem 3.5, $[b(x_i)e(x_i)] = O(N^{-1}\ln N)$. In the proofs of the following lemmas and theorems, we use the above equations. Hence the analysis carried out in [2, §3.5] can be applied immediately with a slight modifications where ever necessary. Therefore, proofs for some lemmas are omitted; for some of the them short proves are given.

Lemma 4.1. *At each mesh point $x_i \in \Omega^N$ and all $x \in \bar{\Omega}_i = [x_{i-1}, x_i]$, we have*

$$\begin{aligned}
|D^-u(x_i) - u'(x)| &\leq CN^{-1}, \quad \textit{for} \quad x_i \leq \sigma_1,\\
|\varepsilon(D^-u(x_i) - u'(x))| &\leq CN^{-1}\ln N, \quad \textit{for} \quad x_i \in (\sigma_1, d),\\
|\varepsilon(D^+u(x_i) - u'(x))| &\leq CN^{-1}\ln N, \quad \textit{for} \quad x_i \in (d, 1-\sigma_2),\\
|D^+u(x_i) - u'(x)| &\leq CN^{-1}, \quad \textit{for} \quad x_i \geq 1-\sigma_2
\end{aligned}$$

where $u(x)$ is the solution of (2.1).

Lemma 4.2. *At each mesh point $x_i \in \Omega^N$,*

$$\begin{aligned}
\max_{0<i\leq N/4} |D^-(V_L - v)(x_i)| &\leq CN^{-1},\\
\max_{N/4<i\leq N/2} |\varepsilon(D^-(V_L - v)(x_i))| &\leq CN^{-1},\\
\max_{N/2<i\leq 3N/4} |\varepsilon(D^+(V_R - v)(x_i))| &\leq CN^{-1},\\
\max_{3N/4<i\leq N} |D^+(V_R - v)(x_i)| &\leq CN^{-1},
\end{aligned}$$

where v and V_L^N, V_R^N are the solutions of (2.2-2.4) and (3.2-3.5) respectively.

Proof. We denote the error and the local truncation error respectively at each mesh point by

$$e(x_i) = V(x_i) - v(x_i) \text{ and } \tau(x_i) = L^N e(x_i).$$

First, we prove that for all $i,\ N/2 \le i \le 3N/4 - 1, \quad |\varepsilon D^+ e_i| \le CN^{-1}$. We have

$$|\varepsilon D^+ e(x_{3N/4-1})| \le C\varepsilon N^{-1}. \tag{4.1}$$

Now we write $\tau(x_i) = L^N e(x_i)$ in the form,

(4.2)

$$\varepsilon D^+ e(x_i) - \varepsilon D^+ e(x_{i-1}) + \frac{1}{2}(x_{i+1} - x_{i-1})(a(x_i) D^+ e(x_i) - b(x_i) e(x_i)) = \frac{1}{2}(x_{i+1} - x_{i-1})\tau(x_i).$$

Summing and rearranging for each $i,\ N/2 \le i \le 3N/4 - 2$, we get

$$\begin{aligned} |\varepsilon D^+ e(x_i)| &\le |\varepsilon D^+ e(x_{3N/4-1})| + \frac{1}{2}\sum_{j=i}^{3N/4-1} (x_{j+1} - x_{j-1})(|\tau(x_j)| + b(x_j)|e(x_j)|) \\ &+ |\frac{1}{2}\sum_{j=i}^{3N/4-1} (x_{j+1} - x_{j-1}) a(x_j) D^+ e(x_j)|. \end{aligned}$$

Using the telescopic effect of the last term, $|e(x_i)| \le CN^{-1}$ and $\| a' \| \le C$, we get $|\varepsilon D^+ (V_R - v)(x_i)| \le CN^{-1}$.

Similarly, one can obtain $|\varepsilon D^- (V_L - v)(x_i)| \le CN^{-1}$, for $N/4 < i \le N/2$.

We can rewrite (4.2) in the form

$$(1 + \rho_j) D^+ e(x_j) = D^+ e(x_{j-1}) + \frac{\rho_j}{a(x_j)}(\tau(x_j) + b(x_j) e(x_j)), \tag{4.3}$$

where $\rho_j = \frac{a(x_j)(x_{j+1} - x_{j-1})}{\varepsilon}$. For $i \ge 3N/4$ use (4.3) to complete the proof. Similarly for $i \le N/4$ we get the required result. ∎

