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IMPROVEMENT OF JENSEN'S INEQUALITY FOR SUPERQUADRATIC FUNCTIONS

S. ABRAMOVICH, B. IVANKOVIĆ, AND J. PEČARIĆ

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA 31905, ISRAEL.
abramos@math.haifa.ac.il

FACULTY OF TRANSPORT AND TRAFIC ENGINEERING, UNIVERSITY OF ZAGREB, VUKELIĆEVA 4, 10000, CROATIA
bozidar.ivankovic@zg.t-com.hr

FACULTY OF TEXTILE, UNIVERSITY OF ZAGREB, PRILAZ BARUNA FILIPOVIĆA 30, 10000 ZAGREB, CROATIA
pecaric@element.hr

ABSTRACT. Since 1907, the famous Jensen's inequality has been refined in different manners. In our paper, we refine it applying superquadratic functions and separations of domains for convex functions.

There are convex functions which are not superquadratic and superquadratic functions which are not convex.

For superquadratic functions which are not convex we get inequalities analogue to inequalities satisfied by convex functions.

For superquadratic functions which are convex (including many useful functions) we get refinements of Jensen's inequality and its extensions.

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1. INTRODUCTION

In this paper we refine results derived from Jensen's inequality and its extensions both for superquadratic and for convex functions. This type of inequalities appear also in [7], [10] and [12].

We start with definitions, notations and theorems which we use in the sequel.

Definition 1.1. A function $f:[0,+\infty\rangle\rightarrow\mathbb{R}$ is called superquadratic if

$$\forall x\in\mathbb{R}^{+}\quad\exists C(x),$$

such that

$$\forall y\in\mathbb{R}^{+}\Rightarrow f(y)-f(x)-f(|y-x|)\geq C(x)(y-x). \tag{1.1}$$

One of the equivalent definition for a convex function is:

Definition 1.2. Let $[a,b]\subset\mathbb{R}$ and $f:[a,b]\rightarrow\mathbb{R}$. If $\forall x\in[a,b]$, $\exists C(x)$ such that

$$\forall y\in[a,b]\Rightarrow f(y)-f(x)\geq C(x)(y-x), \tag{1.2}$$

then f is a convex function.

From these two definitions, it is obvious that every nonnegative superquadratic function is convex.

Jensen's inequality says: Suppose that U is a convex subset of a linear space M, and f is a convex function on U. If $p_1,\ldots,p_n\in[0,1]$, $\sum_{i=1}^{n}p_i=1$ and $x_1,\ldots,x_n\in U$, then

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\leq\sum_{i=1}^{n}p_if(x_i). \tag{1.3}$$

In [1] the authors refined Jensen inequality using superquadratic functions as follows:

Theorem 1.1. *Let $f(x)$ be a superquadratic function and let $x_k\geq0$, $a_k\geq0$, for every $k=1,\ldots,n$.. If $\sum_{k=1}^{n}a_k>0$, then*

$$\sum_{k=1}^{n}a_kf(x_k)-\sum_{k=1}^{n}a_k\cdot f\left(\frac{\sum_{k=1}^{n}a_kx_k}{\sum_{k=1}^{n}a_k}\right)\geq\sum_{k=1}^{n}a_kf\left(\left|x_k-\frac{\sum_{j=1}^{n}a_jx_j}{\sum_{j=1}^{n}a_j}\right|\right).$$

Equality holds in the last inequality when $f(x)=x^2$, $x_k\in\mathbb{R}$, $a_k\geq0$.

Notation . We denote I_n as $I_n=\{1,\ldots,n\}$ and P_n as $P_n=\sum_{i=1}^{n}p_i$ for $p_i\in\mathbb{R}$, $i=1,...,n$.

The index set function $F(I)$ is defined as (see [13, p.87]):

Notation . Let $f:U\rightarrow\mathbb{R}$ be a function defined on a convex subset U in a linear real space. The index set function $F(I)$ for any finite $I\subset\mathbb{N}$ is:

$$F(I)=P_If\left(\frac{1}{P_I}\sum_{i\in I}a_ix_i\right)-\sum_{i\in I}a_if(x_i), \tag{1.4}$$

where $P_I=\sum_{i\in I}a_i$, $a_i\in\mathbb{R}$, and $x_i\in U$ such that $\frac{1}{P_I}\sum_{i\in I}a_ix_i\in U$.

In [13, p. 87, Theorem 3.14], the authors proved a theorem which considers the nondecreasing and nonincreasing features of $F(I)$.

Theorem 1.2. *Let $f : U \to \mathbb{R}$ be a convex function and $I, J \subset \mathbb{N}$ such that $I \cap J = \emptyset$. If $P_I, P_J > 0$, then*

$$F(I \cup J) \leq F(I) + F(J) \tag{1.5}$$

In the case $P_I \cdot P_J < 0$, the inequality is reversed, when we assume that all the expressions are computable.

An immediate consequence of (1.5) is the next corollary given in [13, p. 87, Corollary 3.15].

Corollary 1.3. *Let $f : U \to \mathbb{R}$ be a convex function.*
(a) If $a_i \geq 0 \;\; (i = 1, \ldots, n)$ and $I_k = \{1, \ldots, k\}$, then

$$F(I_n) \leq F(I_{n-1}) \leq \cdots \leq F(I_2) \leq 0 \tag{1.6}$$

and

$$F(I_n) \leq \min_{1 \leq i < j \leq n} \left\{ (a_i + a_j) f\left(\frac{a_i x_i + a_j x_j}{a_i + a_j} \right) - a_i f(x_i) - a_j f(x_j) \right\}. \tag{1.7}$$

(b) If $a_1 > 0$, $a_i \leq 0, i = 2, \ldots, n$ and $P_n > 0$, then the reverse of the inequalities in (1.6) are valid and

$$F(I_n) \geq \max_{1 \leq j \leq n} \left\{ (a_1 + a_j) f\left(\frac{a_1 x_1 + a_j x_j}{a_1 + a_j} \right) - a_1 f(x_1) + a_j f(x_j) \right\}.$$

In [6] the authors obtained results as consequences of Theorem 1.2 by separating the index set I_n for $a_i \geq 0$. The next corollary is somewhat modified from Corollary 2.1 in [6].

