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THE RIEMANN-STIELTJES INTEGRAL ON TIME SCALES

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ABSTRACT. We study the process of integration on time scales in the sense of Riemann-Stieltjes. Analogues of the classical properties are proved for a generic time scale, and examples are given.

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1. INTRODUCTION

The development of integration on time scales is a recent but already well studied subject. Available integration notions on time scales include the Riemann delta integral [2, 11, 12], the Riemann nabla integral [11], the Riemann diamond-alpha integral [13], the Lebesgue delta and nabla integrals [2, 10], and the more general Henstock-Kurzweil delta and nabla integrals [14, 17]. Other studies on time scales are dedicated to improper integrals [3] and to multiple integration [4, 5]. Surprisingly enough, the Stieltjes integral has not received attention in the literature of time scales.

In this paper we study the process of Stieltjes integration on time scales, both in nabla and delta sense. We trust that such integrals will find interesting applications in the study of dynamic equations on time scales, enabling to study more general situations than those treated before (cf. Section 5). Because delta and nabla theories are similar, we avoid repetition by following the approach promoted by Bartosiewicz and Piotrowska [1]: the box symbol $\Box$ is here used to represent the delta operator Δ as well as the nabla operator ∇.

It is assumed that the reader is familiar with the time scale calculus and the notations for delta and nabla differentiation. For an introduction to time scales the reader is referred to the book by Bohner and Peterson [6]. Throughout the paper $\mathbb{T}$ denotes a time scale. Let $a, b \in \mathbb{T}$ and $a < b$. We distinguish $[a, b]$ as a real interval and we define $I = [a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. In this sense, $[a, b] = [a, b]_{\mathbb{R}}$. Along the text I is a nonempty, closed, and bounded interval consisting of points from a time scale $\mathbb{T}$. Moreover, if $I = [a, b]_{\mathbb{T}}$, then we define $I_{\Delta} := [a, \rho(b)]_{\mathbb{T}}$ and $I_{\nabla} := [\sigma(a), b]_{\mathbb{T}}$. By $I_{\Box}$ we denote one of them, where $\Box$ means either ∇ or Δ. Similarly, we use "$\Box$" as a common notation for the two kinds of derivatives on time scales: one can read $f^{\Box}$ either as f^{Δ} or as f^{∇}.

2. THE RIEMANN-STIELTJES $\Box$–INTEGRAL

Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$, $a < b$, and $I = [a, b]_{\mathbb{T}}$. A partition of I is any finite ordered subset

$$P = \{t_0, t_1, \ldots, t_n\} \subset [a, b]_{\mathbb{T}}, \quad \text{where } a = t_0 < t_1 < \ldots < t_n = b\,.$$

Let g be a real-valued increasing function on I. Each partition $P = \{t_0, t_1, \ldots, t_n\}$ of I decomposes I into subintervals $I_{\Box j} = [t_{j-1}, t_j]_{\Box}$, $j = 1, 2, \ldots, n$, such that $I_{\Box j} \cap I_{\Box k} = \emptyset$ for any $k \neq j$. By $\Delta t_j = t_j - t_{j-1}$ we denote the length of the jth subinterval in the partition P; by $\mathcal{P}(I)$ the set of all partitions of I.

Let $P_m, P_n \in \mathcal{P}(I)$. If $P_m \subset P_n$ we call P_n a refinement of P_m. If P_m, P_n are independently chosen, then the partition $P_m \cup P_n$ is a common refinement of P_m and P_n.

Let us now consider a strictly increasing real-valued function g on the interval I. Then, for the partition P of I we define

$$g(P) = \{g(a) = g(t_0), g(t_1), \ldots, g(t_{n-1}), g(t_n) = g(b)\} \subset g(I)$$

and $\Delta g_j = g(t_j) - g(t_{j-1})$. We note that Δg_j is positive and $\sum_{j=1}^{n} \Delta g_j = g(b) - g(a)$. Moreover, $g(P)$ is a partition of $[g(a), g(b)]_{\mathbb{R}}$. In what follows, for the particular case $g(t) = t$ we obtain the Riemann sums for delta integrals studied in [2]. We note that for a general g the image $g(I)$ is not necessarily an interval in the classical sense, even for rd-continuous functions g, because our interval I may contain scattered points. From now on let g be always a strictly increasing real function on the considered interval $I = [a, b]_{\mathbb{T}}$.

Lemma 2.1. *Let $I = [a, b]_{\mathbb{T}}$ be a closed (bounded) interval in $\mathbb{T}$ and let g be continuous on I. For every $\delta > 0$ there is a partition $P_{\delta} = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}(I)$ such that for each*

$j \in \{1, 2, \dots, n\}$ *one has:*

$$\Delta g_j = g(t_j) - g(t_{j-1}) \leq \delta \quad \textit{or} \quad \Delta g_j > \delta \ \wedge \ \rho(t_j) = t_{j-1}.$$

Proof. Let $I = [a, b]_{\mathbb{T}}$ be closed (bounded) interval in $\mathbb{T}$. Firstly, let us observe that if for $t \in I$ $g(t) \in [g(a), g(b)]_{\mathbb{R}}$, then $a \leq t \leq b$ because g is an increasing function on I. We define two families of sets, $B_j = (g(t_{j-1}), g(t_{j-1}) + \delta]_{\mathbb{R}} \cap [g(a), g(b)]_{\mathbb{R}}$ and $A_j = \{t \in I : g(t) \in B_j\}$. Inductively, we construct the partition taking $t_0 = a$ and

$$t_j = \begin{cases} \sup A_j, & \text{if } A_j \neq \emptyset\,, \\ \sigma(t_{j-1}), & \text{if } A_j = \emptyset\,. \end{cases}$$

Then, we get $t_0 < t_1 < \dots < t_n$, where $t_n = b$. It follows that if $t_{j-1} < t_j$ and $t_j = \sigma(t_{j-1})$, then $\rho(t_j) = t_{j-1}$ for any time scale $\mathbb{T}$. ∎

Let f be a real-valued and bounded function on the interval I. Let us take a partition $P = \{t_0, t_1, \dots, t_n\}$ of I. Denote $I_{\Box j} = [t_{j-1}, t_j]_{\Box}$, $j = 1, 2, \dots, n$, and

$$m_{\Box j} = \inf_{t \in I_{\Box j}} f(t), \quad M_{\Box j} = \sup_{t \in I_{\Box j}} f(t)\,.$$

