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SUPERQUADRACITY, BOHR'S INEQUALITY AND DEVIATION FROM A MEAN VALUE

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ABSTRACT. Extensions of Bohr's inequality via superquadracity are obtained, where instead of the power $p = 2$ which appears in Bohr's inequality we get similar results when we deal with $p \geq 2$ and with $p \leq 2$. Also, via superquadracity we extend a bound for deviation from a Mean Value.

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1. INTRODUCTION

Bohr's inequality [8, page 499], states: *for any* $z, w \in \mathbb{C}$ *and for any* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$

$$|z + w|^2 \leq p\,|z|^2 + q\,|w|^2 \tag{1.1}$$

holds with equality iff $w = (p-1)\,z$.

This inequality has many extensions and generalizations, see all the references and their references except [6].

In Section 2 we extend Bohr's theorem for $z, w \in R^+$ by replacing the power 2 with powers $r \geq 2$ and with powers $1 \leq r \leq 2$.

In Section 3 we extend the following theorem proved in [6]: *For integers* $n > 1$, *and for* $x_i \geq 0,\ i = 1, ..., n$ *the inequality* $\max\limits_{1 \leq k \leq n} \{|x_k - a|\} \leq \sqrt{(n-1)(b - a^2)}$ *holds, where* $a = \frac{1}{n}\sum_{i=1}^{n} x_i$ *and* $b = \frac{1}{n}\sum_{i=1}^{n} x_i^2$.

In all the proofs we use the properties of superquadratic functions. Therefore we first quote those definitions and theorems related to superquadracity we use in the sequel:

Definition 1.1. [1, Def2.1] A function f, defined on an interval $I = [0, l]$ or $[0, \infty)$ is superquadratic, if for each x in I, there exists a real number $C(x)$ such that

$$f(y) - f(x) \geq C(x)(y - x) + f(|y - x|) \tag{1.2}$$

for all $y \in I$. If $-f$ is superquadratic then f is called subquadratic.

Theorem 1.1. [1, Th2.3]. *The inequality*

$$f\left(\int g d\mu\right) \leq \int \left(f\left(g\left(s\right)\right) - f\left(\left|g\left(s\right) - \int g d\mu\right|\right)\right) d\mu\left(s\right) \tag{1.3}$$

holds for all probability measures μ *and all non-negative* $\mu-$*integrable functions* g, *if and only if* f *is superquadratic.*

The discrete version that follows from the above theorem is also used in the sequel:

Lemma 1.2. *Suppose that* f *is superquadratic. Let* $x_r \geq 0$, $1 \leq r \leq n$ *and let* $\bar{x} = \sum_{r=1}^{n} \lambda_r x_r$ *where* $\lambda_r \geq 0$ *and* $\sum_{r=1}^{n} \lambda_r = 1$. *Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \bar{x}|).$$

In particular if $n = 2$ we get from Lemma 1.2 that for $0 \leq \alpha \leq 1$, $a, b \geq 0$

$$\begin{aligned} &\alpha f\left(a\right) + \left(1-\alpha\right) f\left(b\right) - f\left(\alpha a + \left(1-\alpha\right) b\right) \\ \geq\ &\alpha f\left(\left(1-\alpha\right)\left|b-a\right|\right) + \left(1-\alpha\right) f\left(\alpha\left|b-a\right|\right). \end{aligned} \tag{1.4}$$

Lemma 1.3. [1, Lemma 2.2] *Let* f *be a superquadratic function with* $C(x)$ *as in Definition 1. Then*

(i) $f(0) \leq 0$.
(ii) If $f(0) = f'(0) = 0$ then $C(x) = f'(x)$ wherever f is differentiable at $x > 0$.
(iii) If $f \geq 0$, then f is convex and $f\left(0\right) = f'(0) = 0$.

The functions $f\left(x\right) = x^p$, $x \geq 0$, are superquadratic for $p \geq 2$, and subquadratic for $0 \leq p \leq 2$, [1].

For the special case $f(x) = x^2$ we get equalities in (1.2), (1.3), and (1.4) and in (1.2) $C(x) = 2x = (x^2)'$.

