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HYERS-ULAM-RASSIAS STABILITY OF A GENERALIZED JENSEN FUNCTIONAL EQUATION

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ABSTRACT. In this paper we obtain the Hyers-Ulam-Rassias stability for the generalized Jensen's functional equation in abelian group $(G, +)$. Furthermore we discuss the case where G is amenable and we give a note on the Hyers-Ulam-stability of the K-spherical $(n \times n)$-matrix functional equation.

Key words and phrases: amenable group, Cauchy equation, Jensen equation, Pexider equation, Matrix equation, K-spherical function, Hyers-Ulam-Rassias stability.

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1. INTRODUCTION

The stability problem of functional equations was posed for the first time by S. M. Ulam [47] in the year 1940. Ulam stated the problem as follows:

> Given a group G_1, a metric group (G_2, d), a number $\varepsilon > 0$ and a mapping f: $G_1 \longrightarrow G_2$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \varepsilon$ for all $x, y \in G_1$, does there exist an homomorphism h: $G_1 \longrightarrow G_2$ and a constant $k > 0$, depending only on G_1 and G_2 such that $d(f(x), h(x)) \leq k\varepsilon$ for all x in G_1?

The first affirmative answer was given by D. H. Hyers [13], under the assumption that G_1 and G_2 are Banach spaces.
In 1978, Th. M. Rassias [32] gave a remarkable generalization of the Hyers's result which allows the Cauchy difference to be unbounded, as follows:

Theorem 1.1. [32] *Let $f : V \longrightarrow X$ be a mapping between Banach spaces and let $p < 1$ be fixed. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for some $\theta \geq 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).
If, in addition, $f(tx)$ is continuous in t for each fixed x, then T is linear.

In 1990, Th. M. Rassias during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for values of p greater or equal to one. Z. Gajda [10] following the same approach as in [32] provided an affirmative solution to Th. M. Rassias'question for p strictly greater than one. However, it was shown independently by Z. Gajda [10] and Th. M. Rassias and P. Šemrl [39] that a similar result for the case of value of p equal to one can not be obtained.

In 1982 J. M. Rassias [30] followed the innovative approach of Rassias's theorem [32] in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^q$ with $p + q \neq 1$.

The concept of the linear mapping, that was introduced for the first time in 1978 by Th. M. Rassias and followed later by several other mathematicians is known today as Hyers-Ulam-Rassias stability. Several papers have been published in this subject and some interesting variants of Ulam's problem have been also investigated by a number of mathematicians. We refer the reader to the following references [3], [7]-[41].

In 1994 P. Gavrută [11] following the spirit of Th. M. Rassias approach [32] obtained a generalization of the Rassias stability Theorem ([32]) by replacing the function :
$(x, y) \longmapsto \theta(\| x \|^p + \| y \|^p)$ by a mapping $\varphi(x, y)$ which satisfies the following condition

$$\sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \quad \text{or} \quad \sum_{n=0}^{\infty} 2^n \varphi(\frac{x}{2^n}, \frac{y}{2^n}) < \infty$$

for every x, y in a Banach space V.

Since then a number of stability results have been obtained for functional equations of the forms

$$f(x+y) = g(x) + h(y), \quad x, y \in G \tag{1.1}$$

$$f(x+y) + f(x-y) = g(x) + h(y), \quad x, y \in G, \tag{1.2}$$

where G is an abelian group. In particular for the classical equations: Cauchy's, Jensen's equations, the quadratic and the Pexider equations. The reader can be refereed to [1]-[41] for a comprehensive account of the subject.
B. Bouikhalene, E. Elqorachi and Th. M. Rassias ([4],[5]) in a very recent study have introduced a new stability approach for the Pexider functional equation.

$$f(x+y) + g(x+\sigma(y)) = h(x) + l(y), \quad x, y \in G, \tag{1.3}$$

where σ is an automorphism of G such that $\sigma \circ \sigma = I$.

The first result of the stability of the Cauchy matrix functional equation

$$f(xy) = f(x)f(y), \quad x, y \in G \tag{1.4}$$

has been obtained by J. Lawrence [24].