The *upper Darboux–Stieltjes* $\Box$*–sum of* f *with respect to the partition* P, denoted by $U_{\Box}(P, f, g)$, is defined by

$$U_{\Box}(P, f, g) = \sum_{j=1}^{n} M_{\Box j} \Delta g_j\,,$$

while the *lower Darboux–Stieltjes* $\Box$*–sum of* f *with respect to the partition* P, denoted by $L_{\Box}(P, f, g)$, is defined by

$$L_{\Box}(P, f, g) = \sum_{j=1}^{n} m_{\Box j} \Delta g_j\,.$$

Example 2.2. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, $f(t) = t$, $g(t) = t^2$, and $I = [0, 1]_{\mathbb{T}}$. For the partition $P = \{0, q^{-n+1}, \dots, q^{-1}, 1\}$, where $t_0 = 0 < q^{-n+1} < \dots < q^{j-n} < \dots < q^{-1} < 1 = t_n$ with $t_j = q^{j-n}$ for $j = 1, \dots, n$, we have $\Delta g_j = t_j^2 - t_{j-1}^2 = q^{2(j-n-1)}(q^2 - 1)$ for $j = 2, \dots, n$, while $\Delta g_1 = t_1^2 - 0 = q^{2-2n}$. Let us read $\Box = \Delta$. In this case $M_{\Delta j} = \rho(t_j)$ and $m_{\Delta j} = t_{j-1}$. For our partition we have that $I_{\Delta j} = [t_{j-1}, \rho(t_j)]_{\mathbb{T}} = t_{j-1}$, $j = 2, \dots, n$. Hence, $m_{\Delta j} = M_{\Delta j} = t_{j-1}$ for $f(t) = t$, $j = 2, \dots, n$, and $m_{\Delta 1} = 0$, $M_{\Delta 1} = \rho(t_1) = q^{-n}$. The lower and upper Darboux–Stieltjes Δ–sums are, respectively,

$$L_{\Delta}(P, f, g) = \sum_{j=2}^{n} t_{j-1} \Delta g_j = \frac{q+1}{q^2+q+1}(1 - q^{3(1-n)})\,,$$

$$U_{\Delta}(P, f, g) = q^{2-3n} + \sum_{j=2}^{n} t_{j-1} \Delta g_j = q^{2-3n} + L(P, f, g) = \frac{q+1+q^{2-3n}}{q^2+q+1}\,.$$

Consider now $\Box = \nabla$. In this case $M_{\nabla j} = t_j$ and $m_{\nabla j} = \sigma(t_{j-1})$. For our partition we have that $I_{\nabla j} = [\sigma(t_{j-1}), t_j]_{\mathbb{T}} = t_j$, $j = 2, \dots, n$. Hence, $m_{\nabla j} = M_{\nabla j} = t_j$ for $f(t) = t$, $j = 2, \dots, n$, and $m_{\nabla 1} = \sigma(0) = 0$, $M_{\nabla 1} = t_1 = q^{1-n}$. The lower and upper Darboux–Stieltjes ∇–sums are, respectively,

$$L_{\nabla}(P, f, g) = \sum_{j=2}^{n} t_j \Delta g_j = q\frac{q+1}{q^2+q+1}(1 - q^{3(1-n)})\,,$$

$$U_{\nabla}(P, f, g) = q^{3-3n} + \sum_{j=2}^{n} t_j \Delta g_j = q^{3-3n} + L_{\nabla}(P, f, g) = q\frac{q+1+q^{2-3n}}{q^2+q+1}\,.$$

For computing the value of a Riemann integral on time scales one uses the fact that $\Delta t_j \geq 0$. Since we assume that g is an increasing function, then $\Delta g_j \geq 0$. As we shall see below, also some other properties of the Riemann-integral (for function $g(t) = t$) are preserved for an arbitrary increasing function g.

Theorem 2.3. *Suppose that f is a bounded function on $I = [a, b]_{\mathbb{T}}$, a, $b \in \mathbb{T}$. Let $P \in \mathcal{P}(I)$ and $m \leq f(t) \leq M$ for all $t \in I$. Then,*

$$m(g(b) - g(a)) \leq L_\Box(P, f, g) \leq U_\Box(P, f, g) \leq M(g(b) - g(a))$$

and

$$L_\Box(P, f, g) \leq L_\Box(Q, f, g) \leq U_\Box(Q, f, g) \leq U_\Box(P, f, g)$$

for any refinement Q of P.

Proof. For $\Box = \Delta$ the proof is an immediate consequence of [2, Lemma 5.2]. For $\Box = \nabla$ the proof is done in similar steps. ∎

Definition 2.1. Let $I = [a, b]_{\mathbb{T}}$, where $a, b \in \mathbb{T}$. The upper Darboux-Stieltjes $\Box$–integral from a to b with respect to function g is defined by

$$\overline{\int_a^b} f(t)\Box g(t) = \inf_{P \in \mathcal{P}(I)} U_\Box(P, f, g)\,;$$

the lower Darboux-Stieltjes $\Box$–integral from a to b with respect to function g is defined by

$$\underline{\int_a^b} f(t)\Box g(t) = \sup_{P \in \mathcal{P}(I)} L_\Box(P, f, g)\,.$$

If $\overline{\int_a^b} f(t)\Box g(t) = \underline{\int_a^b} f(t)\Box g(t)$, then we say that f is $\Box$–integrable with respect to g on I, and the common value of the integrals, denoted by $\int_a^b f(t)\Box g(t) = \int_a^b f\Box g$, is called the Riemann-Stieltjes (or simply Stieltjes) $\Box$–integral of f with respect to g on I.

The set of all functions that are $\Box$–integrable with respect to g in the Riemann-Stieltjes (also Darboux-Stieltjes) sense will be denoted by $\mathcal{R}_\Box(g, I)$.

Theorem 2.4. *If $L_\Box(P, f, g) = U_\Box(P, f, g)$ for some $P \in \mathcal{P}(I)$, then function f is Riemann-Stieltjes $\Box$–integrable and*

$$\int_a^b f\Box g = L_\Box(P, f, g) = U_\Box(P, f, g)\,.$$

Proof. Follows immediately from the definition of Riemann-Stieltjes $\Box$–integral and Theorem 2.3. ∎

Example 2.5. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, and let us continue Example 2.2. Consider functions $f(t) = t$ and $g(t) = t^2$ on the interval $I = [0, 1]_{\mathbb{T}}$. For the partition $P = \{0, q^{-n+1}, \ldots, q^{-1}, 1\}$, where $t_0 = 0 < q^{-n+1} < \ldots < q^{j-n} < \ldots < q^{-1} < 1 = t_n$ with $t_j = q^{j-n}$ for $j = 1, \ldots, n$, we have

$$\overline{\int_a^b} f(t)\Delta g(t) = \inf_{P \in \mathcal{P}(I)} U_\Delta(P, f, g) = \lim_{n\to\infty} \frac{q + 1 + q^{2-3n}}{q^2 + q + 1} = \frac{q + 1}{q^2 + q + 1}$$

and

$$\underline{\int_a^b} f(t)\Delta g(t) = \sup_{P \in \mathcal{P}(I)} L_\Delta(P, f, g) = \lim_{n\to\infty} \frac{q + 1}{q^2 + q + 1}(1 - q^{3(1-n)}) = \overline{\int_a^b} f(t)\Delta g(t)\,.$$