2. BOHR'S TYPE INEQUALITIES AND EXTENTION OF THE PARALLELOGRAM LAW, VIA SUPERQUADRATIC AND SUBQUADRATIC FUNCTIONS

In [5] the following theorems were proved:

Theorem 2.1. [5, Theorem1,Corollary1,Theorem2]:*Let H be a complex separable Hilbert space and $B(\mathcal{H})$ the algebra of all bounded linear operators on H. Let $|X| = (X^*X)^{1/2}$ for any $X \in B(\mathcal{H})$. Then for any $A, B \in B(\mathcal{H})$ and any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

If $1 < p \leq 2$ the inequalities
(i) $|A - B|^2 + |(1-p)A - B|^2 \leq p|A|^2 + q|B|^2$,
and
(ii) $|A - B|^2 + |A - (1-q)B|^2 \geq p|A|^2 + q|B|^2$
hold.
Furthermore, in both (i) and (ii), the equality holds iff $p = q = 2$ or $(1-p)A = B$.

If $p \geq 2$
(iii) $|A - B|^2 + |(1-p)A - B|^2 \geq p|A|^2 + q|B|^2$,
and
(iv) $|A - B|^2 + |A - (1-q)B|^2 \leq p|A|^2 + q|B|^2$
hold.
Furthermore, in both (iii) and (iv), the equality holds iff $(1-p)A = B$.

If $p < 1$
(v) $|A - B|^2 + |(1-p)A - B|^2 \geq p|A|^2 + q|B|^2$,
and
(vi) $|A - B|^2 + |A - (1-q)B|^2 \geq p|A|^2 + q|B|^2$
hold.
Furthermore, in both (v) and (vi), the equality holds iff $(1-p)A = B$.

When $H = \mathbb{C}$ and $B(\mathcal{H}) = \mathbb{C}$, Theorem 2.1 becomes the following theorem:

Theorem 2.2. [5, Theorem1,Corollary1,Theorem2]*: For any $A, B \in \mathbb{C}$ and any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get that (i) to (vi) hold, furthermore,*
(vii) if $p > 1$,

$$|A - B|^2 \leq p|A|^2 + q|B|^2,$$

with equality iff $(1-p)A = B$, which is exactly the classical Bohr's inequality.

In the following we extend Theorem 2.1 for $x \in \mathbb{R}^+$ to powers different than $r = 2$.

Theorem 2.3. *For any $a, b \in \mathbb{R}^+$ and for any $p, q \in \mathbb{R}$ with $\frac{1}{p} + \frac{1}{q} = 1$*
(i) for $r \geq 2$ and $1 < p \leq 2$ $(q \geq 2)$ we get the inequality:

$$pa^r + qb^r \geq \frac{1}{2^{r-2}}((p-1)a + b)^r + \frac{1}{2^{r-2}}|b - a|^r \tag{2.1}$$

and for q, $1 < q \leq 2$, $(p \geq 2)$ we get that

$$pa^r + qb^r \geq \frac{1}{2^{r-2}}(a + (q-1)b)^r + \frac{1}{2^{r-2}}|b - a|^r \tag{2.2}$$

equality holds in (2.1) and (2.2) when $r > 2$ if $p = q = 2$ and $a = b$. Moreover, if $r = 2$ equality holds in (2.1) and (2.2) if $p = q = 2$, which is the parallelogram law.

(ii) for $1 \le r \le 2,$ *if* $p \ge 2,$ $(1 < q \le 2)$, *we get that*

$$pa^r + qb^r \le \frac{1}{2^{r-2}}((p-1)a+b)^r + \frac{1}{2^{r-2}}|b-a|^r \tag{2.3}$$

and analogously if $1 < p \le 2,$ $(q \ge 2)$

$$pa^r + qb^r \le \frac{1}{2^{r-2}}(a+(q-1)b)^r + \frac{1}{2^{r-2}}|b-a|^r. \tag{2.4}$$

For $p = q = 2$, $1 \le r \le 2$ *we get that*

$$(a+b)^r \le 2^{r-1}(a^r + b^r) \le (a+b)^r + (b-a)^r \tag{2.5}$$

which we may consider as an extention of the parallelogram law.