Theorem 1.2. [24] *Let S be a semigroup, let f: $G \longrightarrow M_n(\mathbb{C})$ be a matrix function such that*

$$\| f(xy) - f(x)f(y) \| \leq \delta \tag{1.5}$$

for all $x, y \in S$ and for some $\delta > 0$, then there exists a function $h : S \longrightarrow M_n(\mathbb{C})$ such that $f - h$ is bounded on S and that

$$(h(xy) - h(x)h(y))^2 = 0 \tag{1.6}$$

for all x, $y \in S$.
Furthermore, if S is an abelian group and $n = 2$, there exits $h : S \longrightarrow M_2(\mathbb{C})$ such that $f - h$ is bounded on S and that

$$h(xy) = h(x)h(y) \tag{1.7}$$

for all x, $y \in S$.

The argument used by Lawrence involves a development of the key idea in the proof of Baker's result [3].

In a number of papers ([42]-[46]) H. Stetkær have studied functional equations related to the action by automorphisms on a group G of a compact transformation group K. Writing the action of $k \in K$ on $x \in G$ as $k \cdot x$ and letting dk denote the normalized Haar measure on K the functional equations (1.1) and (1.2) have the form

$$\int_K f(x + k \cdot y)dk = g(x) + h(y), \quad x, y \in G. \tag{1.8}$$

Take $K = \{I\}$, or $K = \{I, -I\}$ (I denote the identity).

In [1] the Hyers-Ulam stability result of the K-quadratical equation

$$\int_K f(x + k \cdot y)dk = f(x) + f(y), \quad x, y \in G \tag{1.9}$$

has been obtained for K a general commutative transformation subgroup of the abelian group $(G, +)$.

Recently, A. Charifi , B. Bouikhalene, and E. Elqorachi [6] proved the Hyers-Ulam stability of the generalized Jensen's functional equation

$$\Sigma_{k\in K}f(x+k\cdot y)=|K|f(x),\ x,y\in G, \tag{1.10}$$

where G is an abelian group, K is a finite abelian subgroup of $Aut(G)$ (the group of automorphisms of G), and $|K|$ denotes the order of K.

The purpose of the present paper is to investigate the Hyers-Ulam-Rassias stability of functional equations (1.10). Furthermore, we prove the stability of the H-spherical $(n\times n)$-matrix functional equation

$$\int_H \phi(x+h\cdot y)dh=\phi(x)\phi(y),\ \ x,y\in G, \tag{1.11}$$

where H is a compact subgroup of automorphisms of G and dh is the normalized Haar measure of H.
The general functional equation

$$\int_H \phi(x+h\cdot y)dh=\phi(x)\phi(y),\ \ x,y\in G, \tag{1.12}$$

in which $\phi : G\longrightarrow \mathbf{B}(\mathcal{H})$ is a continuous and bounded operator function on the Hilbert space $\mathcal{H}$ and $\phi(e)$ is the identity operator $I_{\mathcal{H}}$ was solved by H. Stetkær [44].
Some stability results of equation (1.11) with $n=1$ are obtained by R. Badora [2] and by J. Schwaiger [41].

In this paper, our results are organized as follows: In Section 2, we prove the Hyers-Ulam-Rassias stability theorem of the generalized Jensen's functional equation (1.10). In Section 3, we prove that equation (1.9) and equation (1.10) are stable in amenable groups and for K a subgroup of $Aut(G)$ not necessarily commutative.
In Section 4, we generalized the result obtained by J. Lawrence [24] and T. Iwona [18]. More precisely, we obtain the Lawrence-stability theorem of the matrix equation (1.11).

General Set-Up Let K be a compact transformation group of an abelian group $(G,+)$, acting by automorphisms of G. We let dk denote the normalized Haar measure on K, and the action of $k\in K$ on $x\in G$ is denoted by $k\cdot x$. When K is finite, the normalized Haar measure dk on K is given by

$$\int_K h(k)dk=\frac{1}{|K|}\sum_{k\in K}h(k)$$

for any h: $K\longrightarrow\mathbb{C}$, where $|K|$ denotes the order of K.

