The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 6, Issue 1, Article 6, pp. 1-15, 2009

ULAM STABILITY OF FUNCTIONAL EQUATIONS

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Special Issue in Honor of the 100th Anniversary of S.M. Ulam

Received 4 December, 2008; accepted 25 January, 2009; published 4 September, 2009.

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ABSTRACT. In this survey paper we present some of the main results on Ulam-Hyers-Rassias stability for important functional equations.

Key words and phrases: Ulam stability, Functional equations.

2000 *Mathematics Subject Classification.* Primary 39B82. Secondary 39B62.

ISSN (electronic): 1449-5910

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1. Introduction

The problem of the stability of functional equations was posed by S. M. Ulam. In 1940, during a talk before the Mathematics Club of the University of Wisconsin he presented the following problem:

Let G and (H,d) be a group and a metric group respectively. Given a real number $\varepsilon > 0$, does there exist a positive real number δ such that if $f\colon G \to H$ satisfies the inequality

$$d[f(xy), f(x)f(y)] < \delta \quad \textit{for all } x, y \in G, \tag{1.1}$$

then there exists a homomorphism $F\colon G \to H$ with

$$d[f(x), F(x)] < \varepsilon \quad \textit{for all } x \in G?$$

In other words, we can say that the homomorphism F is "close" to the function f satisfying the inequality (1.1).

2. Additive Cauchy Equation

The first affirmative partial answer to Ulam's question was given by D.H. Hyers in 1941 (see [32]). His result reads as follows.

Theorem 2.1. *Let $f\colon E_1 \to E_2$ be a mapping between Banach spaces E_1, E_2 such that*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \quad \textit{for all } x, y \in E_1, \tag{2.1}$$

for some $\varepsilon > 0$. Then there exists exactly one additive mapping $A\colon E_1 \to E_2$:

$$A(x+y) = A(x) + A(y) \quad \textit{for all } x, y \in E_1,$$

such that

$$\|A(x) - f(x)\| \le \varepsilon \quad \textit{for all } x \in E_1, \tag{2.2}$$

given by the formula

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x), \quad x \in E_1. \tag{2.3}$$

Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then A is a linear mapping.

Ulam's question and the Hyers result became the basis for the so-called stability theory of functional equations in the Ulam-Hyers sense.

Some of the most important results and a wide list of papers devoted to such kinds of stability problems for different types of functional equations can be found for example in the survey papers [20], [26], [31], [46], as well as in the books [33], [9] and others.

It is worth noting that in 1978, Th. M. Rassias [47] weakened the condition for the bound of the norm of the Cauchy difference

$$f(x+y) - f(x) - f(y)$$

and obtained an interesting result. In fact, he proved the following generalization of Hyers' result.

Theorem 2.2. *Let E_1, E_2 be Banach spaces. If $f\colon E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \alpha(\|x\|^p + \|y\|^p) \quad x, y \in E_1, \tag{2.4}$$

for some $\alpha \ge 0$ and some $0 \le p < 1$, then there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le \frac{2\alpha}{2 - 2^p}\|x\|^p, \quad x \in E_1.$$

If, moreover, $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the mapping A is linear.

The result of Th. M. Rassias has stimulated a number of mathematicians working in the area of functional equations and their applications to investigate similar problems for important functional equations in mathematics. Such types of stability are refered to as Ulam-Hyers-Rassias stability of functional equations.

During the last years many results concerning the Ulam-Hyers-Rassias stability of important functional equations have been obtained by several mathematicians. The main goal of this paper is to present a selection of some of the most relevant developments in this subject.