Lemma 4.3. *Let w and W be the solutions of (2.5-2.7) and (3.6-3.10) respectively. Then, we have*

$$\begin{aligned} \max_{0 < i \le N/4} |D^- (W_L - w)(x_i)| &\le CN^{-1}, \\ \max_{N/4 < i < N/2} |\varepsilon (D^- (W_L - w)(x_i))| &\le CN^{-1} \ln N \end{aligned}$$

and

$$\begin{aligned} \max_{N/2 < i < 3N/2} |\varepsilon (D^+ (W_R - w)(x_i))| &\le CN^{-1} \ln N, \\ \max_{3N/2 \le i < N} |D^+ (W_R - w)(x_i)| &\le CN^{-1}. \end{aligned}$$

Proof. Suppose $\sigma_1 = \frac{2\varepsilon}{\alpha} \ln N$ and $\sigma_2 = \frac{2\varepsilon}{\alpha} \ln N$ we have, $|W_L(x_i)| \le CN^{-2}$, for $x_i \le \sigma_1, \quad |W_R(x_i)| \le CN^{-2}$, for $x_i \ge 1 - \sigma_2$ and $|w(x_i)| \le CN^{-2}$. This implies

$$\begin{aligned} \max_{0 < i \le N/4} |D^- (W_L - w)(x_i)| &\le CN^{-1}, \\ \max_{3N/2 \le i < N} |D^+ (W_R - w)(x_i)| &\le CN^{-1}. \end{aligned}$$

For $x_i = 1 - \sigma_2$, we write $L^N W_R(1 - \sigma_2) = 0$ in the form

$$\begin{aligned} \varepsilon D^+ W_R(x_{3N/4-1}) &= (\varepsilon - a(1 - \sigma_2)(h_3 + h_4)) D^+ W_R(1 - \sigma_2) - b(1 - \sigma_2)(h_3 + h_4) W_R(1 - \sigma_2) \\ &\le CN^{-1}. \end{aligned}$$

Similarly one can obtain

$$\begin{aligned}\varepsilon D^- W_L(x_{N/4+1}) &= (\varepsilon - a(\sigma_1)(h_1+h_2))D^- W_L(\sigma_1) - b(\sigma_1)(h_1+h_2)W_L(\sigma_1)\\ &\le CN^{-1}.\end{aligned}$$

Let $\hat{e}(x_i) = (\hat{W}_R - \hat{w})(x_i)$ and $\hat{\tau}(x_i) = L^N \hat{e}(x_i)$. Then on the interval $[d, 1-\sigma_2)$, we write the equation $\hat{\tau}(x_i) = L^N \hat{e}(x_i)$ in the form

$$\varepsilon D^+ \hat{e}(x_j) - \varepsilon D^+ \hat{e}(x_{j-1}) + a(x_j)(\hat{e}(x_{j+1}) - \hat{e}(x_j)) - b(x_j)h_3\hat{e}(x_j) = h_3\hat{\tau}(x_j).$$

Summing from $x_j = x_i > d$ to $x_j = \sigma_2 - h_3$ and rearranging we obtain

$$\begin{aligned}\varepsilon D^+ \hat{e}(x_i) &= \varepsilon D^+ \hat{e}(x_{3N/4-1}) + a(x_{3N/4-1})\hat{e}(x_{3N/4}) - a(x_{i-1})\hat{e}(x_i) - \\ &\quad \sum_{j=i}^{3N/4-1} (a(x_j) - a(x_{j-1}))\hat{e}(x_j) - \varepsilon h_3 \sum_{j=i}^{3N/4-1} [b(x_j)\hat{e}(x_j) + \hat{\tau}(x_j)]\end{aligned}$$

which yields the bound

$$\begin{aligned}\varepsilon D^+ \hat{e}(x_j) &\le CN^{-1}\ln N + Ch_3\sigma_2\varepsilon^{-1}N^{-1} \sum_{j=i}^{3N/4-1} e^{-(j-1)\alpha_2 h_1/\varepsilon}\\ &\le CN^{-1}\ln N.\end{aligned}$$

Finally over the range$(\sigma_1, d]$, we repeat the above procedure to complete the proof. ∎

Theorem 4.4. *Let u be the solution of (2.1) and U the corresponding numerical solution of (3.11). Then for $x \in \overline{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$\begin{aligned}|(D^- U(x_i) - u'(x))| &\le CN^{-1}, \quad 0 < i \le N/4\\ |\varepsilon(D^- U(x_i) - u'(x))| &\le CN^{-1}\ln N, \quad N/4+1 < i \le N/2\end{aligned}$$

and

$$\begin{aligned}|\varepsilon(D^+ U(x_i) - u'(x))| &\le CN^{-1}\ln N, \quad N/2 \le i \le 3N/4-1,\\ |(D^+ U(x_i) - u'(x))| &\le CN^{-1}, \quad 3N/4 \le i \le N-1.\end{aligned}$$