Consequently, $\int_a^b f(t)\Delta g(t) = \frac{q+1}{q^2+q+1}$. We also have that

$$\overline{\int_a^b} f(t)\nabla g(t) = \inf_{P\in\mathcal{P}(I)} U_\nabla(P,f,g) = \frac{q^2+q}{q^2+q+1}$$

and

$$\underline{\int_a^b} f(t)\nabla g(t) = \sup_{P\in\mathcal{P}(I)} L_\nabla(P,f,g) = \lim_{n\to\infty} \frac{q^2+q}{q^2+q+1}(1-q^{3(1-n)}) = \overline{\int_a^b} f(t)\nabla g(t)\,.$$

Hence, $\int_a^b f(t)\nabla g(t) = \frac{q^2+q}{q^2+q+1} = q\int_a^b f(t)\Delta g(t)$.

Theorem 2.6. *Let f be a bounded function on $I=[a,b]_{\mathbb{T}}$, a, $b\in\mathbb{T}$, $m\le f(t)\le M$ for all $t\in I$, and g be a function defined and monotonically increasing on I. Then,*

$$m(g(b)-g(a)) \le \underline{\int_a^b} f(t)\Box g(t) \le \overline{\int_a^b} f(t)\Box g(t) \le M(g(b)-g(a))\,.$$

If $f\in\mathcal{R}_\Box(g,I)$, then

$$m(g(b)-g(a)) \le \int_a^b f(t)\Box g(t) \le M(g(b)-g(a))\,.$$

Proof. The definition of upper and lower Darboux-Stieltjes $\Box$–integral implies that $\underline{\int_a^b} f(t)\Box g(t) \le \overline{\int_a^b} f(t)\Box g(t)$. Thus,

$$\overline{\int_a^b} f(t)\Box g(t) = \inf_{P\in\mathcal{P}(I)} U_\Box(P,f,g) \le U_\Box(P,f,g) \le \sum_{j=1}^n M\Delta g_j = M(g(b)-g(a))\,.$$

Similarly, $m(g(b)-g(a)) \le \underline{\int_a^b} f(t)\Box g(t)$, and the proof is done. ∎

Theorem 2.7. *(Integrability criterion) Let f be a bounded function on $I=[a,b]_{\mathbb{T}}, a,b\in\mathbb{T}$. Then, $f\in\mathcal{R}_\Box(g,I)$ if and only if for every $\varepsilon>0$ there exists a partition $P\in\mathcal{P}(I)$ such that*

$$U_\Box(P,f,g) - L_\Box(P,f,g) < \varepsilon\,. \tag{2.1}$$

Proof. Suppose that $f\in\mathcal{R}_\Box(g,I)$ and let $\varepsilon>0$. Because

$$\int_a^b f\Box g = \inf_{P\in\mathcal{P}(I)} U_\Box(P,f,g) = \sup_{P\in\mathcal{P}(I)} L_\Box(P,f,g)$$

there exist P_1 and $P_2\in\mathcal{P}(I)$ such that

$$\int_a^b f\Box g < U_\Box(P_1,f,g) < \int_a^b f\Box g + \frac{\varepsilon}{2}$$

and

$$\int_a^b f\Box g - \frac{\varepsilon}{2} < L_\Box(P_2,f,g) < \int_a^b f\Box g\,.$$

Hence, $U_\Box(P_1,f,g) - \int_a^b f(t)\Box g(t) < \frac{\varepsilon}{2}$ and $\int_a^b f(t)\Box g(t) - L_\Box(P_2,f,g) < \frac{\varepsilon}{2}$. Let P be a common refinement of P_1 and P_2. Then, Theorem 2.3 implies that $U_\Box(P,f,g) - \int_a^b f(t)\Box g(t) < \frac{\varepsilon}{2}$ and $\int_a^b f(t)\Box g(t) - L_\Box(P,f,g) < \frac{\varepsilon}{2}$. Thus, $U_\Box(P,f,g) - L_\Box(P,f,g) < \varepsilon$.

Conversely, let f be a bounded function on I and g be an increasing function on I. Suppose that for $\varepsilon > 0$ there exists $P \in \mathcal{P}(I)$ such that $U_\Box(P, f, g) - L_\Box(P, f, g) < \varepsilon$. The definition of Riemann-Stieltjes $\Box$–integral and Theorem 2.3 imply that

$$L_\Box(P, f, g) \leq \underline{\int_a^b} f \Box g \leq \overline{\int_a^b} f \Box g \leq U_\Box(P, f, g)\,.$$

It is obvious that $0 \leq \overline{\int_a^b} f \Box g - \underline{\int_a^b} f \Box g < \varepsilon$. Since ε is arbitrary, then $\overline{\int_a^b} f(t) \Box g(t) = \underline{\int_a^b} f \Box g$ and $f \in \mathcal{R}_\Box(g, I)$. ∎

Theorem 2.8. *Suppose that P_δ is a partition as stated in Lemma 2.1. A bounded function f on $I = [a, b]_\mathbb{T}$ is integrable if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$P_\delta \in \mathcal{P}(I) \Rightarrow U_\Box(P_\delta, f, g) - L_\Box(P_\delta, f, g) < \varepsilon\,.$$

Proof. Part "$\Leftarrow$" follows immediately from the integrability criterion (Theorem 2.7).

For the proof of the part "$\Rightarrow$" suppose that $f \in \mathcal{R}_\Box(g, I)$. Let $\varepsilon > 0$. Then, there exist partitions $\mathcal{P}_\delta^1$ and $\mathcal{P}_\delta^2$ such that

$$U(\mathcal{P}_\delta^1, f, g) < \overline{\int_a^b} f \Box g + \frac{\varepsilon}{2} \tag{2.2}$$

and

$$\underline{\int_a^b} f \Box g - \frac{\varepsilon}{2} < L(\mathcal{P}_\delta^2, f, g). \tag{2.3}$$

Let $\mathcal{P}_\delta = \mathcal{P}_\delta^1 \cup \mathcal{P}_\delta^2$. Theorem 2.3 and inequalities (2.2) and (2.3) imply that

$$\underline{\int_a^b} f \Box g - \frac{\varepsilon}{2} < L(\mathcal{P}_\delta, f, g) \leq U(\mathcal{P}_\delta, f, g) < \overline{\int_a^b} f \Box g + \frac{\varepsilon}{2}$$

Because $f \in \mathcal{R}_\Box(g, I)$, then $\underline{\int_a^b} f \Box g = \overline{\int_a^b} f \Box g$. Hence, $U(\mathcal{P}_\delta, f, g) - L(\mathcal{P}_\delta, f, g) < \varepsilon$. ∎