The first important result in this direction is the following:

Theorem 2.3 ([22]). *Let $(G,+)$ be an abelian group, k an integer, $k \geq 2$, $(X, \|\cdot\|)$ a Banach space, $\varphi\colon G \times G \to [0,\infty)$ a mapping such that*

$$\Phi_k(x,y) := \sum_{n=0}^{\infty} k^{-(n+1)} \varphi(k^n x, k^n y) < \infty \quad \text{for all } x, y \in G.$$

Let $f\colon G \to X$ satisfy the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x,y), \quad \text{for all } x, y \in G. \tag{2.5}$$

Then there exists exactly one additive mapping $A\colon G \to X$ such that

$$\|f(x) - A(x)\| \leq \sum_{m=1}^{k-1} \Phi_k(x, mx), \quad x \in G. \tag{2.6}$$

Remark 2.1. In 1991 Rassias asked what the best possible value of k in the estimation (2.6) was. For the function

$$\varphi(x,y) = \theta(\|x\|^p + \|y\|^p),$$

with $\theta \geq 0$, $0 \leq p < 1$, this value is 2. For details, see [22].

Remark 2.2. Theorem 2.2 is true for all $p \in \mathbb{R} \setminus \{1\}$. The proof of this extension can be found in [21], where an example that Theorem 2.2 cannot be proved for $p = 1$ is presented. Further interesting counterexamples can be found in the paper [45].

The problem of stability with special "control function" φ occurring in the inequality (2.5) has been studied by R. Ger. He obtained the following interesting result (see [27]).

Theorem 2.4. *Let X be a real Banach space and let Y be a real normed linear space. Suppose that $\varphi\colon X \to \mathbb{R}$ is a nonnegative subadditive functional on X and $f\colon X \to Y$ is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x) + \varphi(y) - \varphi(x+y)$$

holds true for all $x, y \in X$.

If, moreover, f and φ have a common continuity point or if the function $X \ni x \to \|f(x)\| + \varphi(x)$ is bounded above on a second category Baire set, then there exists a nonnegative constant c such that

$$\|f(x)\| \leq c \cdot \|x\| + \varphi(x), \quad x \in X.$$

We note also the following interesting result of this kind proved by P. Šemrl.

Theorem 2.5 ([50]). *Let $\theta > 0$ and assume that a continuous mapping $f\colon \mathbb{R} \to \mathbb{R}$ satisfies*

$$\left| f\left(\sum_{n=1}^{m} x_n\right) - \sum_{n=1}^{m} f(x_n) \right| \leq \theta \sum_{n=1}^{m} |x_n|, \quad x_1, \dots, x_m \in \mathbb{R},$$

for all $m \in \mathbb{N}$. Then there exists an additive mapping $A\colon \mathbb{R} \to \mathbb{R}$ such that

$$|f(x) - A(x)| \leq \theta |x|, \quad x \in \mathbb{R}.$$

Another generalization of the Ulam-Hyers stability result has been proved by Th. M. Rassias and P. Šemrl [44]. This result reads as follows.

Theorem 2.6. *Let $H\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ be a positively homogeneous mapping of degree p with $p \neq 1$. Given a normed space E_1 and a Banach space E_2 assume that $f\colon E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq H(\|x\|, \|y\|), \quad x, y \in E_1.$$

Then there exists a unique additive mapping $A\colon E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq H(1,1)|2 - 2^p|^{-1}\|x\|^p, \quad x \in E_1.$$

Moreover,

$$A(x) = \begin{cases} \lim_{n\to\infty} 2^{-n} f(2^n x) & \text{for } p < 1, \\ \lim_{n\to\infty} 2^n f(2^{-n} x) & \text{for } p > 1. \end{cases}$$

Remark 2.3. Interesting results about the stability of a generalized additive Cauchy equation can be found in the papers: [3], [6], [38].

Remark 2.4. The historical background and many important results for the Ulam-Hyers-Rassias stability of various functional equations are surveyed in the expository paper [36] (see also [13]).

Remark 2.5. Further results on the stability of the Cauchy equation may be found in: [2], [5], [18], [19], [20], [35], [48], [52].