2. HYERS-ULAM-RASSIAS STABILITY OF THE GENERALIZED JENSEN'S FUNCTIONAL EQUATION (1.10)

In [6] A. Charifi B. Bouikhalene and E. Elqorachi proved the Hyers-Ulam stability of the functional equation

$$\Sigma_{k\in K}f(x+k\cdot y)=|K|f(x),\ x,y\in G. \tag{2.1}$$

By following the ideas established in [6], in the following theorem, we obtain the Hyers-Ulam-Rassias stability of the above equation.

Theorem 2.1. *Let G be an abelian group, let K be a finite abelian subgroup of automorphisms of G. Let $\varphi : G \times G \longrightarrow \mathbb{R}^+$ be a function. We define the sequence functions*

$$\varphi_0 = \varphi,$$

$$\varphi_n(x,y) = \sum_{k\in K\setminus\{I\}} \varphi_{n-1}(x - k\cdot x, y - k\cdot y) \; for\ all\ x,\ y \in G,\ n \in \mathbb{N}^*.$$

Suppose that

$$\widetilde{\varphi}(x,y) = \sum_{n=0}^{+\infty} \frac{1}{|K|^n}\varphi_n(x,y) < +\infty \; for\ all\ x,\ y \in G.$$

If a function $f : G \longrightarrow B$, where B is a Banach space, satisfies the inequality

$$\|\Sigma_{k\in K} f(x + k\cdot y) - |K|f(x)\| \leq \varphi(x,y) \; for\ all\ x, y \in G. \tag{2.2}$$

Then, there exists a unique function $J : G \longrightarrow B$ solution of (2.1) such that $J(e) = f(e)$, and

$$\|f(x) - J(x)\| \leq \frac{1}{|K|}\widetilde{\varphi}(x,x) \; for\ all\ x \in G.$$

Proof. Assume that f: $G \longrightarrow B$ satisfies the inequality (2.2). We use induction on n to prove that the sequence functions

$$f_0(x) = f(x), \tag{2.3}$$

$$f_n(x) = \Sigma_{k\in K} f_{n-1}(x - k\cdot x) \; for\ x \in G \text{ and } n \in \mathbb{N}^* \tag{2.4}$$

satisfy the following conditions

$$f_n(e) = |K|^n f(e) \tag{2.5}$$

$$\|f_n(x) - |K|f_{n-1}(x)\| \leq \varphi_{n-1}(x, -x) \tag{2.6}$$

$$\|f_n(x) - |K|^n f(x)\| \leq \sum_{i=0}^{n-1} |K|^{n-i-1}\varphi_i(x, -x) \tag{2.7}$$

and

$$\|\Sigma_{k\in K} f_n(x + k\cdot y) - |K|f_n(x)\| \leq \varphi_n(x,y). \tag{2.8}$$

By using the definition of f_1 and by letting $x = y = e$, one obtains

$$f_1(e) = \sum_{k\in K} f_0(e - k\cdot e) = |K|f(e).$$

By using inequality (2.2), we get

$$\|f_1(x) - |K|f_0(x)\| = \|\Sigma_{k\in K} f(x - k\cdot x) - |K|f(x)\| \leq \varphi_0(x, -x).$$

Equation (2.4) for $n = 1$ implies that

$$\begin{aligned}
\Sigma_{t\in K} f_1(x + t\cdot y) &= \Sigma_{t\in K}\Sigma_{k\in K} f(x + t\cdot y - k\cdot(x + t\cdot y)) \\
&= |K|f(e) + \Sigma_{k\in K\setminus\{I\}}\Sigma_{t\in K} f(x + t\cdot y - k\cdot(x + t\cdot y)) \\
&= f_1(e) + \Sigma_{k\in K\setminus\{I\}}\Sigma_{t\in K} f(x - k\cdot x + t\cdot(y - k\cdot y)).
\end{aligned}$$

However,

$$|K|f_1(x) = |K|\Sigma_{k\in K} f(x - k\cdot x) = f_1(e) + |K|\Sigma_{k\in K\setminus\{I\}} f(x - k\cdot x),$$

hence, we deduce that

$$\|\Sigma_{t\in K} f_1(x + t\cdot y) - |K|f_1(x)\|$$

$$= \|\Sigma_{k\in K\setminus\{I\}}\Sigma_{t\in K} f(x-k\cdot x+t\cdot(y-k\cdot y)) - |K|\Sigma_{k\in K\setminus\{I\}} f(x-k\cdot x)\|$$
$$\leq \Sigma_{k\in K\setminus\{I\}}\|\Sigma_{t\in K} f(x-k\cdot x+t\cdot(y-k\cdot y)) - |K| f(x-k\cdot x)\|$$
$$\leq \varphi_1(x,y).$$