(iii) for $r = 2$ *if* $1 < p \le 2,$ $(q \ge 2)$ *we get that*

$$((p-1)a+b)^2 + |b-a|^2 \le pa^2 + qb^2 \le (a+(q-1)b)^2 + |b-a|^2,$$

and if $2 \le p$, $(1 < q \le 2)$

$$(a+(q-1)b)^2 + |b-a|^2 \le pa^2 + qb^2 \le ((p-1)a+b)^2 + |b-a|^2.$$

Proof. From (1.4) we get for the superquadratic function $f(x) = x^r,$ $r \ge 2,$ $0 \le \alpha \le 1,$ $\beta = 1-\alpha$ that

$$\begin{aligned} \alpha a^r + \beta b^r &\ge (\alpha a + \beta b)^r + (\alpha\beta^r + \beta\alpha^r)|b-a|^r \\ &= (\alpha a + \beta b)^r + \alpha\beta\left(\beta^{r-1} + \alpha^{r-1}\right)|b-a|^r. \end{aligned} \tag{2.6}$$

By the substitution $\frac{1}{\alpha} = q$ $\frac{1}{1-\alpha} = p$ we get for $0 < \alpha < 1$ after some manipulations that (2.6) is equivalent to

$$pa^r + qb^r \ge \frac{1}{(p-1)p^{r-2}}((p-1)a+b)^r + \left(\alpha^{r-1} + \beta^{r-1}\right)|b-a|^r. \tag{2.7}$$

As $r \ge 2$ we get that $\alpha^{r-1} + \beta^{r-1} \ge \frac{1}{2^{r-2}}$ and for $1 \le p \le 2$ we get $\frac{1}{(p-1)p^{r-2}} \ge \frac{1}{2^{r-2}}$.

Hence for $r \ge 2$ and $1 < p \le 2$ $(q \ge 2)$ we get the inequality (2.1), and for $1 < q \le 2,$ $(p \ge 2)$ we get (2.2). It is obvious that equality holds in (2.1) and (2.2) when $r > 2$ if $p = q = 2$ and $a = b$. It is also clear from (2.6) and (2.7) that if $r = 2$ equality holds in (2.1) and (2.2) if $p = q = 2$ and in this case we get the parallelogram law.

Now, for $1 \le r \le 2$, $f(x) = x^r$ is a subquadratic function. Hence we get for $0 < \alpha < 1$ $\beta = (1-\alpha)$, $a, b \ge 0,$ $\alpha = \frac{1}{q},$ $\beta = \frac{1}{p}$

$$pa^r + qb^r \le \frac{1}{(p-1)p^{r-2}}((p-1)a+b)^r + \left(\alpha^{r-1} + \beta^{r-1}\right)|b-a|^r. \tag{2.8}$$

As $1 \le r \le 2$ we get that $\alpha^{r-1} + \beta^{r-1} \le 2^{2-r}$ and if $p > 2,$ $\frac{1}{(p-1)p^{r-2}} < \frac{1}{2^{r-2}}$ and hence if $p \ge 2,$ $(1 < q \le 2)$, we get from (2.8) that (2.3) holds and analogously if $1 < p \le 2,$ $(q \ge 2)$, $1 \le r \le 2$, we get from (2.8) that (2.4) holds. For $p = q = 2$, $1 \le r \le 2$ we get that

$$(a+b)^r \le 2^{r-1}(a^r + b^r) \le (a+b)^r + (b-a)^r$$

which we may consider as an extention of the parallelogram law. The left hand side of this is Inequality 4 in [10]

In case $r = 2$ we get from (2.1) and (2.4) that for $1 \le p \le 2$

$$((p-1)a+b)^2 + |b-a|^2 \le pa^2 + qb^2 \le (a+(q-1)b)^2 + |b-a|^2$$

holds and if $2 \le p$, $(1 < q \le 2)$

$$(a+(q-1)b)^2 + |b-a|^2 \le pa^2 + qb^2 \le ((p-1)a+b)^2 + |b-a|^2.$$

∎

Remark 2.1. For $x \geq 0,$ Theorem 2.3 generalize Theorem 2.1. Indeed, for $r = 2$ (2.1) is the same as Theorem 2.1(i), and (2.2) is the same as Theorem 2.1(iv), (2.3) is the same as Theorem 2.1(iii) and (2.4) is the same as Theorem 2.1(ii).