3. MULTIPLICATIVE CAUCHY EQUATION

Let $(G, \cdot)$ be a semigroup and E a normed algebra with the norm satisfying the condition

$$\|x \cdot y\| = \|x\|\|y\|, \quad \text{for } x, y \in E.$$

A mapping $f\colon G \to E$ is said to be multiplicative iff f satisfies the equation (called the multiplicative Cauchy equation):

$$f(x \cdot y) = f(x)f(y), \quad x, y \in G. \tag{3.1}$$

For this equation the situation differs from that of the additive Cauchy equation.

Namely, we have (see [4]):

Theorem 3.1. *Let G be a semigroup and E a normed algebra with the multiplicative norm. Given $\delta > 0$, let $f\colon G \to E$ satisfy the inequality*

$$\|f(x \cdot y) - f(x)f(y)\| \leq \delta \quad \text{for } x, y \in G. \tag{3.2}$$

Then either $\|f(x)\| \leq \frac{1}{2}\left(1 + (1 + 4\delta)^{\frac{1}{2}}\right)$ or f is multiplicative.

We say that in this case we have the superstability.

It is known that the assumption of multiplicativity for the norm is essential in this theorem.

For the space $M_2(\mathbb{C})$ of all 2×2 matrices with the usual norm we have (see [42]):

Theorem 3.2. *If G is an abelian group and a mapping $f\colon G \to M_2(\mathbb{C})$ satisfies the inequality (3.2) for some $\delta \geq 0$, then there exists a multiplicative mapping $m\colon G \to M_2(\mathbb{C})$ such that $f - m$ is bounded.*

In other words, in this case we have the Ulam-Hyers stability instead of the superstability phenomenon for the multiplicative Cauchy equation.

The reader may find interesting results concerning the superstability problem in [30] by R. Ger and [25] by R. Ger and P. Šemrl.

Now we state the result of P. Găvruţă [23] on so called mixed stability which answers the problem of Rassias-Tabor.

Theorem 3.3. *Let $\varepsilon, s > 0$ and $\delta = [2^s + (2^{2s} + 8\varepsilon)^{\frac{1}{2}}]/2$. Let B be a normed algebra with multiplicative norm and X be a real normed space. If $f\colon X \to B$ satisfies the inequality*

$$\|f(x+y) - f(x)f(y)\| \leq \varepsilon(\|x\|^s + \|y\|^s) \tag{3.3}$$

for all $x, y \in X$, then either

$$\|f(x)\| \leq \delta\|x\|^s \quad \text{for all } x \in X \text{ with } \|x\| \geq 1,$$

or

$$f(x+y) = f(x)f(y) \qquad \text{for all } x, y \in X.$$

The last result of this section is a very general and interesting theorem due to L. Székelyhidi [53].

Theorem 3.4. *Let G be a semigroup and V a right invariant vector space of complex-valued mappings on G. If $f, m\colon G \to \mathbb{C}$ are mappings such that the mapping*

$$f(xy) - f(x)m(y)$$

in variable x belongs to V for each $y \in G$, then either $f \in V$ or m is multiplicative.

4. JENSEN'S AND PEXIDER'S FUNCTIONAL EQUATIONS

Let us start from the following result for the Ulam-Hyers stability of Jensen's functional equation due to Z. Kominek [40].

Theorem 4.1. *Let D be a subset of $\mathbb{R}^n$ with non-empty interior and Y be a real Banach space. Assume that there exists an x_0 in the interior of D such that $D_0 = D - x_0$ fulfills the condition $\frac{D}{2} \subset D_0$. Let a mapping $f\colon D \to Y$ satisfy the inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \delta,$$

for some $\delta \geq 0$ and for all $x, y \in D$. Then there exist a mapping $F\colon \mathbb{R}^n \to Y$ and a constant $K > 0$ such that

$$2F\left(\frac{x+y}{2}\right) = F(x) + F(y) \tag{4.1}$$

for all $x, y \in \mathbb{R}^n$ and

$$\|f(x) - F(x)\| \leq K, \quad x \in D.$$

The equation (4.1) is called Jensen's functional equation.

The next result concerning the Ulam-Hyers-Rassias stability has been proved by S.-M. Jung [37] and was applied to study the asymptotic behavior of additive mappings.