Proof. Following the method of proof adopted in [2, Theorem 3.17], using the Lemmas 4.2 and 4.3 we get the required result. ∎

Remark 4.1. Since $\bar{U}$ is a linear function in the open interval $\Omega_i = (x_i, x_{i+1})$ for each $i, 0 \le i \le N-1$, we have $\bar{U}'(x) = D^+U(x_i)$ for all $x \in \Omega_i$. It then follows, from Theorem 4.4, that $\bar{U}'$ is an $\varepsilon-$uniform approximation to $u'(x)$ for each $x \in (x_i, x_{i+1})$. We now show that this approximation can be extended in a natural way to the entire domain $\overline{\Omega}$. We define the piecewise constant function $\bar{D}^+U$ on $[0,1)$ by

$$\bar{D}^+U(x) = D^+U(x_i), \quad \text{for} \quad x \in [x_i, x_{i+1}),\ i = 0, ..., N-1$$

and at the point $x = 1$ by

$$\bar{D}^+U(1) = D^+U(x_{N-1}).$$

Then, from the above theorem, $\bar{D}^+U$ is an $\varepsilon-$uniform global approximation to u' in the sense that

$$\sup_{0<\varepsilon\le 1} \parallel \varepsilon(\bar{D}^+U - u') \parallel_{\overline{\Omega}} \le CN^{-1}\ln N.$$

5. NUMERICAL RESULTS

In this section, an example is given to illustrate the numerical method discussed in this paper.

$$\begin{aligned} &\varepsilon u''(x) + a(x)u'(x) - b(x)u(x) = f(x), \quad x \in (0,1), \\ &-u'(0) = 1, \quad u(1) + \varepsilon u'(1) = 0, \end{aligned} \tag{5.1}$$

where,

$$a(x) = \begin{cases} x-2, \; x \le 0.5, \\ x+1, \; x \ge 0.5, \end{cases} \qquad b(x) = 3, \; 0 < x < 1, \quad f(x) = \begin{cases} -\exp(x), \; x \le 0.5, \\ x-3, \; x \ge 0.5. \end{cases}$$

For all integers N, satisfying $N,\ 2N \in\ R_N = [128, 256, 512, 1024]$ and for a finite set of values $\varepsilon \in R_\varepsilon = [2^{-19}, 2^{-2}]$, we compute the maximum pointwise two-mesh differences

$$E_\varepsilon^N = \| U^N - \overline{U}^{2N} \|_{\Omega^N},$$

where U^N and $\overline{U}^{2N}$ denote respectively, the numerical solutions obtained using N and $2N$ mesh intervals. From these values the $\varepsilon-$uniform maximum pointwise two-mesh difference $E^N = \max\limits_{\varepsilon \in R_\varepsilon} E_\varepsilon^N$ are formed for each available value of N satisfying $N,\ 2N \in R_N$. Approximations of $\varepsilon-$uniform order of local convergence are defined, for all $N,\ 4N \in R_N$, by

$$p^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

We compute the maximum pointwise two-mesh difference for the first derivative of the solution by

$$D_\varepsilon^N = \begin{cases} \max |(D^- U^N - \bar{D}^- U^{2N})(x_i)|, & \text{for} \quad 1 \le i \le N/4 \\ \max |\varepsilon(D^- U^N - \bar{D}^- U^{2N})(x_i)|, & \text{for} \quad N/4+1 \le i \le N/2 \\ \max |\varepsilon(D^+ U^N - \bar{D}^+ U^{2N})(x_i)|, & \text{for} \quad N/2+1 \le i \le 3N/4-1 \\ \max |(D^+ U^N - \bar{D}^+ U^{2N})(x_i)|, & \text{for} \quad 3N/4 \le i \le N-1. \end{cases}$$

From these values the $\varepsilon-$uniform maximum pointwise two-mesh difference $D^N = \max\limits_{\varepsilon \in R_\varepsilon} D_\varepsilon^N$ and the $\varepsilon-$uniform order of local convergence $dp^N = \log_2(\frac{D^N}{D^{2N}})$ are formed for each available value of N satisfying $N,\ 2N \in R_N$. In Fig 1 and Fig 2, the solution and its first scaled derivative at the mesh points for the problem (5.1) are plotted as function of N and ε respectively. Table 5.1, presents the values of E^N and p^N for the solution u. In the case of derivative, we present the values of D^N and dp^N in Table 5.2.

Table 5.1: Values of E^N and p^N for the solution u.