Lemma 2.3 and the properties of the Riemann delta (nabla) integral imply the following:

Theorem 2.9. *Let $I = [a, b]_\mathbb{T}$, $a, b \in \mathbb{T}$. Then, the condition $f \in \mathcal{R}_\Box(g, I)$ is equivalent to each one of the following items:*

i) *f is a monotonic function on I;*
ii) *f is a continuous function on I;*
iii) *f is regulated on I;*
iv) *f is bounded and has a finite number of discontinuity points on I.*

Theorem 2.10. *Let f be bounded on $I = [a, b]_\mathbb{T}$, $a < b$. Then,*

i) *If there exists an $\varepsilon > 0$ and a partition $P^* \in \mathcal{P}(I)$ such that the inequality (2.1) in Theorem 2.7 is satisfied, then (2.1) is satisfied for every refinement P of P^*.*
ii) *If inequality (2.1) is satisfied for the partition P given by $t_0 = a < t_1 < \ldots < t_n = b$, and ξ_j, $\tau_j \in I_{\Box j} = [t_{j-1}, t_j]_\Box$ for $j = 1, 2, \ldots, n$, then*

$$\sum_{j=1}^{n} \left| f(\xi_j) - f(\tau_j) \right| \Delta g_j < \varepsilon\,.$$

iii) *If* $f \in \mathcal{R}_\Box(g, I)$ *and inequality (2.1) is satisfied for the partition* P *given by* $t_0 = a < t_1 < \ldots < t_n = b$ *and* $\xi_j \in I_{\Box j} = [t_{j-1}, t_j]_\Box$ *for* $j = 1, 2, \ldots, n$*, then*

$$\left|\sum_{j=1}^{n} f(\xi_j)\Delta g_j - \int_a^b f(t)\Box g(t)\right| < \varepsilon\,.$$

Proof. Part i) follows from Theorem 2.3. Let us now prove part ii). Let f be a real-valued and bounded function on the interval I. Let us take a partition $P = \{t_0, t_1, \ldots, t_n\}$ of I. Let ξ_j, $\tau_j \in I_{\Box j} = [t_{j-1}, t_j]_\Box$ for $j = 1, 2, \ldots, n$. Then, $M_{\Box j} - f(\xi_j) \geq 0$, $f(\tau_j) - m_{\Box j} \geq 0$, and

$$\sum_{j=1}^{n} \left|f(\xi_j) - f(\tau_j)\right| \Delta g_j \leq \sum_{j=1}^{n} \left|\big(f(\xi_j) - f(\tau_j)\big) + \big(M_{\Box j} - f(\xi_j)\big) + \big(f(\tau_j) - m_{\Box j}\big)\right| \Delta g_j\,.$$

It follows that

$$\sum_{j=1}^{n} \left|f(\xi_j) - f(\tau_j)\right| \Delta g_j \leq \sum_{j=1}^{n} \left|M_{\Box j} - m_{\Box j}\right| \Delta g_j = U_\Box(P, f, g) - L_\Box(P, f, g) < \varepsilon\,.$$

Finally, let us prove part iii). Let the inequality (2.1) be satisfied for the partition P given by $t_0 = a < t_1 < \ldots < t_n = b$. Let $\xi_j \in I_{\Box j} = [t_{j-1}, t_j]_\Box$ for $j = 1, 2, \ldots, n$. Then, $\sum_{j=1}^{n} f(\xi_j)\Delta g_j \leq U_\Box(P, f, g)$ and $L_\Box(P, f, g) \leq \int_a^b f(t)\Box g(t)$. Hence,

$$\left|\sum_{j=1}^{n} f(\xi_j)\Delta g_j - \int_a^b f(t)\Box g(t)\right| \leq \left|U_\Box(P, f, g) - L_\Box(P, f, g)\right| < \varepsilon\,.$$

∎

Theorem 2.11 gives a comparison between the two kinds of Riemann–Stieljes integrals on time scales and the Riemann–Stieljes integral on the real interval. In the particular case $g(t) = t$ one gets the result as stated in [1].

Theorem 2.11. *Let* $a, b \in \mathbb{T}$ *and* $g : [a, b] \to \mathbb{R}$ *be a strictly increasing function on* $[a, b]$*. Let* $f : [a, b] \to \mathbb{R}$*. Denote by* $f|_\mathbb{T}$ *and* $g|_\mathbb{T}$ *the restrictions of functions* f *and* g *to the time scale* $\mathbb{T}$*. Then,*

i) $\int_a^b f|_\mathbb{T}(t)\Delta\, g|_\mathbb{T}(t) \leq \int_a^b f(t)dg(t) \leq \int_a^b f|_\mathbb{T}(t)\nabla\, g|_\mathbb{T}(t)$ *if* f *is strictly increasing on* $[a, b]$*;*

ii) $\int_a^b f|_\mathbb{T}(t)\nabla\, g|_\mathbb{T}(t) \leq \int_a^b f(t)dg(t) \leq \int_a^b f|_\mathbb{T}(t)\Delta\, g|_\mathbb{T}(t)$ *if* f *is strictly decreasing on* $[a, b]$.

Proof. We prove here the item i). The proof of ii) is similar. If f is strictly increasing on $[a, b]$, then $f|_\mathbb{T}$ is regulated on $[a, b]_\mathbb{T}$ and all integrals exist for an increasing function g. Let $P \in \mathcal{P}([a, b]_\mathbb{T})$ and $P = \{t_0, \ldots, t_n\}$, where $a = t_0 < t_1 < \ldots < t_n = b$. Then, $\int_a^b f(t)dg(t) \leq \sum_{j=1}^{n} f(t_j)\Delta g_j = U_\nabla(P, f, g)$. Taking the infimum of the right-hand side over all partitions from $\mathcal{P}([a, b]_\mathbb{T})$ we get $\int_a^b f(t)dg(t) \leq \overline{\int_a^b} f|_\mathbb{T}(t)\nabla\, g|_\mathbb{T}(t)$. Similarly, $\int_a^b f(t)dg(t) \geq \sum_{j=1}^{n} f(t_{j-1})\Delta g_j = L_\Delta(P, f, g)$. Taking now the supremum of the right-hand side we get that $\int_a^b f(t)dg(t) \geq \underline{\int_a^b} f|_\mathbb{T}(t)\Delta\, g|_\mathbb{T}(t)$. For $f|_\mathbb{T} \in R_\Box(g|_\mathbb{T}, [a, b]_\mathbb{T})$, then also $\int_a^b f(t)dg(t) \leq \int_a^b f|_\mathbb{T}(t)\nabla\, g|_\mathbb{T}(t)$ and $\int_a^b f(t)dg(t) \geq \int_a^b f|_\mathbb{T}(t)\Delta\, g|_\mathbb{T}(t)$. ∎

Corollary 2.12. *Let* $a, b \in \mathbb{T}$*,* $I = [a, b]_\mathbb{T}$*,* $g : I \to \mathbb{R}$ *be a strictly increasing function, and* $f : I \to \mathbb{R}$*. Then,*

i) $\int_a^b f(t)\Delta g(t) \leq \int_a^b f(t)\nabla g(t)$ *if* f *is strictly increasing on* I*;*

ii) $\int_a^b f(t)\nabla g(t) \leq \int_a^b f(t)\Delta g(t)$ *if* f *is strictly decreasing on* I.