Theorem 4.2. *Let $p > 0$, $p \neq 1$, $\delta \geq 0$, $\theta \geq 0$, be given. Let X and Y be a real normed space and a real Banach space, respectively. Assume that $f\colon X \to Y$ satisfies the inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \leq \delta + \theta(\|x\|^p + \|y\|^p) \tag{4.2}$$

for all $x, y \in X$. Further, assume $f(0) = 0$ and $\delta = 0$ in (4.2) for the case of $p > 1$. Then there exists exactly one additive mapping $F\colon X \to Y$ such that

$$\|f(x) - F(x)\| \le \delta + \|f(0)\| + \frac{\theta}{2^{1-p} - 1}\|x\|^p \qquad (\textit{for } p < 1)$$

or

$$\|f(x) - F(x)\| \le \frac{2^{p-1}}{2^{p-1} - 1}\theta\|x\|^p \qquad (\textit{for } p > 1)$$

for all $x \in X$.

The proof of this theorem is similar to the proof of the results presented in the previous sections and such a method of proof is called the direct method or Hyers' method.

Now we shall present a result on the stability of the Jensen's functional equation called a stability on a restricted domain.

Theorem 4.3 ([37]). *Let $d > 0$ and $\delta \ge 0$ be given real numbers. Suppose that $f\colon X \to Y$ (X, Y as in Theorem 4.2) satisfies the inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\| \le \delta$$

for all $x, y \in X$ with $\|x\| + \|y\| \ge d$. Then there exists exactly one additive mapping $F\colon X \to Y$ such that

$$\|f(x) - F(x)\| \le 5\delta + \|f(0)\|, \quad x \in X.$$

Let us remark that for convex functions the problem of stability is much more complicated and one cannot prove the analogue of Hyers' result for convex functions.

The reader may find more information on this problem in [7] and [33].

Finally, we state the result about the stability of the Pexider's functional equation

$$f(x + y) = g(x) + h(y) \tag{4.3}$$

with unknown functions f, g, h proved by K. Nikodem [43].

Theorem 4.4. *Let S be an abelian semigroup with zero and let Y be a sequentially complete topological vector space over $\mathbb{Q}$ (the set of rational numbers). Suppose that V is a non-empty Q-convex symmetric and bounded subset of Y. If, moreover, $f, g, h\colon S \to V$ satisfy the condition*

$$f(x + y) - g(x) - h(x) \in V, \quad x, y \in S,$$

then there exist functions $f_1, g_1, h_1\colon S \to Y$ fulfilling the Pexider's equation (4.3) for all $x, y \in S$ and such that

$$f_1(x) - f(x) \in 3\, \textit{seqcl}\, V,$$

$$g_1(x) - g(x) \in 4\, \textit{seqcl}\, V,$$

$$h_1(x) - h(x) \in 4\, \textit{seqcl}\, V,$$

for all $x \in S$.

5. D'ALEMBERT'S AND LOBACZEVSKI'S FUNCTIONAL EQUATIONS

In this section we shall discuss the D'Alembert functional equation (called also the cosine equation)

$$f(x+y)+f(x-y)=2f(x)f(y). \tag{5.1}$$

For this equation we discover a different kind of stability, usually called superstability (i.e. if f satisfies some inequality, than f satisfies the equation). The first result for this equation has been proved by J. Baker [4]. Here we will present the result proved by Găvruţă [24] because of the simplicity of the proof.

Theorem 5.1. *Let $\delta > 0$ and G be an abelian group and $f\colon G \to \mathbb{C}$ be a function satisfying the inequality*

$$|f(x+y)+f(x-y)-2f(x)f(y)| \le \delta \quad \textit{for all } x,y \in G. \tag{5.2}$$

Then either f is bounded or satisfies the D'Alembert functional equation (5.1).

Recently, the author of this paper has proved a very general criterium on the stability of the D'Alembert functional equation (see [17]).