	Number of mesh points N			
	128	256	512	1024
E^N	7.7516e-3	3.8910e-3	1.9493e-3	9.7561e-4
p^N	9.9435e-1	9.9718e-1	9.9858e-1	-

Table 5.2: Values of D^N and dp^N for the derivative of the solution u on $(0, x_{N/4}]$, $(x_{N/4}, d]$, $[d, x_{3N/4})$ and $[x_{3N/4}, 1)$ respectively.

	Number of mesh points N			
	128	256	512	1024
D^N	1.1667e-2	5.8445e-3	2.9248e-3	1.4630e-3
dp^N	9.9728e-1	9.9874e-1	9.9941e-1	-
D^N	1.1840e-2	9.8403e-3	7.5267e-3	5.1422e-3
dp^N	2.6689e-1	3.8668e-1	5.4963e-1	-
D^N	1.2148e-2	9.9441e-3	7.5538e-3	5.1514e-3
dp^N	2.8881e-1	3.9664e-1	5.5224e-1	-
D^N	1.6538e-3	9.5453e-4	5.7041e-4	3.2504e-4
dp^N	7.9292e-1	7.4279e-1	8.1138e-1	-

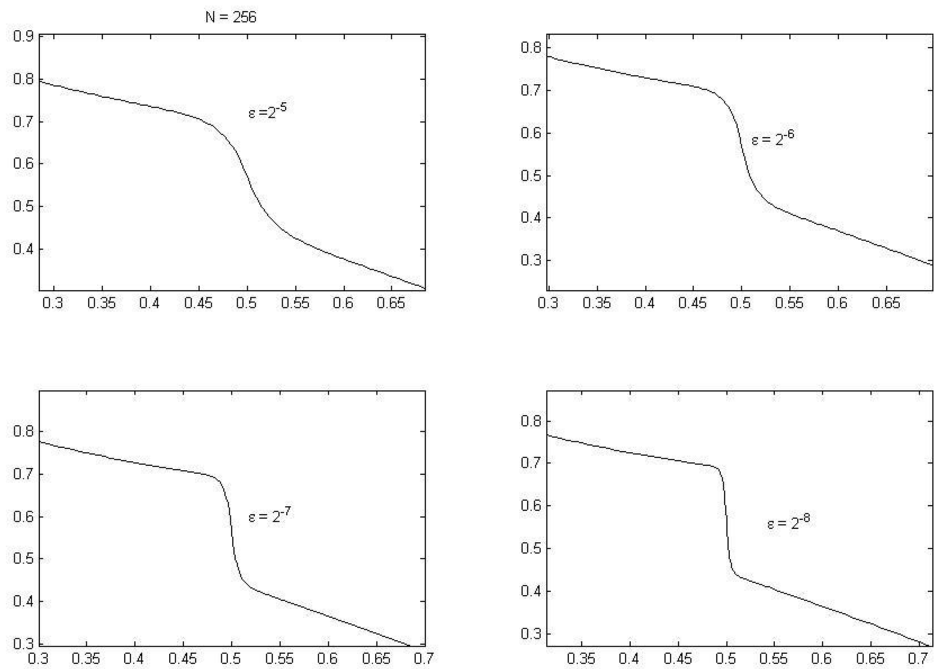

Figure 1: Graphs of the numerical solution of problem 5.1 for $\varepsilon \in [2^{-8}, 2^{-5}]$ with $N = 256$.