Corollary 1.4. *Let $f : U \to \mathbb{R}$ be a convex function on the convex subset U of the linear space M, $x_i \in U$, $p_i > 0$, $i \in I_n = \{1, \ldots, n\}$, with $\sum_{i=1}^{n} p_i = 1$. If $I \subset I_n$, then:*

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i \right) \geq \sum_{i \in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i \in I} p_i x_i}{P_I} \right) \geq 0.$$

The next lemma is an immediate consequence of the Corollary 1.4 by choosing $p_i = \dfrac{a_i}{\sum_{i=1}^{n} a_i}$.

Lemma 1.5. *Let $f : U \to \mathbb{R}$ be a convex function on the convex subset U of the linear space M, $x_i \in U$, $a_i \geq 0, i \in I_n = \{1, \ldots, n\}$. $\sum_{i=1}^{n} a_i > 0$ and $\sum_{i \in I} a_i > 0$ If $I \subseteq I_n$, then:*

$$\sum_{i=1}^{n} a_i f(x_i) - \sum_{i=1}^{n} a_i \cdot f\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i} \right) \geq \sum_{i \in I} a_i f(x_i) - \sum_{i \in I} a_i \cdot f\left(\frac{\sum_{i \in I} a_i x_i}{\sum_{i \in I} a_i} \right) \geq 0.$$

In [6, Theorem 2.1] the authors proved:

Theorem 1.6. *Let $f : C \to \mathbb{R}$ be a convex function on the convex subset C of the linear space M, $x_i \in C$, $p_i > 0$, $i \in I_n = \{1, ..., n\}$, $n \geq 3$ with $\sum_{i=1}^{n} p_i = 1$. If $I =$*

$\{I \subset I_n, I \neq I_n, |I| \geq 2\}$, *then:*

$$\begin{aligned} f\left(\sum_{i=1}^{n} p_i x_i\right) &\leq \min_I \left[P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i)\right] \\ &\leq \frac{1}{2^n - n - 2}\left[\sum_{I\subset I_n} P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \left(2^{n-1} - n\right)\sum_{i=1}^{n} p_i f(x_i)\right] \\ &\leq \max_I \left[P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) + \sum_{i\in I_n\setminus I} p_i f(x_i)\right] \\ &\leq \sum_{i=1}^{n} p_i f(x_i). \end{aligned}$$

In [3, Theorem 1] it was proved that:

Theorem 1.7. *Let $F(x_1, ..., x_n)$ be a complex function in n variables and let*

$$|F(x_1, ..., x_n)| \leq |F(|x_1|, ..., |x_n|)|.$$

Let also $|F(x_1, ..., x_n)|$ be a concave function for $(x_1, ..., x_n) \in \mathbb{R}^n$.If $f_i(t)$, $i = 1, ..., n$, $w(t)$ are complex functions of real variables, and $f_i w(t)$, $w(t)$ are integrable on $[a, b]$, then

$$\begin{aligned} A(c) &= \left|\int_a^c w(t) F(f_1(t), ..., f_n(t))\, dt\right| \\ &\quad + \int_c^b |w(t)|\, dt \cdot \left|F\left(\frac{\int_c^b |w(t) f_1(t)|\, dt}{\int_c^b |w(t)|\, dt}, ..., \frac{\int_c^b |w(t) f_n(t)|\, dt}{\int_c^b |w(t)|\, dt}\right)\right| \end{aligned}$$

is a decreasing function in c, $a \leq c \leq b$. Analogous results for the discrete case are obvious.

The following Corollary 1.8 appears in [9, Corollary 2]. It is also an immediate result of Theorem 1.6 and of the analogous discrete case of Theorem 1.7 by using the convexity of f and by replacing x_k by $x_k - \sum_{i=1}^{n} \frac{p_i x_i}{P_n}$.

Corollary 1.8. *Let f be a convex function on $\mathbb{R}$ and $p_i > 0$. Then for every nonempty subset of $I \subseteq I_n$ we have*

$$\begin{aligned} &\sum_{k=1}^{n} p_k f\left(x_k - \sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) \\ \geq\ & P_I f\left(\frac{P_{\overline{I}}}{P_n}\left(\sum_{i\in I} \frac{p_i x_i}{P_I} - \sum_{i\in\overline{I}} \frac{p_i x_i}{P_{\overline{I}}}\right)\right) + P_{\overline{I}} f\left(\frac{P_I}{P_n}\left(\sum_{i\in\overline{I}} \frac{p_i x_i}{P_{\overline{I}}} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right)\right) \\ \geq\ & P_n f(0), \end{aligned} \tag{1.8}$$

where $\overline{I} = I_n / I$.

We need also the following notations:

Notation . For a function $f(x)$ defined on $\mathbb{R}^+$, $x_k \geq 0$, $a_k > 0$, $k \in I_n$ and for every $I \subseteq I_n$ we denote:

$$\Delta_I = \sum_{k\in I} a_k f\left(\left|\frac{x_k}{a_k} - \frac{\sum_{j\in I} x_j}{\sum_{j\in I} a_j}\right|\right), \tag{1.9}$$

$$\Delta_n = \sum_{k=1}^{n} a_k f\left(\left|\frac{x_k}{a_k} - \frac{\sum_{j=1}^{n} x_j}{\sum_{j=1}^{n} a_j}\right|\right) \tag{1.10}$$

and

$$\Delta_2(i,j) = a_i f\left(\left|\frac{x_i}{a_i} - \frac{x_i + x_j}{a_i + a_j}\right|\right) + a_j f\left(\left|\frac{x_j}{a_j} - \frac{x_i + x_j}{a_i + a_j}\right|\right) \tag{1.11}$$

for $1 \le i \le j \le n.$

The next notations are analogue to those given in Notation .