Let G be a group and $\mathbb{C}$ the set of complex numbers. Let $f, g\colon G \to \mathbb{C}$, then we denote

$$\begin{aligned} g_a(x) &:= g(x+a), \quad x,a \in G,\\ A(f)(x,y) &:= f(x+y)+f(x-y)-2f(x)f(y), \quad x,y \in G,\\ U^1 &:= \{g | g\colon G \to \mathbb{C}\},\\ U^2 &:= \{H | H\colon G\times G \to \mathbb{C}\}. \end{aligned}$$

If U^1 has the property: if $g \in U^1$ then $g_a \in U^1$ for $a \in G$, then we say that U^1 has the "translation property".

We have the following.

Theorem 5.2 ([17]). *Let $f\colon G \to \mathbb{C}$ be a function. Let U^1 be a linear space over $\mathbb{C}$ with the "translation property". If, moreover,*

(i) *for every $u \in G$, $A(f)(\cdot,u) \in U^1$,*

then

(ii) *$f \in U^1$ or $A(f) = 0$.*

Remark 5.1. Linear spaces U^1, U^2 satisfying the conditions

(iii) $A(f) \in U^2$,

(iv) for every $u \in G$, $A(f)(\cdot,u) \in U^1$,

are called A-conjugate spaces.

Remark 5.2. The famous Baker superstability result for the D'Alembert functional equation (see [4]) is obtained for

$$\begin{aligned} U^1 &= B(G;\,\mathbb{C}),\\ U^2 &= B(G\times G;\,\mathbb{C}), \end{aligned}$$

where U^1 and U^2 denote the A-conjugate linear spaces of bounded functions on G and $G \times G$ respectively with values on complex numbers.

For the Lobaczevski's functional equation

$$f^2\left(\frac{x+y}{2}\right) = f(x)f(y) \tag{5.3}$$

we have the following result [24].

Theorem 5.3. *Let $\delta > 0$ and G be an abelian 2-divisible group. Let $f\colon G \to \mathbb{C}$ be such that*

$$\left| f^2\left(\frac{x+y}{2}\right) - f(x)f(y) \right| \le \delta \quad \textit{for all } x, y \in G.$$

Then either

$$|f(x)| \le \frac{1}{2}\left[|f(0)| + (|f(0)|^2 + 4\delta)^{\frac{1}{2}}\right] \quad \textit{for all } x \in G,$$

or f satisfies the Lobaczevski's functional equation (5.3).

The equation

$$f(xy)f(xy^{-1}) = f^2(x) - f^2(y) \tag{5.4}$$

is called the sine functional equation. For this equation P. Cholewa has proved the following superstability result.

Theorem 5.4. *Let G be an abelian group uniquely divisible by 2. Then, every unbounded mapping $f\colon G \to \mathbb{C}$ satisfying the inequality*

$$|f(xy)f(xy^{-1}) - f^2(x) + f^2(y)| \le \delta, \quad x, y \in G,$$

for some $\delta \ge 0$, is a solution of the sine equation (5.4).

Another interesting equation

$$f(xy) = yf(x) + xf(y) \tag{5.5}$$

is known as the functional equation of derivation. For this equation we have the P. Šemrl [49] superstability result.

Theorem 5.5. *Let E be a Banach space, let A be an algebra of operators on E and let $B(E)$ be the algebra of all bounded operators on E. Assume that $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a function with the property*

$$\lim_{t\to\infty}\left[\frac{1}{t}\varphi(t)\right] = 0.$$

If, moreover, $f\colon A \to B(E)$ satisfies the inequality

$$\|f(xy) - yf(x) - xf(y)\| \le \varphi(\|x\|\|y\|), \quad x, y \in A,$$

then f satisfies the derivation equation (5.5).

For further interesting information about the Ulam-Hyers-Rassias stability of some generalizations of the cosine and sine functional equations, the reader is referred to the paper of R. Ger [29].

6. Equation of Homogeneous Mappings

The problem of Ulam-Rassias type stability for homogeneous mappings was originated by the author of this survey paper. The first result was presented in [11].