So that the inductive assumptions (2.5), (2.6), (2.7) and (2.8) are trues for $n=1$. Now, the inductive assumptions must be demonstrated to hold true for the next positive integer $n+1$. It follows from (2.4), (2.2) and the induction assumptions that

$$f_{n+1}(e) = \Sigma_{k\in K} f_n(e-k\cdot e) = |K| f_n(e) = |K|^{n+1} f(e),$$
$$\|f_{n+1}(x) - |K| f_n(x)\| = \|\Sigma_{k\in K} f_n(x-k\cdot x) - |K|\Sigma_{k\in K} f_{n-1}(x-k\cdot x)\|$$
$$= \|f_n(e) + \Sigma_{k\in K\setminus\{I\}} f_n(x-k\cdot x) - |K| f_{n-1}(e) - |K|\Sigma_{k\in K\setminus\{I\}} f_{n-1}(x-k\cdot x)\|$$
$$\leq \Sigma_{k\in K\setminus\{I\}}\|f_n(x-k\cdot x) - |K| f_{n-1}(x-k\cdot x)\|$$
$$\leq \Sigma_{k\in K\setminus\{I\}}\varphi_{n-1}(x-k\cdot x, -x+k\cdot x) = \varphi_n(x,-x).$$

$$\begin{aligned}\|f_{n+1}(x) - |K|^{n+1} f(x)\| &= \|\sum_{i=0}^{n} |K|^{n-i}(f_{i+1}(x) - |K| f_i(x))| \\ &\leq \sum_{i=0}^{n} |K|^{n-i}\|f_{i+1}(x) - |K| f_i(x))\| \\ &\leq \sum_{i=0}^{n} |K|^{n-i}\varphi_i(x,-x).\end{aligned}$$

Now, for all $x, y \in G$ we get

$$\|\Sigma_{t\in K} f_{n+1}(x+t\cdot y) - |K| f_{n+1}(x)\|$$
$$= \||K| f_n(e) + \Sigma_{t\in K}\Sigma_{k\in K\setminus\{I\}} f_n(x+t\cdot y-k\cdot(x+ty)) - |K| f_n(e) - |K|\Sigma_{k\in K\setminus\{I\}} f_n(x-k\cdot x)\|$$
$$\leq \Sigma_{k\in K\setminus\{I\}}\|\Sigma_{t\in K} f_n(x-k\cdot x+t\cdot(y-k\cdot y)) - |K| f_n(x-k\cdot x)\|$$
$$\leq \Sigma_{k\in K\setminus\{I\}}\varphi_n(x-k\cdot x, y-k\cdot y) = \varphi_{n+1}(x,y),$$

which proves that the inductive assumptions are true for any positive integer n.
The inequality (2.6), implies that the sequence functions

$$g_n(x) = \frac{f_n(x)}{|K|^n}$$

is a Cauchy sequence for every fixed x in G. Since B is a Banach space, the limit of this sequence exists, and this limit is in B. Define $J : G \longrightarrow B$ by

$$J(x) = \lim_{n\to+\infty}\frac{f_n(x)}{|K|^n}. \tag{2.9}$$

Clearly, from (2.5), we get $J(e) = f(e)$.
Dividing (2.7) by $|K|^n$ to obtain $\|f(x) - \frac{f_n(x)}{|K|^n}\| \leq \sum_{i=0}^{n-1} |K|^{-i-1}\varphi_i(x,-x)$. As $n \longmapsto +\infty$, then the last inequality becomes $\|f(x) - J(x)\| \leq \frac{1}{|K|}\widetilde{\varphi}(x,-x)$ for all $x \in G$. From (2.8), we can easily verified that J: $G \longrightarrow B$ satisfies the generalized Jensen's functional equation (2.1). Suppose that there exists another mapping, H: $G \longrightarrow B$ solution of the generalized Jensen's functional equation

$$\Sigma_{k\in K} H(x+k\cdot y) = |K| H(x) \text{ for all } x, y \in G \tag{2.10}$$

which satisfies $H(e) = f(e)$ and $\|f(x) - H(x)\| \leq \frac{1}{|K|}\widetilde{\varphi}(x, -x)\ for\ all\ x \in G$.
First we make the induction assumption

$$\|f_n(x) - |K|^n H(x)\| \leq |K|^{n-1} \sum_{p=n}^{+\infty} \frac{1}{|K|^p} \varphi_p(x, -x). \tag{2.11}$$