Notation . For a function $f(x)$ defined on $\mathbb{R}$, $x_k \ge 0$, $a_k > 0$, $k \in I_n$, and for every $I \subseteq I_n$ we denote:

$$d_I = \sum_{k\in I} a_k f\left(\frac{x_k}{a_k}\right) - \left(\sum_{k\in I} a_k\right) f\left(\frac{\sum\limits_{k\in I} x_k}{\sum\limits_{k\in I} a_k}\right) \tag{1.12}$$

and

$$d_n = \sum_{k=1}^{n} a_k f\left(\frac{x_k}{a_k}\right) - \left(\sum_{k=1}^{n} r^n a_k\right) f\left(\frac{\sum\limits_{k=1}^{n} x_k}{\sum\limits_{k=1}^{n} a_k}\right) \tag{1.13}$$

Note that $d_I = -F(I)$ from (1.4), after the change of variables $x_k \longrightarrow \frac{x_k}{a_k}$ $k \in I_n$.
Therefore it is clear from Theorem 1.2, Corollary 1.3 and Corollary 1.4 that:

Corollary 1.9. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a convex function. If $I \subseteq J \subseteq I_n$, and $x_k \ge 0$, $a_k > 0$, $k \in I_n$, then*

$$0 \;\;\le\;\; d_I \le d_J \le d_n \tag{1.14}$$

and in particular

$$d_n \ge \max_{1\le i<j\le n} \left\{ a_i f(\frac{x_i}{a_i}) + a_j f(\frac{x_j}{a_j}) - (a_i + a_j) f\left(\frac{x_i + x_j}{a_i + a_j}\right)\right\}.$$

We get the following proposition from Theorem 1.1 by replacing I_n with I and x_k with $\frac{x_k}{a_k}$.

Proposition 1.10. *Let $f(x)$ be a superquadratic function and let $x_k \ge 0$. If $a_k > 0,\ k \in I_n$, then for every $I \subseteq I_n$*

$$d_I \ge \Delta_I. \tag{1.15}$$

In particular

$$d_n \ge \Delta_n. \tag{1.16}$$

Example 1.11. It is easy to verify that $f(x) = x^p$ for every $p \ge 2$ are superquadratic, and especially $f(x) = x^2$ is superquadratic. In case $p = 2$, (1.16) is satisfied with equality, for $x_i \in \mathbb{R}$ and $a_i \ge 0$, $i \in I_n$, $\sum_{i=1}^{n} a_i > 0$.

The following inequalities were discussed in [4, Theorem 2] for convex functions and in [2, Theorem 4 and Theorem 5] for convex and superquadratic functions. Here we quote the part of those theorems used for our refinements:

Theorem 1.12. *Let* $a = (a_1, ..., a_n)$ *, where* $0 \leq \sum_{j=1}^{i} a_j \leq 1,\ i = 1, ..., n,\ \sum_{i=1}^{n} a_i = 1$*, and* $b = (b_1, ..., b_n)$ *,* $0 < \sum_{j=1}^{i} b_j < 1,\ i = 1, ..., n-1,\ \sum_{i=1}^{n} b_i = 1$*, and* $a \neq b$*. Denote*

$$m_i = \frac{\sum_{j=1}^{i} a_j}{\sum_{j=1}^{i} b_j}, \qquad \overline{m}_i = \frac{\sum_{j=i+1}^{n} a_j}{\sum_{j=i+1}^{n} b_j}, \qquad i = 1, ..., n, \tag{1.17}$$

and

$$m^* = \min_{1 \leq i \leq n} \{m_i, \overline{m_i}\}, \tag{1.18}$$

If $x = (x_1, ..., x_n)$ *is any increasing n-tuple in* T^n*, where* T *is an interval in* $\mathbb{R}$*, then*

$$\sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right) - m^* \left(\sum_{i=1}^{n} b_i f(x_i) - f\left(\sum_{i=1}^{n} b_i x_i\right)\right) \geq 0 \tag{1.19}$$

where $f : T \longrightarrow \mathbb{R}$ *is a convex function on the interval* T*.*

If $f(x)$ be a nonnegative superquadratic function (and therefore also convex) on T where T is $[0, a]$ or $[0, \infty)$, then

$$\begin{aligned} &\sum_{i=1}^{n} a_i f(x_i) - f\left(\sum_{i=1}^{n} a_i x_i\right) - m^* \left(\sum_{i=1}^{n} b_i f(x_i) - f\left(\sum_{i=1}^{n} b_i x_i\right)\right) \\ \geq\ & nf\left(\frac{\sum_{i=1}^{n} (a_i - m^* b_i) \left|x_i - \sum_{j=1}^{n} a_j x_j\right| + m^* \left|\sum_{i=1}^{n} (a_i - b_i) x_i\right|}{n}\right). \end{aligned} \tag{1.20}$$

2. RESULTS

Our main results is obtained in Theorem 2.1. For superquadratic functions we get in (2.1) an inequality analogue to inequality (1.14) which holds for convex functions. For nonnegative superquadratic functions (which are therefore also convex) we get in (2.1), (2.2) and (2.3) refinements of (1.14). All the other theorems and examples in this paper follow from this theorem

Theorem 2.1. *Let* $x_k \geq 0$*,* $p_k > 0$*,* $k = 1, ..., n$*,* $I \subseteq I_n$ *and* $\overline{I} = I_n / I$*.*

a. *If* $f(x)$ *is superquadratic on* $\mathbb{R}^+$*, then*

$$d_n \geq d_I + P_I f\left(\left|\sum_{j=1}^{n} \frac{x_j}{P_n} - \sum_{i \in I} \frac{x_i}{P_I}\right|\right) + \sum_{i \in \overline{I}} p_i f\left(\left|\frac{x_i}{p_i} - \sum_{j=1}^{n} \frac{x_j}{P_n}\right|\right). \tag{2.1}$$

b. *If* $f(x)$ *is also nonnegative on* $\mathbb{R}^+$ *then*

$$d_n \geq d_I + P_I f\left(\frac{P_{\overline{I}}}{P_n}\left|\sum_{i \in I} \frac{x_i}{P_I} - \sum_{i \in \overline{I}} \frac{x_i}{P_{\overline{I}}}\right|\right) + P_{\overline{I}} f\left(\frac{P_I}{P_n}\left|\sum_{i \in I} \frac{x_i}{P_I} - \sum_{i \in \overline{I}} \frac{x_i}{P_{\overline{I}}}\right|\right), \tag{2.2}$$

and for $n \geq 2$

$$d_n \geq \max_{I \subseteq I_n} (\Delta_I) \geq \underset{1 \leq i < j \leq n}{max} \Delta_2(i, j). \tag{2.3}$$