P. M. Vasić and J. D. Kečkić [13] generalize (1.1) to the following: For $z_1, z_2, ..., z_n \in \mathbb{C}$, $w_1, w_2, ...w_n > 0$ and $p > 1,$ we have

$$\left|\sum_{i=1}^{n} z_i\right|^p \leq \left(\sum_{i=1}^{n} w_i^{1/(1-p)}\right)^{p-1} \sum_{i=1}^{n} w_i \left|z_i\right|^p . \tag{2.9}$$

Equality holds iff $w_1 \left|z_1\right| = w_2 \left|z_2\right| = ... = w_n \left|z_n\right|$ and $z_k \overline{z}_j \geq 0,$ $k, j = 1, 2, ..., n$.

In the following theorem we refine (2.9) when $z_i \geq 0,$ $i = 1, ..., n$ for $p \geq 2,$ and get a companion inequality of (2.9) for $1 < p \leq 2$.

Theorem 2.4. *Let $x_i \geq 0$ and $w_i > 0,$ $i = 1, ..., n$. Then for $p \geq 2$*

$$\begin{aligned}\left(\sum_{i=1}^{n} x_i\right)^p \leq & \left(\sum_{i=1}^{n} w_i^{1/(1-p)}\right)^{p-1} \sum_{i=1}^{n} w_i x_i^p \\ & - \left(\sum_{i=1}^{n} w_i^{1/(1-p)}\right)^{-1} \sum_{j=1}^{n} w_j \left| x_j \sum_{i=1}^{n} w_i^{1/(1-p)} - \sum_{i=1}^{n} x_i w_i^{1/(1-p)}\right| .\end{aligned} \tag{2.10}$$

For $0 < p \leq 2$ the inequality is reversed.

Proof. For $p \geq 2$ and $x \geq 0$ $f(x) = x^p$ is superquadratic, therefore from Lemma 1.2 we get that

$$\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i^p \geq \left(\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i\right)^p + \frac{1}{A_n}\sum_{i=1}^{n} a_i \left(\left|x_i - \frac{1}{A_n}\sum_{j=1}^{n} a_j x_j\right|\right)^p, \tag{2.11}$$

where $a_i > 0$, $x_i \geq 0,$ $i = 1, ..., n$ and $A_n = \sum_{i=1}^{n} a_i$.

Substituting $x_i \to \frac{x_i}{a_i},$ $i = 1, ..., n,$ we get from (2.11)

$$\frac{1}{A_n}\sum_{i=1}^{n} a_i^{(1-p)} x_i^p \geq \frac{1}{A_n^p}\left(\sum_{i=1}^{n} x_i\right)^p + \frac{1}{A_n}\sum_{i=1}^{n} a_i \left(\left|\frac{x_i}{a_i} - \frac{1}{A_n}\sum_{j=1}^{n} x_j\right|\right)^p . \tag{2.12}$$

The substitution $a_i^{(1-p)} = w_i,$ $i = 1, ..., n$ in (2.12) leads to

$$\begin{aligned} & \frac{1}{\sum_{i=1}^{n} w_i^{1/(1-p)}} \sum_{i=1}^{n} w_i x_i^p \\ \geq & \frac{1}{\left(\sum_{i=1}^{n} w_i^{1/(1-p)}\right)^p} \left(\sum_{i=1}^{n} x_i\right)^p \\ & + \frac{1}{\left(\sum_{i=1}^{n} w_i^{1/(1-p)}\right)^{p+1}} \sum_{j=1}^{n} w_j \left| x_j \sum_{i=1}^{n} w_i^{1/(1-p)} - \sum_{i=1}^{n} x_i w_i^{1/(1-p)}\right| \end{aligned}$$

which is equivalent to (2.10). This is a refinement of (2.9) for $p \geq 2$ and $x \geq 0$.