We shall present here the following result.

Theorem 6.1. *Let E be a real linear space and F a real Banach space. Let $f, g\colon E \to F$ and $\varphi\colon \mathbb{R} \to \mathbb{R}$ and $h\colon \mathbb{R} \times E \to \mathbb{R}_+$ be given functions. Assume that*

$$\|f(\alpha x) - \varphi(\alpha)g(x)\| \le h(\alpha, x)$$

for all $(\alpha, x) \in \mathbb{R} \times E$ *and* $\varphi(1) = 1$. *Suppose that there exists* $\beta \in \mathbb{R}$ *such that* $\varphi(\beta) \neq 0$ *and the series*

$$\sum_{n=1}^{\infty} |\varphi(\beta)|^{-n} H(\beta, \beta^n x)$$

converges pointwise for all $x \in E$, *and*

$$\liminf_{n\to\infty} |\varphi(\beta)|^{-n} H(\alpha, \beta^n x) = 0,$$

for all $(\alpha, x) \in \mathbb{R} \times E$, *where*

$$H(\alpha, x) := h(\alpha, x) + |\varphi(\alpha)| h(1, x).$$

Then there exists exactly one φ*-homogeneous function* $A\colon E \to F$*:*

$$A(\alpha x) = \varphi(\alpha) A(x), \quad (\alpha, x) \in \mathbb{R} \times E,$$

such that

$$\|A(x) - f(x)\| \leq \sum_{n=1}^{\infty} |\varphi(\beta)|^{-n} H(\beta, \beta^{n-1} x),$$

$$\|A(x) - f(x)\| \leq \sum_{n=1}^{\infty} |\varphi(\beta)|^{-n} G(\beta, \beta^{n-1} x),$$

for $x \in E$, *where*

$$G(\alpha, x) := h(\alpha, x) + h(1, \alpha x), \quad (\alpha, x) \in \mathbb{R} \times E.$$

Let us note that the function A is given as the limit of the generalized Hyers-Rassias sequence

$$A_n(x) := \varphi^{-n}(\beta) f(\beta^n x), \quad x \in E,\ n \in \mathbb{N}.$$

Next we present a result about the stability of the homogeneous mappings in topological spaces (see [10]).

Theorem 6.2. *Assume that* X *is a real linear space and* Y *a sequentially complete locally convex linear topological space. Let* $K \subset X$ *be a cone and* $f, g\colon K \to Y$ *be given functions. Suppose that there exist an* $A \subset [1, \infty)$, $1 \in A$, *int* $A \neq 0$ *and a bounded subset* $U \subset Y$ *such that*

$$\alpha^{-1} f(\alpha x) - g(x) \in U, \quad \alpha \in A, x \in K.$$

Then there exists exactly one positively homogeneous function $F\colon K \to Y$ *such that*

$$F(x) - f(x) \in s(s-1)^{-1} cl\, conv(U - U), \quad x \in K$$

and

$$F(x) - g(x) \in s(s-1)^{-1} cl\, conv[(U - U) \cup U \cup (-U)], \quad x \in K,$$

where $s = \sup\ A < \infty$. *If* $s = \infty$, *then*

$$F(x) - f(x) \in cl\, conv(U - U), \quad x \in K$$

and

$$F(x) - g(x) \in cl\, conv[(U - U) \cup U \cup (-U)], \quad x \in K.$$

Moreover,

$$f(x) = \lim_{n\to\infty} \alpha^{-n} f(\alpha^n x) = \lim_{n\to\infty} \alpha^{-n} g(\alpha^n x), \quad x \in K$$

for $\alpha \in A \setminus \{1\}$ *and the convergence is uniform on* K.

For further information concerning results on the stability of the homogenity condition, one may see for example [34], [39], [41] and [54].

7. Quadratic Functional Equation

The equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{7.1}$$

is called the quadratic functional equation. We define any solution of the equation (7.1) to be a quadratic function.

The Hyers-Ulam stability of the quadratic functional equation was studied by F. Skof [51] and P. Cholewa [7], whereas the Rassias type stability was studied by the present author in [12].