Proof. The identity

$$\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) - \left(\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)\right) \tag{2.4}$$
$$= \sum_{i\in \bar{I}} p_i f(x_i) + P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right),$$

and the inequality

$$\sum_{i\in \bar{I}} p_i f(x_i) + P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right) - \left(\sum_{i=1}^{n} p_i\right) f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) \tag{2.5}$$
$$\geq P_I f\left(\left|\sum_{j=1}^{n} \frac{x_j p_j}{P_n} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right|\right) + \sum_{i\in \bar{I}} p_i f\left(\left|x_i - \sum_{j=1}^{n} \frac{p_j x_j}{P_n}\right|\right),$$

which results from Theorem 1.1, as $f : [0, \infty\rangle \to \mathbb{R}$ is a superquadratic function, leads to:

$$\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) - \left(\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)\right) \tag{2.6}$$
$$\geq P_I f\left(\left|\sum_{j=1}^{n} \frac{x_j p_j}{P_n} - \sum_{i\in I} \frac{x_i p_i}{P_I}\right|\right) + \sum_{i\in \bar{I}} p_i f\left(\left|x_i - \sum_{j=1}^{n} \frac{x_j p_j}{P_n}\right|\right).$$

Replacing in (2.6) x_k by $\frac{x_k}{p_k}$, $k = 1, ..., n$ we get (2.1).

If f is also nonnegative on $\mathbb{R}^+$, then $f(x)$ is also convex for $x \geq 0$ and $f(0) = f'(0) = 0$ (see [1]). Defining

$$f^*(x) = f(|x|),$$

we get that f^* is convex and symmetric on $\mathbb{R}$. Hence

$$P_I f\left(\left|\sum_{j=1}^{n} \frac{p_j x_j}{P_n} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right|\right) + \sum_{i\in \bar{I}} p_i f\left(\left|x_i - \sum_{j=1}^{n} \frac{p_j x_j}{P_n}\right|\right) \tag{2.7}$$
$$= P_I f^*\left(\sum_{j=1}^{n} \frac{p_j x_j}{P_n} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right) + \sum_{i\in \bar{I}} p_i f^*\left(x_i - \sum_{j=1}^{n} \frac{p_j x_j}{P_n}\right)$$
$$\geq P_I f^*\left(\sum_{j=1}^{n} \frac{p_j x_j}{P_n} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right) + P_{\bar{I}} f^*\left(\sum_{i\in \bar{I}} \frac{p_i x_i}{P_{\bar{I}}} - \sum_{j=1}^{n} \frac{p_j x_j}{P_n}\right).$$

The last inequality follows from the convexity of f^*.

As

$$\sum_{j=1}^{n} \frac{p_j x_j}{P_n} - \sum_{i\in I} \frac{p_i x_i}{P_I} = \frac{P_{\bar{I}}}{P_n}\left(\sum_{i\in \bar{I}} \frac{p_i x_i}{P_{\bar{I}}} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right) \tag{2.8}$$

$$\sum_{i\in \bar{I}} \frac{p_i x_i}{P_{\bar{I}}} - \sum_{j=1}^{n} \frac{p_j x_j}{P_n} = \frac{P_I}{P_n}\left(\sum_{i\in \bar{I}} \frac{p_i x_i}{P_{\bar{I}}} - \sum_{i\in I} \frac{p_i x_i}{P_I}\right), \tag{2.9}$$

from (2.6), (2.7), (2.8), (2.9), the symmetry of f^* and from $f^* = f$ for $x \geq 0$ we get that

$$\begin{aligned} &\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) - \left(\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)\right) \\ \geq\ & P_I f\left(\frac{P_{\overline{I}}}{P_n}\left|\sum_{i\in I}\frac{p_i x_i}{P_I} - \sum_{i\in \overline{I}}\frac{p_i x_i}{P_{\overline{I}}}\right|\right) + P_{\overline{I}} f\left(\frac{P_I}{P_n}\left|\sum_{i\in I}\frac{p_i x_i}{P_I} - \sum_{i\in \overline{I}}\frac{p_i x_i}{P_{\overline{I}}}\right|\right). \end{aligned} \tag{2.10}$$

Replacing in (2.10) $x_k \geq 0$ by $\frac{x_k}{p_k} \geq 0$, $k = 1, ..., n$ leads to (2.2).

The function f is positive, therefore, from (2.1), $d_n \geq d_I$. Because $f(x)$ is superquadratic

$$d_I \geq \Delta_I.$$

As d_n is not changed under any permutation of indices, we get that (2.3) holds, hence the proof of the theorem is completed. ∎

Remark 2.1. From Corollary 1.8 and Theorem 1.1, for nonnegative superquadratic functions (which are therefore convex and satisfy $f(0) = f'(0) = 0$), the same reasoning which leads to (2.10), leads also to

$$\begin{aligned} &\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) \\ \geq\ & \sum_{k=1}^{n} p_k f\left(\left|x_k - \sum_{i=1}^{n}\frac{p_i x_i}{P_n}\right|\right) \\ \geq\ & P_I f\left(\frac{P_{\overline{I}}}{P_n}\left|\sum_{i\in I}\frac{p_i x_i}{P_I} - \sum_{i\in \overline{I}}\frac{p_i x_i}{P_{\overline{I}}}\right|\right) + P_{\overline{I}} f\left(\frac{P_I}{P_n}\left|\sum_{i\in \overline{I}}\frac{p_i x_i}{P_{\overline{I}}} - \sum_{i\in I}\frac{p_i x_i}{P_I}\right|\right) \\ \geq\ & P_n f(0) = 0. \end{aligned} \tag{2.11}$$

Comparing (2.10) and (2.11) we realize that none of these inequalities is necessarily stronger than the other.