When $0 < p \leq 2$ and $x \geq 0$ $f(x) = x^p$ is subquadratic, hence, in this case we get the reverse inequality of (2.10). This can be seen as a companion inequality of (2.9) when $0 < p \leq 2$ and $x \geq 0$. ∎

In the following Theorem 2.5 we get a reversal of Bohr's type inequality for superquadratic functions by using the reversal of Jensen's inequality that was established in [2, Theorem 3 and Remark 3] , see also [3]:

Let $(a_1, ..., a_n)$ be a real n-tuple such that

$$a_1 > 0, \quad a_i \leq 0, \quad i = 2, ..., n, \quad A_n = \sum_{i=1}^{n} a_i > 0. \tag{2.13}$$

If $x_i \geq 0$, $i = 2, ..., n$ and $\overline{x} = \frac{1}{A_n}\sum_{i=1}^{n} a_i x_i \geq 0$, then for a superquadratic function $f : [0, \infty) \to \mathbb{R}$ the following inequality

$$f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i\right) \geq \frac{1}{A_n}\sum_{i=1}^{n} a_i f(x_i) + f(|\overline{x} - x_1|) - \frac{1}{A_n}\sum_{i=2}^{n} a_i f(|x_i - x_1|) \tag{2.14}$$

holds. Moreover when f in a nonnegative superquadratic function, the following refinement of the reversal of Jensen's inequality is obtained:

$$\begin{aligned} & f\left(\frac{1}{A_n}\sum_{i=1}^{n} a_i x_i\right) \\ \geq\ & \frac{1}{A_n}\sum_{i=1}^{n} a_i f(x_i) + f(|\overline{x} - x_1|) - \frac{1}{A_n}\sum_{i=1}^{n} a_i f(|x_i - x_1|) \geq \frac{1}{A_n}\sum_{i=1}^{n} a_i f(x_i). \end{aligned} \tag{2.15}$$

To get the reversal of Bohr's inequality we chose $f(x) = x^p$, $x \geq 0, p \geq 2$ in (2.15) and we get:

Theorem 2.5. *Let* $z_1 \geq 0$, $z_i \leq 0$, $w_1 > 0$, $w_i \leq 0$, $i = 2, ..., n$, $\sum_{i=1}^{n} z_i > 0$, $\sum_{i=1}^{n} w_i > 0$, $p \geq 2$*, then*

$$\begin{aligned} & \left(\sum_{i=1}^{n} z_i\right)^p \\ \geq\ & \left(\sum_{i=1}^{n} w_i |w_i|^{\frac{p}{1-p}}\right)^{p-1} \sum_{i=1}^{n} w_i |z_i|^p \\ & + \left|\left(\left(\sum_{i=1}^{n} z_i\right) - \frac{\sum_{i=1}^{n} w_i |w_i|^{\frac{p}{1-p}}}{w_1^{\frac{1}{1-p}}} z_1\right)\right|^p \\ & - \left(\sum_{i=1}^{n} w_i |w_i|^{\frac{p}{1-p}}\right)^{p-1} \sum_{i=2}^{n} w_i \left|z_i - \frac{\sum_{i=1}^{n} w_i |w_i|^{\frac{p}{1-p}}}{w_1^{\frac{1}{1-p}}} z_1\right|^p \end{aligned} \tag{2.16}$$

holds.