3. ALGEBRAIC PROPERTIES OF THE RIEMANN-STIELTJES □–INTEGRAL

In this section we prove some algebraic properties of the Riemann-Stieltjes integral on time scales. The properties are valid for an arbitrary time scale $\mathbb{T}$ with at least two points. We define $\int_a^a f(t)\Box g(t) = 0$ and $\int_a^b f(t)\Box g(t) = -\int_b^a f(t)\Box g(t)$ for $a > b$.

Theorem 3.1. *Let* $I = [a,b]_{\mathbb{T}}$, $a, b \in \mathbb{T}$. *Every constant function* $f : \mathbb{T} \to \mathbb{R}$, $f(t) \equiv c$, *is Stieltjes* □*–integrable with respect to* g *on* I *and*

$$\int_a^b c\Box g(t) = c\,(g(b) - g(a))\,.$$

Proof. Let $P \in \mathcal{P}(I)$ and $P = \{t_0, \ldots, t_n\}$. Then,

$$\begin{aligned} L_\Box(P,f,g) = U_\Box P,f,g) &= c\sum_{j=1}^{n} \Delta g_j \\ &= g(t_1) - g(a) + g(t_2) - g(t_1) + \cdots + g(b) - g(t_{n-1}) \\ &= g(b) - g(a). \end{aligned}$$

Hence, $\underline{\int_a^b} f\Box g = \overline{\int_a^b} f\Box g = c\,(g(b) - g(a))$. ∎

Theorem 3.2. *Let* $t \in \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$. *If* f *is Riemann-Stieltjes* Δ*–integrable with respect to* g *from* t *to* $\sigma(t)$, *then*

$$\int_t^{\sigma(t)} f(\tau)\Delta g(\tau) = f(t)\,(g^\sigma(t) - g(t))\ , \tag{3.1}$$

where $g^\sigma = g \circ \sigma$. *Moreover, if* g *is* Δ*–differentiable at* t, *then*

$$\int_t^{\sigma(t)} f(\tau)\Delta g(\tau) = \mu(t)f(t)g^\Delta(t)\,. \tag{3.2}$$

Proof. For $\sigma(t) = t$ both equations (3.1) and (3.2) hold. When $\sigma(t) > t$, only one partition for $I = [t, \sigma(t)]_{\mathbb{T}}$ is possible. In that case $P = \{t, \sigma(t)\}$ and we have one set $I_{\Delta 1} = [t, \rho(\sigma(t))]_{\mathbb{T}} = \{t\}$. Hence, $m_{\Delta 1} = M_{\Delta 1} = f(t)$, and

$$\underline{\int_t^{\sigma(t)}} f\Delta g = \overline{\int_t^{\sigma(t)}} f\Delta g = U_\Delta(P,f,g) = L_\Delta(P,f,g) = f(t)(g(\sigma(t)) - g(t))\,.$$

As $\sigma(t) > t$, then also $\mu(t) \neq 0$ and $g^\sigma(t) - g(t) = \mu(t)g^\Delta(t)$. ∎

Theorem 3.3. *Let* $t \in \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$. *If* f *is Riemann-Stieltjes* ∇*–integrable with respect to* g *from* $\rho(t)$ *to* t, *then*

$$\int_{\rho(t)}^{t} f(\tau)\nabla g(\tau) = f(t)\,(g(t) - g^\rho(t))\ , \tag{3.3}$$

where $g^\rho = g \circ \rho$. *Moreover, if* g *is* ∇*–differentiable at* t, *then*

$$\int_{\rho(t)}^{t} f(\tau)\nabla g(\tau) = \nu(t)f(t)g^\nabla(t)\,. \tag{3.4}$$

Proof. For $\rho(t) = t$ both equations (3.3) and (3.4) hold. When $t > \rho(t)$, only one partition for $I = [\rho(t), t]_{\mathbb{T}}$ is possible. In that case $P = \{\rho(t), t\}$ and we have one set $I_{\nabla 1} = [\sigma(\rho(t)), t]_{\mathbb{T}} = \{t\}$. Hence, $m_{\nabla 1} = M_{\nabla 1} = f(t)$, and

$$\underline{\int_{\rho(t)}^{t}} f\nabla g = \overline{\int_{\rho(t)}^{t}} f\nabla g = U_{\nabla}(P, f, g) = L_{\nabla}(P, f, g) = f(t)(g(t) - g(\rho(t)))\,.$$

As $\rho(t) < t$, then also $\nu(t) > 0$ and $g(t) - g^{\rho}(t) = \nu(t)g^{\nabla}(t)$. ∎

Corollary 3.4. *Let $a, b \in \mathbb{T}$ and $a < b$.*

i) *If $\mathbb{T} = \mathbb{R}$, then a bounded function f on $[a, b]_{\mathbb{T}}$ is Riemann-Stieltjes $\Box$–integrable with respect to the increasing function g from a to b if and only if f is Riemann-Stieltjes integrable on I in the classical sense. Moreover, then $\int_a^b f(t)\Box g(t) = \int_a^b f(t)dg(t)$, where the integral on the right hand side is the classical Riemann-Stieltjes integral (see, e.g.,* [7, Chapter 4]*).*

ii) *If $\mathbb{T} = \mathbb{Z}$, then each function $f : \mathbb{Z} \to \mathbb{R}$ is Riemann-Stieltjes $\Box$–integrable from a to b with respect to an arbitrarily increasing function $g : \mathbb{Z} \to \mathbb{R}$. Moreover, $\int_a^b f(t)\Delta g(t) = \sum_{t=a}^{b-1} f(t)(g(t+1) - g(t))$, $\int_a^b f(t)\nabla g(t) = \sum_{t=a+1}^{b} f(t)(g(t) - g(t-1))$.*