From (2.11) we get also that

$$\begin{aligned} &\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) \\ \geq\ & \sum_{k=1}^{n} p_k f\left(\left|x_k - \sum_{i=1}^{n}\frac{p_i x_i}{P_n}\right|\right) \\ \geq\ & \max_{I\subseteq I_n}\left\{P_I f\left(\frac{P_{\overline{I}}}{P_n}\left|\sum_{i\in I}\frac{p_i x_i}{P_I} - \sum_{i\in \overline{I}}\frac{p_i x_i}{P_{\overline{I}}}\right|\right) + P_{\overline{I}} f\left(\frac{P_I}{P_n}\left|\sum_{i\in \overline{I}}\frac{p_i x_i}{P_{\overline{I}}} - \sum_{i\in I}\frac{p_i x_i}{P_I}\right|\right)\right\} \\ \geq\ & P_n f(0) = 0. \end{aligned} \tag{2.12}$$

Again, Inequality (2.10) is not necessarily stronger than (2.12).

But note that (2.10) gives a stronger result than the following inequality derived from (2.11):

$$\begin{aligned}&\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right)\\ \geq\ & P_I f\left(\frac{P_{\bar{I}}}{P_n}\left|\sum_{i\in I}\frac{p_i x_i}{P_I} - \sum_{i\in\bar{I}}\frac{p_i x_i}{P_{\bar{I}}}\right|\right) + P_{\bar{I}} f\left(\frac{P_I}{P_n}\left|\sum_{i\in\bar{I}}\frac{p_i x_i}{P_{\bar{I}}} - \sum_{i\in I}\frac{p_i x_i}{P_I}\right|\right).\end{aligned}$$

Example 2.2. A special case of (2.3) in Theorem 2.1 for $f(x) = x^p$, $p = 2$, appears in [11, Theorem 3]. As mentioned in Example 1.11 $f(x) = x^2$ is superquadratic and satisfies (1.16) with equality for every $x \in \mathbb{R}$ (not only for $x \in \mathbb{R}^+$).

The following examples show that in some cases $d_n \geq \Delta_n$ is better then $d_n \geq \underset{1\leq i<j\leq n}{max} \Delta_2(i,j)$, and in other cases, $d_n \geq \underset{1\leq i<j\leq n}{max} \Delta_2(i,j)$ is better than $d_n \geq \Delta_n$.

Example 2.3. Let $n = 3$, $f(x) = x^p$, $p = 3$, $x_1 = 1$, $x_2 = x_3 = 2$, $a_1 = a_2 = a_3 = 1$ then

$$\Delta_3 > \underset{1\leq i\leq j\leq 3}{max} \Delta_2(i,j),$$

and hence in this case, inequality $d_3 \geq \Delta_3$, is better than $d_3 \geq \underset{1\leq i<j\leq 3}{max} \Delta_2(i,j)$, which means that $d_3 \geq \max_{I\subseteq I_3}(\Delta_I) = \Delta_3$.

Indeed,

$$\begin{aligned}\Delta_3 &= a_1 f\left(\left|\frac{x_1}{a_1} - \frac{x_1+x_2+x_3}{a_1+a_2+a_3}\right|\right)\\ &\quad + a_2 f\left(\left|\frac{x_2}{a_2} - \frac{x_1+x_2+x_3}{a_1+a_2+a_3}\right|\right) + a_3 f\left(\left|\frac{x_3}{a3} - \frac{x_1+x_2+x_3}{a_1+a_2+a_3}\right|\right)\end{aligned}$$

$$\Delta_3 = \frac{10}{27}$$

$$\underset{1\leq i\leq j\leq 3}{Max} \Delta_2(i,j) = \left|1 - \frac{3}{2}\right|^3 + \left|2 - \frac{3}{2}\right|^3 = \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3 = \frac{1}{4}.$$

Example 2.4. Let $n = 3$, $f(x) = x^p$, $p = 3$, $a_1 = 3$, $a_2 = 2$, $a_3 = 1$, $x_1 = 3$, $x_2 = 4$, $x_3 = 3$ then

$$\begin{aligned}\Delta_3 &= 3\left|1 - \frac{10}{6}\right|^3 + 2\left|2 - \frac{10}{6}\right|^3 + 1\left|3 - \frac{10}{6}\right|^3\\ &= \frac{10}{3} = 3\frac{1}{3}\end{aligned}$$

$$\underset{1\leq i\leq j\leq 3}{max} \Delta_2(i,j) = \Delta_2(1,3) = 3\left|1 - \frac{6}{4}\right|^3 + 1\left|3 - \frac{6}{4}\right|^3 = 3\frac{3}{4},$$

therefore, $\underset{1\leq i\leq j\leq 3}{max} \Delta_2(i,j) > \Delta_3$.

This means that in this case $d_3 \geq \underset{1\leq i<j\leq 3}{max} \Delta_2(i,j)$ is better than $d_3 \geq \Delta_3$, which means that $d_3 \geq \max_{I\subseteq I_3}(\Delta_I) = \underset{1\leq i<j\leq 3}{max} \Delta_2(i,j)$.