Proof. The function $f(x) = x^p$ $p \geq 2$, $x \geq 0$ is a non-negative superquadratic function and therefore, satisfies (2.15). For this function we make the following change of variables

$$x_i a_i = z_i, \quad i = 1, ..., n. \tag{2.17}$$

For $x_i \geq 0$ and a_i, $i = 1, ..., n$ which satisfies (2.13), we get that $z_1 \geq 0$, $z_i \leq 0$, $i = 2, ..., n$, $\sum_{i=1}^n z_i \geq 0$ and $\left|\frac{z_i}{a_i}\right| = \frac{z_i}{a_i}$ $i = 1, ..., n$. Therefore, inserting (2.17) in (2.15) we get

$$\begin{aligned} & A_n^{-p}\left(\sum_{i=1}^n z_i\right)^p \\ \geq\ & A_n^{-1}\sum_{i=1}^n a_i\left|a_i\right|^{-p}\left|z_i\right|^p + A_n^{-p}\left|\left(\left(\sum_{i=1}^n z_i\right) - \frac{\sum_{i=1}^n a_i}{a_1}z_1\right)\right|^p \\ & -A_n^{-1}\sum_{i=2}^n a_i\left|a_i\right|^{-p}\left|z_i - \frac{a_i}{a_1}z_1\right|^p . \end{aligned} \tag{2.18}$$

Replacing in (2.17) $a_i\left|a_i\right|^{-p} = w_i$ and therefore $a_i = w_i\left|w_i\right|^{\frac{p}{1-p}}$, $i = 1, ..., n$ we get that (2.16) holds. ∎

3. Upper bounds for deviations from a Mean Value

In [6] A. Cipu proved the following theorem:

Theorem 3.1. *Let $n > 1$ be an integer and $x_1, x_2, ..., x_n$ be positive real numbers. Denote $a = \frac{1}{n}\sum_{i=1}^n x_i$ and $b = \frac{1}{n}\sum_{i=1}^n x_i^2$, then $\max_{1\leq k\leq n}\{|x_k - a|\} \leq \sqrt{(n-1)(b-a^2)}$.*

We extend this theorem in the following two theorems:

Theorem 3.2. *Let $n > 1$ be an integer and $x_1, x_2, ..., x_n$ be positive real numbers. Denote $a = \sum_{i=1}^n \alpha_i x_i$ and $c = \sum_{i=1}^n \alpha_i x_i^p$, where $0 < \alpha_i < 1$, $i = 1, ..., n$ $\sum \alpha_i = 1$, $p \geq 2$, then*

$$\max_{1\leq k\leq n}\{|x_k - a|\} \leq T\left(c - a^p\right)^{\frac{1}{p}} \tag{3.1}$$

where

$$T = \frac{\left(1-\alpha_0\right)^{1-\frac{1}{p}}}{\alpha_0^{\frac{1}{p}}\left(\alpha_0^{p-1} + \left(1-\alpha_0\right)^{p-1}\right)^{\frac{1}{p}}}, \qquad \alpha_0 = \min_{1\leq k\leq n}\left(\alpha_k\right).$$

Remark 3.1. It is easy to see that Theorem 3.1 is a special case of Theorem 3.2 where $p = 2$, $\alpha_k = \frac{1}{n}$, $k = 1, ..., n$.

Theorem 3.3. *Let f be a positive superquadratic function on $[0, \infty)$, which satisfies $f(AB) \leq f(A)f(B)$ for $A > 0$, $B > 0$.*

Let $x_1, x_2, ..., x_n$ be positive real numbers. Denote $a = \sum_{i=1}^n \frac{x_i}{n}$ and $d = \frac{1}{n}\sum_{i=1}^n f(x_i)$, then

$$\max_{1\leq k\leq n}\left(f\left(|x_k - a|\right)\right) \leq \frac{f(n-1)\,n}{n-1+f(n-1)}\left(d - f(a)\right). \tag{3.2}$$

Remark 3.2. For $\alpha_i = \frac{1}{n}$, i=1,...,n Theorem 3.2 is a special case of Theorem 3.3.

Proof. (of Theorem 3.2) $f(x) = x^p$ $p \geq 2$ is superquadratic and symmetric, therefore applying Lemma 1.2

$$\sum_{i=1}^n \alpha_k x_i^p - \left(\sum_{i=1}^n \alpha_k x_i\right)^p \geq \sum_{i=1}^n \alpha_k\left(|x_i - a|\right)^p \tag{3.3}$$

holds.