Proof. i) Notice that for $\mathbb{T} = \mathbb{R}$ Definition 2.1 coincides with the classical definition of the Riemann-Stieltjes integral. Moreover, since for $\mathbb{T} = \mathbb{R}$ the $\Box$–differential coincides with the standard differential, it follows that

$$\int_a^b f(t)\Box g(t) = \int_a^b f(t)g'(t)dt = \int_a^b f(t)dg(t)\,.$$

ii) Let $I = [a, b]_{\mathbb{Z}}$, $a < b$, $a, b \in \mathbb{Z}$. Consider the partition $P_\delta \in \mathcal{P}(I)$ given by $a = t_0 < t_1 < \ldots < t_n = b = a + n$, where $t_j = a + j$ for $j = 0, \ldots, n$. Notice that $[t_{j-1}, \rho(t_j)]_{\mathbb{T}} = \{t_{j-1}\}$ for each $j = 1, \ldots, n$ and $\Delta g(t_j) = g(t_j) - g(t_j - 1)$. A direct calculation shows that $L_\Delta(P_\delta, f, g) = U_\Delta(P_\delta, f, g) = \sum_{j=1}^{n} f(t_{j-1})\Delta g(t_j) = \sum_{t=a}^{b-1} f(t)(g(t+1) - g(t))$. Hence, $U_\Delta(P_\delta, f, g) = L_\Delta(P_\delta, f, g)$ and Theorem 2.4 imply the desired formula for the Δ–case. Similarly proof holds for the ∇–case. ∎

Theorem 3.5. *(Linearity) Let functions f_1 and f_2 be Riemann-Stieltjes $\Box$–integrable on the interval $[a, b]_{\mathbb{T}}$ with respect to g, and c be a constant. Then,*

i) *$cf_1 \in \mathcal{R}_{\Box}(g, I)$ and $\int_a^b cf_1\Box g = c\int_a^b f_1\Box g$;*

ii) *$f_1 + f_2 \in \mathcal{R}_{\Box}(g, I)$ and $\int_a^b (f_1 + f_2)\Box g = \int_a^b f_1\Box g + \int_a^b f_2\Box g$.*

Proof. Let $I_j = (t_{j-1}, t_j]_{\mathbb{T}}$, and $p \in \mathcal{P}(I)$ be any partition of the interval I. Denote $I_j = [t_{j-1}, t_j]_{\Box}$ and

$$m_j^{f_1} = \inf_{t \in I_{\Box j}} f_1(t)\,, \qquad M_j^{f_1} = \sup_{t \in I_{\Box j}} f_1(t)\,,$$

$$m_j^{f_2} = \inf_{t \in I_{\Box j}} f_2(t)\,, \qquad M_j^{f_2} = \sup_{t \in I_{\Box j}} f_2(t)\,.$$

i) Let us notice that for $c = 0$ function $cf_1 \in \mathcal{R}_{\Box}(g, I)$. Suppose that $c \neq 0$ and denote

$$\mathfrak{m}_j = \inf_{t \in I_{\Box j}} (cf_1)(t) \quad \text{and} \quad \mathfrak{M}_j = \sup_{t \in I_{\Box j}} (cf_1)(t)\,.$$

Then,

$$\mathfrak{m}_j = cm_j^{f_1} \quad \text{and} \quad \mathfrak{M}_j = cM_j^{f_1} \quad \text{for } c > 0;$$

$$\mathfrak{m}_j = cM_j^{f_1} \quad \text{and} \quad \mathfrak{M}_j = cm_j^{f_1} \quad \text{for } c < 0\,.$$

This implies that

$$L_\Box(P, cf_1, g) = \begin{cases} cL_\Box(P, f_1, g), & \text{for } c > 0 \\ cU_\Box(P, f_1, g), & \text{for } c < 0 \end{cases}$$

and

$$U_\Box(P, cf_1, g) = \begin{cases} cU_\Box(P, f_1, g), & \text{for } c > 0 \\ cL_\Box(P, f_1, g), & \text{for } c < 0. \end{cases}$$

Thus,

$$U_\Box(P, cf_1, g) - L_\Box(P, cf_1, g) = |c|(U_\Box(P, f_1, g) - L_\Box(P, f_1, g)) \,. \tag{3.5}$$

Because $f_1 \in \mathcal{R}_\Box(g, I)$, then for any $\varepsilon > 0$ there exists a partition P such that $U_\Box(P, cf_1, g) - L_\Box(P, cf_1, g) < \frac{\varepsilon}{|c|}$. Together with (3.5) this leads to $U_\Box(P, cf_1, g) - L_\Box(P, cf_1, g) < \varepsilon$. Hence, $cf_1 \in \mathcal{R}_\Box(g, I)$. Moreover,

$$\underline{\int_a^b} cf_1 \Box g = \sup_{P \in \mathcal{P}(I)} L_\Box(P, cf_1, g) = c \sup_{P \in \mathcal{P}(I)} L_\Box(P, f_1, g) = c \underline{\int_a^b} f_1 \Box g$$

and

$$\overline{\int_a^b} cf_1 \Box g = \inf_{P \in \mathcal{P}(I)} U_\Box(P, cf_1, g) = c \inf_{P \in \mathcal{P}(I)} U_\Box(P, f_1, g) = c \overline{\int_a^b} f_1 \Box g \,.$$

Since $f_1 \in \mathcal{R}_\Box(g, I)$, then $\overline{\int_a^b} f_1 \Box g = \underline{\int_a^b} f_1 \Box g$, which proves the intended conclusion.

ii) Let

$$\mathfrak{m}_j = \inf_{t \in I_{\Box j}} (f_1 + f_2)(t) \quad \text{and} \quad \mathfrak{M}_j = \sup_{t \in I_{\Box j}} (f_1 + f_2)(t) \,.$$

Then, $\mathfrak{m}_j \geq m_j^{f_1} + m_j^{f_2}$ and $\mathfrak{M}_j \leq M_j^{f_1} + M_j^{f_2}$. This implies

$$L_\Box(P, f_1, g) + L_\Box(P, f_2, g) \leq L_\Box(P, f_1 + f_2, g)$$
$$U_\Box(P, f_1, g) - U_\Box(P, f_2, g) \leq U_\Box(P, f_1 + f_2, g) \,.$$