As a result of Theorem 2.1 we get the following theorem:

Theorem 2.5. *Let* $a_1, a_2, ..., a_n > 0$, $n \geq 2$ *and* $s = \sum_{k=1}^{n} a_k$. *Then for* $p \geq 1$,

$$\sum_{k=1}^{n} \frac{a_k}{(s-a_k)^p} \geq \frac{1}{s^{p-1}} \left(\frac{n}{n-1}\right)^p + \Delta$$

where

$$\Delta = \max_{1 \leq i < j \leq n} \left(\frac{a_i a_j^{p+1} \left|a_j - a_i\right|^{p+1}}{(s-a_i)^p \left(s(a_i + a_j) - (a_i^2 + a_j^2)\right)^{p+1}} + \frac{a_j a_i^{p+1} \left|a_j - a_i\right|^{p+1}}{(s-a_j)^p \left(s(a_i + a_j) - (a_i^2 + a_j^2)\right)^{p+1}} \right).$$

In particular for $p = 1$ *we get that*

$$\sum_{k=1}^{n} \frac{a_k}{s-a_k} \geq \left(\frac{n}{n-1}\right) + \max_{1 \leq i < j \leq n} \left(\frac{a_i a_j \left|a_j - a_i\right|^2}{(s-a_i)(s-a_j)\left(s(a_i+a_j) - (a_i^2 + a_j^2)\right)} \right).$$

Proof. We use the inequality $d_n \geq \max_{1 \leq i < j \leq n} \Delta_2(i,j)$ for the superquadratic function $f(x) = x^{p+1}$, $p \geq 1$, $x \geq 0$.

$$\sum_{k=1}^{n} \frac{a_k}{(s-a_k)^p} = \sum_{k=1}^{n} a_k (s-a_k) \left(\frac{a_k}{a_k(s-a_k)}\right)^{p+1}.$$

As $p+1 \geq 2$, $f(x) = x^{p+1}$ is superquadratic and by Theorem 2.1, part *b)*, we get that

$$\begin{aligned}\sum_{k=1}^{n} \frac{a_k}{(s-a_k)^p} \geq & \left(\sum_{k=1}^{n} a_k (s-a_k)\right) \left(\frac{\sum_{k=1}^{n} a_k}{\sum_{k=1}^{n} a_k (s-a_k)}\right)^{p+1} \\ & + \max_{i \leq i < j \leq n} \left\{ a_i (s-a_i) \left| \frac{1}{s-a_i} - \frac{a_i + a_j}{a_i(s-a_i) + a_j(s-a_j)} \right|^{p+1} \right. \\ & \left. + a_j (s-a_j) \left| \frac{1}{s-a_j} - \frac{a_i + a_j}{a_i(s-a_i) + a_j(s-a_j)} \right|^{p+1} \right\}\end{aligned}$$

therefore

$$\sum_{k=1}^{n} \frac{a_k}{(s-a_k)^p} \geq \left(\sum_{k=1}^{n} a_k (s-a_k)\right)^{-p} \left(\sum a_k\right)^{p+1} + \Delta$$

$$\sum_{k=1}^{n} \frac{a_k}{(s-a_k)^p} \geq \left(s^2 - \sum_{k=1}^{n} a_k^2\right)^{-p} s^{p+1} + \Delta$$

$$\sum_{k=1}^{n} \frac{a_k}{(s-a_k)^p} \geq \frac{s^{p+1}}{\left(s^2 - \frac{s^2}{n}\right)^p} + \Delta = \frac{1}{s^{p-1}} \left(\frac{n}{n-1}\right)^p + \Delta$$

holds. ∎

Remark 2.2. The particular case $p = 1$ in Theorem 2.5 is [11, Theorem 4].

Remark 2.3. A more general result is obtained for superqudratic and positive, i.e. for a convex function f

$$\sum_{k=1}^{n} a_k (s-a_k) f\left(\frac{1}{s-a_k}\right) \geq s^2 \left(\frac{n-1}{n}\right) f\left(\frac{n}{s(n-1)}\right) + \Delta_2^*(i,j).$$

$\Delta_2^*(i,j)$ is $\Delta_2(i,j)$ where we replace in $\Delta_2(i,j)$ x_i by a_i and a_i by $a_i(s-a_i)$.
Additional result of Theorem 2.1 is a refinement of Hölder's inequality:

Theorem 2.6. *Let $x_k \geq 0$, $y_k > 0$, $k = 1, ...n$ and $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, then for $n \geq 2$*

$$\begin{aligned} &\left(\sum_{i=1}^{n} x_i^p\right)\left(\sum_{i=1}^{n} y_i^q\right)^{p-1} \\ \geq\ &\left(\sum_{i=1}^{n} x_i y_i\right)^p + \left(\sum_{i=1}^{n} y_i^q\right)^{p-1} \max_{1\leq j\leq k\leq n} \frac{\left(y_j^p + y_k^p\right)\left|x_k y_j^{q-1} - x_j y_k^{q-1}\right|^p}{\left(y_k^q + y_j^q\right)^p}. \end{aligned}$$

Proof. Because of the superquadracity of $f(x) = x^p$, $p \geq 2$, we get from (2.3)

$$\begin{aligned} \sum_{i=1}^{n} x_i^p &= \sum_{i=1}^{n} y_i^q \left(\frac{x_i y_i}{y_i^q}\right)^p \geq \left(\sum_{i=1}^{n} y_i^q\right)^{1-p}\left(\sum_{i=1}^{n} y_i x_i\right)^p \\ &\quad + \underset{1\leq j\leq k\leq n}{Max}\left\{ y_k^q \left|\frac{x_k y_k}{y_k^q} - \frac{x_i y_i + x_k y_k}{y_k^q + y_j^q}\right| + y_j^q \left|\frac{x_j y_j}{y_j^q} - \frac{x_j y_j + x_k y_k}{y_k^q + y_j^q}\right|^p\right\}. \end{aligned}$$

Multiplying both sides by $\left(\sum_{i=1}^{n} y_i^q\right)^{p-1}$ and simplifying, the theorem is proved. ∎

This refinement is a generalization of [11, Theorem 5] where the case $p = 2$ is discussed.
Now we show a refinement of Lemma 1.5 and Theorem 2.1 by using the above Theorem 1.12 from [2] and [4].