Here we shall present some main results and ideas obtained by S. Czerwik [14].

Let X be a commutative semigroup with zero and the following law of cancelation:

$$a+c=b+c \quad \text{implies} \quad a=b \quad \text{for all } a,b,c\in X.$$

Moreover, we assume that in X is defined a multiplication by nonnegative real numbers with the usual properties. Assume that in X is a metric d such that

$$\begin{aligned} d(x+y,x+z)&=d(y,z) \quad \text{for all } x,y,z\in X,\\ d(tx,ty)&=td(x,y) \quad \text{for all } x,y\in X,\ t\in\mathbb{R}_+. \end{aligned}$$

Denote

$$\|x\|:=d(x,0), \quad x\in X.$$

A commutative semigroup with zero satisfying all the above conditions will be called a quasi-normed space.

Now we can present the main results on stability of the quadratic functional equation due to S. Czerwik [14].

Let E be a group and $h\colon E\times E\to\mathbb{R}_+$ a given function. We denote

$$\begin{aligned} H(x,y):=&h(x,y)+h(x,0)+h(y,0)+h(0,0),\\ K(x,y):=&2h(x,y)+h(x+y,0)+h(x-y,0) \end{aligned}$$

for $x,y\in E$.

Theorem 7.1 ([14]). *Let X be an abelian group and Y a complete quasi-normed space. Let $F,G\colon X\to Y$ satisfy the inequality*

$$d[F(x+y)+F(x-y),G(x)+G(y)]\le h(x,y) \quad \textit{for all } x,y\in X. \tag{7.2}$$

Let for some integer $k\ge 2$ and $m=1,\dots,k-1$ the series

$$\sum_{s=0}^{\infty} h(mk^s x,k^s x)k^{-2s}$$

and

$$\sum_{s=0}^{\infty} h(k^s x,0)k^{-2s}$$

be convergent for all $x\in X$. If, moreover,

$$\liminf_{n\to\infty} h(k^n x,k^n y)k^{-2n}=0 \quad \textit{for all } x,y\in X,$$

then there exists exactly one quadratic mapping $A\colon X \to Y$ such that

$$d[A(x)+F(0),F(x)] \le k^{-2}\sum_{m=1}^{k-1}\sum_{s=0}^{\infty}(k-m)H(mk^s x,k^s x)k^{-2s},$$
$$d[2A(x)+G(0),G(x)] \le k^{-2}\sum_{m=1}^{k-1}\sum_{s=0}^{\infty}(k-m)K(mk^s x,k^s x)k^{-2s},$$

for all $x \in X$.

Similarly we can obtain the following result.

Theorem 7.2 ([14]). *Let X be an abelian group divisible by $k \in \mathbb{N}$, $k \ge 2$ and Y a complete quasi-normed space. Let $F, G\colon X \to Y$ satisfy the inequality (7.2). Suppose that for $m = 1,\dots,k-1$ the series*

$$\sum_{s=0}^{\infty} h(mk^{-s}x,k^{-s}x)k^{2s} \tag{7.3}$$

and

$$\sum_{s=0}^{\infty} h(k^{-s}x,0)k^{2s} \tag{7.4}$$

are convergent for all $x \in X$. If, moreover,

$$\liminf_{n\to\infty} h(k^{-n}x,k^{-n}y)k^{2n} = 0 \quad \textit{for all } x,y \in X, \tag{7.5}$$

and $F(0) = 0$, then there exists exactly one quadratic mapping $B\colon X \to Y$ such that

$$d[B(x),F(x)] \le k^{-2}\sum_{m=1}^{k-1}\sum_{s=1}^{\infty}(k-m)H(mk^{-s}x,k^{-s}x)k^{2s},$$
$$d[B(x),G(x)] \le k^{-2}\sum_{m=1}^{k-1}\sum_{s=1}^{\infty}(k-m)K(mk^{-s}x,k^{-s}x)k^{2s},$$

for all $x \in X$.