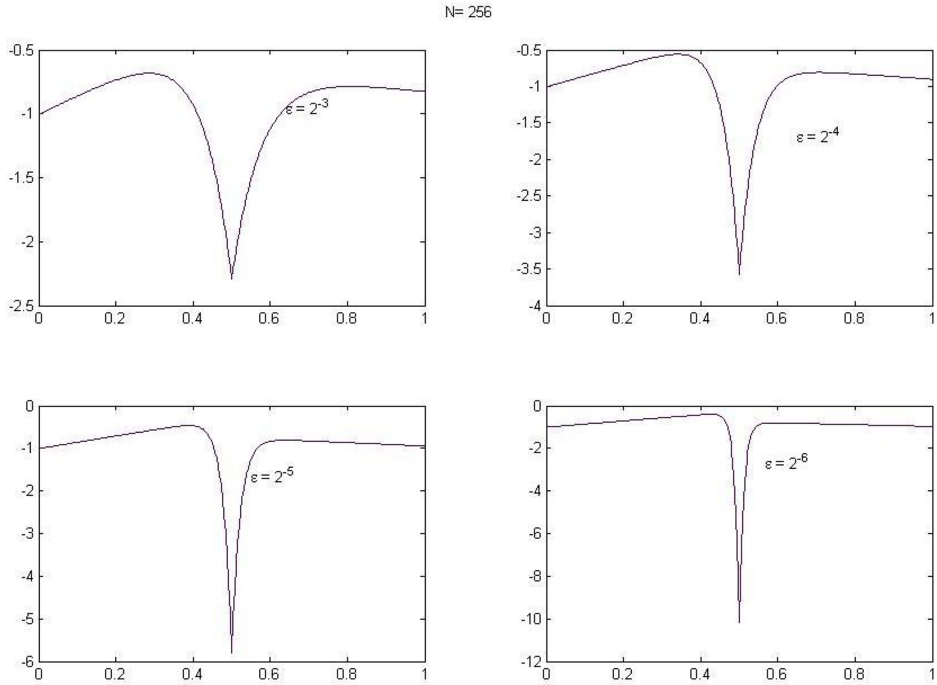

Figure 2: Graphs of the numerical derivative of the solution of problem (5.1) for $\varepsilon \in [2^{-6}, 2^{-3}]$ *with* $N = 256$.

REFERENCES

[1] R. E. O'MALLEY, Jr *Introduction to Singular Perturbations* , Academic Press, New York, 1974.

[2] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, G. I. SHISHKIN, *Robust Computational Techniques for Boundary Layers*, Chapman Hall/ CRC, Boca Raton, 2000.

[3] J. J. H. MILLER, E. O'RIORDAN, G. I. SHISHKIN, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific Publishing Company Private Ltd., 1996.

[4] N. KOPTEVA, M. STYNES, Approximation of derivatives in a convection-diffusion two-point bondary value problem, *Applied Numerical Mathematics*, **39** (2001), pp. 47–60.

[5] A. R. ANSARI, A. F. HEGARTY, Numerical solution of a convection diffusion problem with Robin boundary conditions, *Jouranl of Computational and Applied Mathematics*, **156**, (2003), pp. 221–238.

[6] J. JAYAKUMAR, N. RAMANUJAM, A numerical method for singular perturbation problems arising in chemical reactor theory, *Comp. Math. Appl.*, **27**(5) (1994), pp. 83–99.

[7] S. NATESAN, N. RAMANUJAM, Initial-value technique for singularly perturbed boundary-value problems for second-order ordinary differential equations arising in chemical reactor theory, *Journal of Optimization and Application*, **97** (2) (1998), pp. 455–470.

[8] S. NATESAN, N. RAMANUJAM, A "Booster method" for singular perturbation problems arising in chemical reactor theory, *Applied Mathematics and Computation*, **100** (1999), pp. 27–48.

[9] Z. CEN, A hybrid difference scheme for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient, *Applied Mathematics and Computation*, **169** (2005), pp. 689–699.

[10] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, G. I. SHISHKIN, Global maximum parameter-uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient, *Mathematical And Computer Modelling*, **40**, (2004), pp. 1375–1392.

[11] V. SHANTHI, N. RAMANUJAM, Asymptotic numerical methods for singularly perturbed fourth-order ordinary differential equations with a weak interior layer, *Applied Mathematics and Computation*, **172** (2006), pp. 252–266.

[12] T. VALANARASU, N. RAMANUJAM, An asymptotic initial value method for second order singularly perturbation problems of convection-diffusion type with a discontinuous source term, *Journal of Applied Mathematics and Computing*, **23** (2008), pp. 141–152.

[13] P. A. FARRELL, A. F. HEGARTY, J. J. H. MILLER, E. O'RIORDAN, G. I. SHISHKIN, Singularly perturbed convection- diffusion problem with boundary and weak interior layers,*Journal Of Computational And Applied Mathematics*, **166** (1) (2004), pp. 133–151.

[14] R. M. PRIYADHARSHINI and N. RAMANUJAM, Approximation of derivative for a singularly perturbed second-order ordinary differential equation of Robin type with discontinuous convection coefficient and source term, *Numerical Mathematics: Theory, Methods and Applications*, **2** (1) (2009), pp.100-118.

[15] R. M. PRIYADHARSHINI and N. RAMANUJAM, Approximation of derivative to a singularly perturbed reaction-convection-diffusion problem with two parameters, *J. Appl. Math. & Informatics,* **27** (3-4) (2009), pp.517-529.

[16] R. K. DUNNE, E. O'RIORDAN, Interior layers arising in linear singularly perturbed differential equations with discontinuous coefficients, *Proceedings of the fourth International Conference on finite difference methods: Theory and Applications*, Lozenetz, Bulgaria, August 26–29, 2006 (I. Farago, p. Vabishchevich and L. Vulkov eds.), Rousse University, Bulgaria (2007), pp. 29–38.