Theorem 2.7. *Let T be an interval in $\mathbb{R}$ Let $\left(\frac{x_1}{p_1}, \frac{x_2}{p_2}, ..., \frac{x_n}{p_n}\right)$ be any increasing n-tuple in T^n, where $p_i > 0$, $i = 1, ..., n$, and let*

(2.13)

$$m_{n,I} = \min_{1\leq i\leq n}\left(\frac{\sum_{j=1}^{i} p_j}{\sum_{j=1}^{i} p_j \chi_I(j)}, \frac{\sum_{j=i+1}^{n} p_j}{\sum_{j=i+1}^{n} p_j \chi_I(j)}\right), \quad \textit{where } \chi_I(j) = \begin{cases} 1, & j \in I \\ 0, & j \in \overline{I} \end{cases}$$

and

$$m_{3,2} = \min_{1\leq i\leq 3}\left(\frac{\sum_{j=1}^{i} p_j}{\sum_{j=1}^{i} p_j \chi_I(j)}, \frac{\sum_{j=i+1}^{3} p_j}{\sum_{j=i+1}^{3} p_j \chi_I(j)}\right), \quad |I| = 2.$$

a. *Note $m_{n,I} \geq 1$. If $f(x)$ is convex on T, then*

(2.14) $$d_n \geq m_{n,I} d_I,$$

(2.15) $$d_n \geq d_{n-1} \geq, ..., \geq d_3 \geq m_{3,2} d_2 \geq d_2$$

and

(2.16) $$d_n \geq \max_{1\leq i<j\leq n}\left\{ p_i f\left(\frac{x_i}{p_i}\right) + p_j f\left(\frac{x_j}{p_j}\right) - (p_i + p_j) f\left(\frac{x_i + x_j}{p_i + p_j}\right)\right\}.$$

b. *If $f(x)$ is a nonnegative superquadratic function on T, $T=[0,a]$ or $[0,\infty)$, then (2.14), (2.15), (2.16),*

$$\begin{aligned} & d_n - m_{n,I} d_I \\ \geq\ & P_n n f\left(\left(\sum_{i=1}^{n} p_i \left|\frac{x_i}{p_i} - \sum_{j=1}^{n} \frac{x_j}{P_n}\right| \right.\right. \\ & \left.\left. + m_{n,I} P_I \left|\sum_{i=1}^{n} \frac{x_i}{P_n} - \frac{\sum_{i\in I} x_i}{P_I}\right| - m_{n,I} \sum_{i\in I} p_i \left|\frac{x_i}{p_i} - \sum_{j=1}^{n}\frac{x_j}{P_n}\right|\right) \frac{1}{P_n n}\right), \end{aligned} \tag{2.17}$$

and

$$d_n \geq m_{n,I} d_I \geq m_{n,I} \Delta_I \tag{2.18}$$

and (2.3) hold.

Proof. We apply the results of Theorem 1.12 to the case where $p_i \geq 0$, $i=1,\dots,n$, $\sum_{i=1}^{n} p_i = P_n > 0$, and $\sum_{i\in I} p_i = P_I > 0$, for $I \subseteq I_n$. In this case we allow part of the $q_i - th$ to be equal to zero. We define

$$q_i = \begin{cases} p_i, & i \in I \\ 0, & i \in \overline{I} \end{cases} .$$

If $p_j = 0$, and $q_j = 0$ we replace $\frac{p_j}{q_j}$ by 1. If $p_k \neq 0$ and $q_k = 0$ we replace $\frac{p_k}{q_k}$ by ∞, we get that

$$m^* = m_{n,I} \frac{P_I}{P_n},$$

where $m_{n,I}$ is as in (2.13), and we get from Theorem 1.12 for the convex functions $f(x)$ where $x_{i+1} \geq x_i$, $i=1,\dots,n-1$,

$$\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) \geq m_{n,I}\left(\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)\right),$$

and after replacing x_i by $\frac{x_i}{p_i}$, $i=1,\dots,n$ we get (2.14). It is obvious that in this case $\frac{P_n}{P_I} \geq m_{n,I} \geq 1$, therefore, (2.15) and (2.16) hold.

If $f(x)$ is nonnegative and superquadratic (therefore also convex), we get for any increasing n-tuple in T^n, $x=(x_1,\dots,x_n)$ where T is an interval $[0,a]$ or $[0,\infty)$, that

$$\begin{aligned} & \sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\sum_{i=1}^{n} \frac{p_i x_i}{P_n}\right) - m_{n,I}\left(\sum_{i\in I} p_i f(x_i) - P_I f\left(\frac{\sum_{i\in I} p_i x_i}{P_I}\right)\right) \\ \geq\ & P_n n f\left(\left(\sum_{i=1}^{n} p_i \left| x_i - \sum_{j=1}^{n} \frac{p_j}{P_n} x_j\right| \right.\right. \\ & \left.\left. + m_{n,I} P_I \left|\sum_{i=1}^{n} \frac{p_i x_i}{P_n} - \frac{\sum_{i\in I} p_i x_i}{P_I}\right| - m_{n,I} \sum_{i\in I} p_i \left| x_i - \sum_{j=1}^{n}\frac{p_j}{P_n} x_j\right|\right) \frac{1}{P_n n}\right), \end{aligned}$$

and we get (2.17) after replacing x_i by $\frac{x_i}{p_i}$ $i=1,\dots,n$, for a nonnegative superquadratic functions.