Finally, for the case $F(0) \ne 0$, we have

Theorem 7.3 ([14]). *Let X be an abelian group divisible by $k \in \mathbb{N}$, $k \ge 2$ and Y a Banach space. Let the mappings $F, G\colon X \to Y$ satisfy the inequality*

$$\|F(x+y)+F(x-y)-G(x)-G(y)\| \le h(x,y) \quad \textit{for all } x,y \in X.$$

Assume, additionally, that the series (7.3) and (7.4) are convergent for all $x \in X$ and the condition (7.5) is satisfied. Then there exists exactly one quadratic mapping $C\colon X \to Y$ such that

$$\|C(x)+F(0)-F(x)\| \le k^{-2}\sum_{m=1}^{k-1}\sum_{s=1}^{\infty}(k-m)H(mk^{-s}x,k^{-s}x)k^{2s},$$
$$\|2C(x)+G(0)-G(x)\| \le k^{-2}\sum_{m=1}^{k-1}\sum_{s=1}^{\infty}(k-m)K(mk^{-s}x,k^{-s}x)k^{2s},$$

for all $x \in X$.

Now we shall consider the Ulam-Hyers stability problem in L^p–spaces.

Assume that (Y, μ) is a measure space. If $f\colon Y \to \mathbb{R}$ is non-negative function, then we define its upper integral (after J. Tabor)

$$\int_Y^{+} f\, d\mu := \inf \left\{ \int_Y \varphi\, d\mu : \ \varphi \in L_1(Y, \mathbb{R}),\ f(x) \le \varphi(x),\ x \in Y \right\}$$

or

$$\int_Y^{+} f\, d\mu := +\infty$$

if the corresponding set is empty.

For $p > 0$ and a metric abelian group E we set

$$\|x\| := d(x, 0), \quad x \in E,$$

$$L_p^{+}(Y, E) := \left\{ f\colon Y \to E : \ \int_Y^{+} \|f(x)\|^p\, d\mu < +\infty \right\}.$$

Theorem 7.4 ([16]). *Let (X, μ) be an abelian complete measurable group and E be a metric abelian group without elements of order two. Let $f\colon X \to E$ be a such that $Qf \in L_P^{+}(X \times X, E)$, for some $p > 0$. If, moreover, $\mu(X) = +\infty$, then*

$$Qf(x, y) \overset{\mu\times\mu}{=} 0,$$

where $Qf(x, y) := 2f(x) + 2f(y) - f(x + y) - f(x - y)$.

In other words, Theorem 7.4 says that under the assumption that the quadratic difference of f is bounded by an integrable function, the function f satisfies the quadratic equation almost everywhere, that is, in this case the phenomenon of superstability occurs.

For more details about similar results, the reader is referred to [15], [55].

To end, we shall present a result on the stability of the generalized quadratic functional equation on topological spaces.

Theorem 7.5 ([1]). *Let G be an abelian 2-divisible group and let $B \subset X$ be a nonempty bounded set, where X is a sequentially complete locally convex linear topological space. If the functions $f, g\colon G \to X$ satisfy*

$$f(x + y) + f(x - y) - g(x) - g(y) \in B, \quad x, y \in G,$$

then there exists exactly one quadratic function $Q\colon G \to X$ such that

$$Q(x) + f(0) - f(x) \in \frac{2}{3} cl\, conv(B - B), \quad x \in G,$$

$$2Q(x) + g(0) - g(x) \in \frac{2}{3} cl\, conv(B - B), \quad x \in G.$$

Moreover, the function Q is given by the formula

$$Q(x) = \lim_{n\to\infty} f_n(x) = \frac{1}{2} \lim_{n\to\infty} g_n(x), \quad x \in G$$

where

$$f_n(x) = \frac{1}{2^{2n}} f(2^n x), \quad g_n(x) = \frac{1}{2^{2n}} g(2^n x), \quad n \in \mathbb{N},\ x \in G$$

and convergence is uniform on G.

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