The integrability criterion implies the existence of partitions $P_1 \in \mathcal{P}(I)$ and $P_2 \in \mathcal{P}(I)$ such that the inequalities

$$U_\Box(P_1, f, g) - L_\Box(P_1, f, g) < \frac{\varepsilon}{2} \quad \text{and} \quad U_\Box(P_2, f, g) - L_\Box(P_2, f, g) < \frac{\varepsilon}{2} \tag{3.6}$$

also hold on their common refinement $P_\varepsilon = P_1 \cup P_2$. From (3.6) follows that $U_\Box(P_\varepsilon, f_1 + f_2, g) - L_\Box(P_\varepsilon, f_1 + f_2, g) < \varepsilon$. Hence, $f_1 + f_2 \in \mathcal{R}_\Box(g, I)$. Additionally,

$$\begin{aligned} \underline{\int_a^b} (f_1 + f_2) \Box g &= \sup_{P \in \mathcal{P}(I)} L_\Box(P, f_1 + f_2, g) \\ &\leq \sup_{P \in \mathcal{P}(I)} L_\Box(P, f_1, g) + \sup_{P \in \mathcal{P}(I)} L_\Box(P, f_2, g) = \underline{\int_a^b} f_1 \Box g + \underline{\int_a^b} f_2 \Box g \end{aligned}$$

and

$$\begin{aligned} \overline{\int_a^b} (f_1 + f_2) \Box g &= \inf_{P \in \mathcal{P}(I)} U_\Box(P, f_1 + f_2, g) \\ &\geq \inf_{P \in \mathcal{P}(I)} U_\Box(P, f_1, g) + \inf_{P \in \mathcal{P}(I)} U_\Box(P, f_2, g) = \overline{\int_a^b} f_1 \Box g + \overline{\int_a^b} f_2 \Box g \,, \end{aligned}$$

so that

$$\int_a^b (f_1+f_2)\Box g \ge \overline{\int_a^b} f_1\Box g + \overline{\int_a^b} f_2\Box g \quad \text{and} \quad \int_a^b (f_1+f_2)\Box g \ge \underline{\int_a^b} f_1\Box g + \underline{\int_a^b} f_2\Box g\,.$$

Because $f_1 \in \mathcal{R}_\Box(g,I)$ and $f_2 \in \mathcal{R}_\Box(g,I)$, then $\overline{\int_a^b} f_1\Box g = \underline{\int_a^b} f_1\Box g$ and $\underline{\int_a^b} f_2\Box g = \overline{\int_a^b} f_2\Box g$. ∎

Similarly to the proof of Theorem 3.5 item i), one can show the following:

Theorem 3.6. *Let $f \in \mathcal{R}_\Box(g_1,I)$ and $f \in \mathcal{R}_\Box(g_2,I)$, where g_1 and g_2 are increasing functions on $[a,b]_\mathbb{T}$. Then, $f \in \mathcal{R}_\Box(g_1+g_2,I)$ and $\int_a^b f\Box\,(g_1+g_2) = \int_a^b f\Box g_1 + \int_a^b f\Box g_2$.*

Theorem 3.7. *Let a, b, $c \in \mathbb{T}$ with $a<b<c$. If f is bounded on $[a,c]_\mathbb{T}$ and g is monotonically increasing on $[a,c]_\mathbb{T}$, then*

$$\int_a^c f\Box g = \int_a^b f\Box g + \int_b^c f\Box g\,.$$

Proof. There exist partitions $P_1 \in \mathcal{P}([a,b]_\mathbb{T})$ and $P_2 \in \mathcal{P}([b,c]_\mathbb{T})$ such that for $\varepsilon > 0$

$$U_\Box(P_1,f,g) - L_\Box(P_1,f,g) < \frac{\varepsilon}{2} \quad \text{and} \quad U_\Box(P_2,f,g) - L_\Box(P_2,f,g) < \frac{\varepsilon}{2}\,.$$

Then, there is a partition $P \in \mathcal{P}([a,c]_\mathbb{T})$ such that $U_\Box(P,f,g) = U_\Box(P_1,f,g) + U_\Box(P_2,f,g)$ and $L_\Box(P,f,g) = L_\Box(P_1,f,g) + L_\Box(P_2,f,g)$ so that $U_\Box(P,f,g) - L_\Box(P,f,g) < \varepsilon$. Hence, f is $\Box$–integrable with respect to g and

$$\begin{aligned}\int_a^c f\Box g &\le U_\Box(P_1,f,g) + U_\Box(P_2,f,g)\\ &\le L_\Box(P_1,f,g) + L_\Box(P_2,f,g) + \varepsilon \le \int_a^b f\Box g + \int_b^c f\Box g + \varepsilon\,.\end{aligned}$$

Similarly,

$$\int_a^c f\Box g \ge \int_a^b f\Box g + \int_b^c f\Box g - \varepsilon\,.$$

∎

Theorem 3.8. *(Integration by substitution) Let $\tilde{\mathbb{T}}$ be a time scale. Assume that $\varphi : \tilde{\mathbb{T}} \to \mathbb{R}$ is a strictly increasing continuous function and $\mathbb{T} = \varphi(\tilde{\mathbb{T}})$ is a time scale. Moreover, φ maps an interval $[A,B]_{\tilde{\mathbb{T}}}$ onto $[a,b]_\mathbb{T}$. Let g be monotonically increasing on $[a,b]_\mathbb{T}$ and $f \in \mathcal{R}_\Box(g,[a,b]_\mathbb{T})$. Then, $f\circ\varphi \in \mathcal{R}_\Box(g\circ\varphi,[A,B]_{\tilde{\mathbb{T}}})$ and*

$$\int_a^b f(t)\Box g(t) = \int_A^B f(\varphi(s))\Box g(\varphi(s)).$$

Proof. Since φ is a strictly increasing continuous function, there is a one-to-one correspondence between the partition $Q = \{s_0,s_1,\ldots,s_n\} \subset [A,B]_{\tilde{\mathbb{T}}}$, where $A = s_0 < s_1 < \ldots < s_n = B$, and the partition $P = \{t_0,t_1,\ldots,t_n\} \subset [a,b]_\mathbb{T}$, where $a = t_0 = \varphi(s_0) < t_1 = \varphi(s_1) < \ldots < t_n = \varphi(s_n) = b$. Since $f([t_{j-1},t_j]_\mathbb{T}) = f([\varphi(s_{j-1},\varphi(t_j)]_{\tilde{\mathbb{T}}})$ for each j, then $L_\Box(P,f,g) = L_\Box(Q,f\circ\varphi,g\circ\varphi)$ and $U_\Box(P,f,g) = U_\Box(Q,f\circ\varphi,g\circ\varphi)$. The result follows from Theorem 2.7. ∎

4. FROM THE RIEMANN-STIELTJES TO RIEMANN □–INTEGRAL

Theorem 4.3 below establishes a relation between the Riemann-Stieltjes □–integral and the Riemann □–integral for a function g being □–differentiable on the interval I of integration. We begin by noting that one can easily reformulate to nabla version, with the interval opened from the left side, the delta mean value theorem [2, Theorem 1.14]:

Theorem 4.1. *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is ∇ – differentiable on $(a,b]$. Then there exist $\xi, \tau \in (a,b]$ such that*

$$f^{\nabla}(\xi) \leq \frac{f(b)-f(a)}{b-a} \leq f^{\nabla}(\tau)\,.$$

In Corollary 4.2 we write together [2, Theorem 1.14] and Theorem 4.1 with our "□"–notation:

Corollary 4.2. *Let f be a continuous function on $[a,b]_{\mathbb{T}}$ that is $\Box$ – differentiable on $[a,b]_{\Box}$. Then there exist $\xi, \tau \in [a,b]_{\Box}$ such that*

$$f^{\Box}(\xi) \leq \frac{f(b)-f(a)}{b-a} \leq f^{\Box}(\tau)\,.$$

Theorem 4.3. *Let $I=[a,b]_{\mathbb{T}}, a,b \in \mathbb{T}$. Suppose that g is an increasing function such that $g^{\Box}$ is continuous on $(a,b)_{\mathbb{T}}$ and f is a real bounded function on I. Then, $f \in \mathcal{R}_{\Box}(g,I)$ if and only if $fg^{\Box} \in \mathcal{R}_{\Box}(g,I)$. Moreover,*

$$\int_a^b f(t)\Box g(t) = \int_a^b f(t) g^{\Box}(t)\Box t\,.$$

Proof. Let $\varepsilon > 0$. Since $g^{\Box}$ is continuous, then $g^{\Box} \in \mathcal{R}_{\Box}(t,I)$, i.e., $g^{\Box}$ is Riemann–Stieltjes □–integrable on I. Hence, there exists a partition $P = \{a = t_0, t_1, \ldots, t_n = b\} \in \mathcal{P}(I)$ such that $U_{\Box}(P, g^{\Delta}, t) - L_{\Box}(P, g^{\Delta}, t) < \frac{\varepsilon}{M}$, where $M = \sup_{t \in I} |f(t)|$. From the □–version of the mean value theorem on time scales (see Corollary 4.2), for each $j = 1, \ldots, n$ there are $\xi_j, \tau_j \in I_j = [t_{j-1}, t_j]_{\Box}$ such that

$$g^{\Box}(\tau_j)\Delta t_j \leq \Delta g_j \leq g^{\Box}(\xi_j)\Delta t_j\,.$$

From Theorem 2.10 part ii), for any $s_j \in I_j$, $j = 1, \ldots, n$,

$$\sum_{j=1}^{n} \left| g^{\Box}(\xi_j) - g^{\Box}(s_j) \right| \Delta t_j < \varepsilon\,.$$

Then, $\sum_{j=1}^{n} f(s_j)\Delta g_j \leq \sum_{j=1}^{n} f(s_j) g^{\Box}(\xi_j)\Delta t_j$ and

$$\left| \sum_{j=1}^{n} f(s_j)\Delta g_j - \sum_{j=1}^{n} f(s_j) g^{\Box}(s_j)\Delta t_j \right| \leq \left| \sum_{j=1}^{n} f(s_j) g^{\Box}(\xi_j)\Delta t_j - \sum_{j=1}^{n} f(s_j) g^{\Box}(s_j)\Delta t_j \right|\,.$$

Hence,

$$\left| \sum_{j=1}^{n} f(s_j)\Delta g_j - \sum_{j=1}^{n} f(s_j) g^{\Box}(s_j)\Delta t_j \right| \leq M \left| \sum_{j=1}^{n} \left(g^{\Box}(\xi_j) - g^{\Box}(s_j) \right) \Delta t_j \right| < \varepsilon \tag{4.1}$$

for any $s_j \in I_j$. Thus, $\sum_{j=1}^{n} f(s_j)\Delta g_j \leq U_{\Box}(P, fg^{\Box}, t) + \varepsilon$ and $U_{\Box}(P,f,g) \leq U_{\Box}(P, fg^{\Box}, t) + \varepsilon$. Inequality (4.1) implies that $U_{\Box}(P, fg^{\Box}, t) \leq U_{\Box}(P,f,g) + \varepsilon$. Thus,

$$\left| U_{\Box}(P,f,g) - U_{\Box}(P, fg^{\Box}, t) \right| \leq \varepsilon\,.$$

Moreover,

$$\left|\overline{\int_a^b} f(t)\Box g(t) - \overline{\int_a^b} f(t)g^{\Box}(t)\Box t\right| \leq \varepsilon \,.$$

Since ε is arbitrary, we conclude that

$$\overline{\int_a^b} f(t)\Box g(t) = \overline{\int_a^b} f(t)g^{\Box}(t)\Box t.$$

In a similar way one prove that $\underline{\int_a^b} f(t)\Box g(t) = \underline{\int_a^b} f(t)g^{\Box}(t)\Box t$. ∎

Theorem 4.4. *(Delta integration by parts) Let $I = [a,b]_{\mathbb{T}}, a, b \in \mathbb{T}$. Suppose that g is an increasing function such that g^{Δ} is continuous on $(a,b)_{\mathbb{T}}$ and f is a real bounded function on I. Then,*

$$\int_a^b f\Delta g = [fg]_a^b - \int_a^b g^{\sigma}\Delta f \,,$$

where, as usual, g^{σ} means $g \circ \sigma$.

Proof. Theorem 4.3 and integration by parts for the Riemann Δ–integral on time scales (see [2]) imply that $\int_a^b f(t)\Delta g(t) = \int_a^b f(t)g^{\Delta}(t)\Delta t$ and $\int_a^b g^{\sigma} f^{\Delta}\Delta t + \int_a^b f\Delta t = [fg]_a^b$. Hence, $\int_a^b f\Delta g = [fg]_a^b - \int_a^b g^{\sigma}\Delta f$. ∎

Theorem 4.5. *(Nabla integration by parts) Let $I = [a,b]_{\mathbb{T}}, a, b \in \mathbb{T}$. Suppose that g is an increasing function such that g^{∇} is continuous on $(a,b)_{\mathbb{T}}$ and f is a real bounded function on I. Then,*

$$\int_a^b f\nabla g = [fg]_a^b - \int_a^b g^{\rho}\nabla f \,,$$

where $g^{\rho} = g \circ \rho$.

Proof. Theorem 4.3 and integration by parts for the Riemann ∇–integral on time scales implies that $\int_a^b f(t)\nabla g(t) = \int_a^b f(t)g^{\nabla}(t)\nabla t$ and $\int_a^b g^{\rho} f^{\nabla}\nabla t + \int_a^b f\nabla t = [fg]_a^b$. Hence, $\int_a^b f\nabla g = [fg]_a^b - \int_a^b g^{\rho}\nabla f$. ∎

5. CONCLUSION AND FUTURE PERSPECTIVES

This article is about the concept of Riemann-Stieltjes delta integration on time scales. The results of the paper may be used, e.g., to generalize the $\mathbb{T} = \mathbb{R}$ inequalities proved in [8, 9] to a general time scale $\mathbb{T}$. Then, as the particular case $g(t) = t$, one would obtain the previous inequalities proved for the Riemann integral on time scales [15, 16].

Another interesting line of research is to investigate the possibility of extending all previous notions of integration on time scales by putting together our present results with the Henstock-Kurzweil integrals introduced by Peterson and Thompson in [14, 17]. Such Henstock-Kurzweil-Stieltjes integrals on time scales are under study and will be addressed elsewhere.

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