In this case the superquadratic function f is nonnegative and therefore also convex, hence, from (2.17), as $m_{n,I} \geq 1$, we get that (2.18) and as in Theorem 2.1, that (2.3) holds. ∎

3. COMMENTS

Comment 1. *The main results in* [11] *is*

Theorem 3.1. *If* $x_k \in \mathbb{R},\ a_k > 0,\ (1 \le k \le n)$, *then*

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \cdots + \frac{x_n^2}{a_n} \ge \frac{(x_1 + x_2 + \cdots + x_n)^2}{a_1 + a_2 + \cdots + a_n} + \max_{1 \le i < j \le n} \frac{(a_i x_j - a_j x_i)^2}{a_i a_j (a_i + a_j)}. \tag{3.1}$$

This theorem is a consequence of part (a) of Corollary C Multiplying it with (-1), *one gets*

$$-F(I_n) \ge \max_{1 \le i < j \le n} \left\{ a_i f(x_i) + a_j f(x_j) - (a_i + a_j) f\left(\frac{a_i x_i + a_j x_j}{a_i + a_j} \right) \right\}.$$

If $I_n = \{1, \ldots, n\}$, *then* $P_I = \sum_{i=1}^n a_i = a_1 + a_2 + \cdots + a_n$ *and we have*

$$\sum_{i=1}^n a_i f(x_i) - \sum_{i=1}^n a_i \cdot f\left(\frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i} \right)$$
$$\ge \max_{1 \le i < j \le n} \left\{ a_i f(x_i) + a_j f(x_j) - (a_i + a_j) f\left(\frac{a_i x_i + a_j x_j}{a_i + a_j} \right) \right\}.$$

Substitution $x_i \to \dfrac{x_i}{a_i}$ leads to

$$\sum_{i=1}^n a_i f\left(\frac{x_i}{a_i} \right) - \sum_{i=1}^n a_i \cdot f\left(\frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n a_i} \right)$$
$$\ge \max_{1 \le i < j \le n} \left\{ a_i f\left(\frac{x_i}{a_i} \right) + a_j f\left(\frac{x_j}{a_j} \right) - (a_i + a_j) f\left(\frac{x_i + x_j}{a_i + a_j} \right) \right\}.$$

Specifically, for the convex function $f(x) = x^2$, by some treatment of the expression in the brackets we get the inequality

$$\sum_{i=1}^n \frac{x_i^2}{a_i} \ge \frac{\left(\sum_{i=1}^n x_i\right)^2}{\sum_{i=1}^n a_i} + \max_{1 \le i < j \le n} \left\{ \frac{(a_j x_i - a_i x_j)^2}{a_i a_j (a_i + a_j)} \right\}$$

This inequality is the same as (3.1).

This is also a special case of Theorem 2.1 for $f(x) = x^2$.

Comment 2. *Examples 2.2 and 2.3 have shown that a lemma analogue to Corollary 1.9 does not exist. Namely,* $I \subseteq I_n$ *does not ensure* $\Delta_I \le \Delta I_n$.

However in the case of the superquadratic function $f(x) = x^2$ *as* $d_i = \Delta_i$ *we get that* $d_n = \Delta_n \ge \Delta_{n-1} \ge, ..., \Delta_{n-2} \ge, ..., \ge \Delta_2$, *so in this case* $I \subseteq I_n$ *ensures* $\Delta_I \le \Delta I_n$.

It may be of interest to find what additional conditions on the superquadratic function or on the a_i-*th and the* x_i-*th are needed so that* $I \subseteq I_n$ *leads to* $\Delta_I \le \Delta I_n$.

Comment 3. *It is possible to show that (2.1) in Theorem 2.1 results from the main theorems in* [2]. *A similar result for convex functions was proved in* [8, Theorem 1] *and* [4, Theorem C].

In [2, Theorem 3] it was proved that for a superquadratic function $f : [0, \infty) \to \mathbb{R}$ the following holds

$$\sum_{i=1}^n a_i f(x_i) - f\left(\sum_{i=1}^n a_i x_i \right) - m \left(\sum_{i=1}^n b_i f(x_i) - f\left(\sum_{i=1}^n b_i x_i \right) \right) \tag{3.2}$$
$$\ge m f\left(\left| \sum_{i=1}^n (a_i - b_i) x_i \right| \right) + \sum_{i=1}^n (a_i - m b_i) f\left(\left| x_i - \sum_{j=1}^n a_j x_i \right| \right),$$

where $a_i \geq 0,\ b_i > 0,\ i = 1, ..., n$, $\sum_1^n a_i = \sum_1^n b_i = 1$, and $m = \min_{1\leq i\leq n}\left(\frac{a_i}{b_i}\right)$.

Inserting $a_i = \frac{p_i}{P_n},\ b_i = \frac{q_i}{Q_n}$, $p_i \geq 0$, $q_i > 0$, $P_n = \sum_{i=1}^n p_i > 0$, $Q_n = \sum_{i=1}^n q_i$ and $\widetilde{m} = \frac{P_n}{Q_n}m$, we get from (3.2) that for a superquadratic function $f : [0,\infty\rangle \to \mathbb{R}$

$$\sum_{i=1}^{n} p_i f(x_i) - P_n f\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right) - \widetilde{m}\left(\sum_{i=1}^{n} q_i f(x_i) - Q_n f\left(\sum_{i=1}^{n} q_i x_i\right)\right) \tag{3.3}$$
$$\geq \widetilde{m} Q_n f\left(\left|\sum_{i=1}^{n}\left(\frac{p_i}{P_n} - \frac{q_i}{Q_n}\right)x_i\right|\right) + \sum_{i=1}^{n}(p_i - \widetilde{m}q_i) f\left(\left|x_i - \sum_{j=1}^{n}\frac{p_j}{P_n}x_i\right|\right)$$

Holds.

We deal now with (3.3) in a special case. In this case as in Theorem 2.7 we allow part of the $q_i - th$ to be equal to zero under the condition that if $p_j = 0$, so is $q_j = 0$ and we consider in this case that $\frac{p_j}{q_j} = 1$. If $p_k \neq 0$ and $q_k = 0$ we replace $\frac{p_k}{q_k}$ by ∞.

Therefore it is obvious that (3.3) holds and that $\widetilde{m} = \min_{1\leq i\leq n}\left(\frac{p_i}{q_i}\right)$.

Now we choose

$$q_i = \begin{cases} p_i, & i \in I \\ 0, & i \in \overline{I} \end{cases}.$$

Hence $Q_n = \sum_{i=1}^n q_i = \sum_{i\in I} p_i = P_I$, and in this case $\widetilde{m} = 1$, and from (3.3) we get that (2.6) holds. Hence for a superquadratic function $f : [0,+\infty) \to \mathbb{R}$ (2.1) holds.

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