Denote y$_i = x_i - a,\ i = 1, ..., n.$ Then from $\sum_{i=1}^{n} \alpha_i y_i = 0$ we get from Hölder's inequality

$$\begin{aligned}(\alpha_n |y_n|)^p &= \left(\left|\sum_{i=1}^{n-1} \alpha_i y_i\right|\right)^p \leq \left(\sum_{i=1}^{n-1} \alpha_i |y_i|\right)^p \\ &= \left(\sum_{i=1}^{n-1} \alpha_i^{1-\frac{1}{p}} (\alpha_i |y_i|^p)^{\frac{1}{p}}\right)^p \leq \left(\sum_{i=1}^{n-1} \alpha_i\right)^{p-1} \sum_{i=1}^{n-1} \alpha_i |y_i|^p ,\end{aligned}$$

and

$$(\alpha_n |y_n|)^p \leq (1-\alpha_n)^{p-1} \left(\sum_{i=1}^{n} \alpha_i |y_i|^p - \alpha_n |y_n|^p\right).$$

Therefore from (3.3)

$$\alpha_n \left(\alpha_n^{p-1} + (1-\alpha_n)^{p-1}\right) |y_n|^p \leq (1-\alpha_n)^{p-1} (c - a^p)$$

which by taking into consideraation that $\frac{(1-\alpha)^{p-1}}{\alpha\left(\alpha^{p-1}+(1-\alpha)^{p-1}\right)}$ is decreasing for $0 < \alpha < 1$ leads to (3.1). ∎

Proof. (of Theorem 3.3) The function f is superquadratic on $[0, \infty)$, therefore from Lemma 1.2 for $\lambda_i = \frac{1}{n},\ i = 1, ..., n$

$$\sum_{i=1}^{n} \frac{f(x_i)}{n} - f\left(\frac{\sum_{i=1}^{n} x_i}{n}\right) \geq \frac{1}{n} \sum_{i=1}^{n} f\left(x_i - \frac{\sum_{i=1}^{n} x_i}{n}\right) \tag{3.4}$$

holds.

In other words for $y_i = x_i - a$, $i = 1, ..., n$,

$$\frac{1}{n} \sum_{i=1}^{n} f(|y_i|) \leq d - f(a) \tag{3.5}$$

holds.

From $\sum_{i=1}^{n} y_i = 0$ we get that $|y_n| = \left|-\sum_{i=1}^{n-1} y_i\right|$. As f is positive, according to Lemma 1.3 it is convex and also increasing, and therefore

$$\begin{aligned}f(|y_n|) &= f\left(\left|-\sum_{i=1}^{n-1} y_i\right|\right) \leq f\left(\sum_{i=1}^{n-1} |y_i|\right) \\ &= f\left((n-1)\sum_{i=1}^{n-1} \frac{f^{-1}(f(|y_i|))}{n-1}\right) \\ &\leq f\left((n-1) f^{-1}\left(\frac{\sum_{i=1}^{n-1} f(|y_i|)}{n-1}\right)\right).\end{aligned} \tag{3.6}$$

Indeed, the left side inequality is because f is increasing and the right side inequality follows because f^{-1} is concave and f is increasing.

As f satisfies also $f(AB) \leq f(A) f(B)$ we get that

$$f\left((n-1) f^{-1}\left(\frac{\sum_{i=1}^{n-1} f(|y_i|)}{n-1}\right)\right) \leq \frac{f(n-1)}{n-1} \sum_{i=1}^{n-1} f(|y_i|) \tag{3.7}$$

and from (3.5), (3.6) and (3.7) we obtain

$$f\left(|y_n|\right) \leq \frac{f\left(n-1\right)}{n-1}\left(\sum_{i=1}^{n-1} f\left(|y_i|\right) - f\left(|y_n|\right)\right).$$

From the last inequality as $f(x)$ is positive and increasing, together with (3.5)

$$\frac{n-1+f\left(n-1\right)}{n-1} f\left(|y_n|\right) \leq \frac{f\left(n-1\right)}{n-1}\sum_{i=1}^{n} f\left(|y_i|\right) \leq \frac{f\left(n-1\right)}{n-1} n\left(d-f\left(a\right)\right)$$

holds, which is equivalent to (3.2). ∎

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