In view of (2.10), (2.4) and the condition $H(e) = f(e)$, we get

$$\begin{aligned} \|f_1(x) - |K|H(x)\| &= \|\Sigma_{k\in K} f(x - k \cdot x) - \Sigma_{k\in K} H(x - k \cdot x) \\ &= \|f(e) + \Sigma_{k\in K\setminus\{I\}} f(x - k \cdot x) - H(e) - \Sigma_{k\in K\setminus\{I\}} H(x - k \cdot x)\| \\ &\leq \Sigma_{k\in K\setminus\{I\}} \|f(x - k \cdot x) - H(x - k \cdot x)\| \\ &\leq \Sigma_{k\in K\setminus\{I\}} \frac{1}{|K|}\widetilde{\varphi}(x - k \cdot x, -x + k \cdot x) = \Sigma_{k\in K\setminus\{I\}} \frac{1}{|K|} \sum_{n=0}^{+\infty} \frac{1}{|K|^n} \varphi_n(x - k \cdot x, -x + k \cdot x) \\ &= \sum_{n=0}^{+\infty} \frac{1}{|K|^{n+1}} \varphi_{n+1}(x, -x) = \sum_{n=1}^{+\infty} \frac{1}{|K|^n} \varphi_n(x, -x) \end{aligned} \tag{2.12}$$

Assuming that (2.11) is true for all integers $i \leq n$, hence we have

$$\begin{aligned} \|f_{n+1}(x) - |K|^{n+1} H(x)\| &= \|\Sigma_{k\in K} f_n(x - k \cdot x) - |K|^n \Sigma_{k\in K} H(x - k \cdot x) \\ &= \||K|^n f(e) + \Sigma_{k\in K\setminus\{I\}} f_n(x - k \cdot x) - |K|^n H(e) - |K|^n \Sigma_{k\in K\setminus\{I\}} H(x - k \cdot x) \\ &\leq \Sigma_{k\in K\setminus\{I\}} \|f_n(x - k \cdot x) - |K|^n H(x - k \cdot x) \\ &\leq \Sigma_{k\in K\setminus\{I\}} |K|^{n-1} \sum_{p=n}^{+\infty} \frac{1}{|K|^p} \varphi_p(x - k \cdot x, -x + k \cdot x) = |K|^n \sum_{p=n}^{+\infty} \frac{1}{|K|^{p+1}} \varphi_{p+1}(x, -x) \\ &= |K|^n \sum_{p=n+1}^{+\infty} \frac{1}{|K|^p} \varphi_p(x, -x) \end{aligned} \tag{2.13}$$

By letting $n \longrightarrow +\infty$ we get from inequality $\|\frac{f_n(x)}{|K|^n} - H(x)\| \leq \frac{1}{|K|} \sum_{p=n}^{+\infty} \frac{1}{|K|^p} \varphi_p(x, -x)$ that $J = H$. This completes the proof of Theorem 2.1. ∎

Corollary 2.2. [6] *Let $\delta > 0$ and $p < 1$, let G be a normed space, let $K = \{I, \sigma\}$ where σ is an involution of G. If a function $f : G \longrightarrow B$, where B is a Banach space, satisfies the inequality*

$$\|f(x + y) + f(x + \sigma(y)) - 2f(x)\| \leq \delta(\|x\|^p + \|y\|^p)\ for\ all\ x, y \in G. \tag{2.14}$$

Then, there exists a unique function $q : G \longrightarrow B$ solution of equation

$$f(x + y) + f(x + \sigma(y)) = 2f(x), \quad x, y \in G \tag{2.15}$$

such that $J(e) = f(e)$ and

$$\|f(x) - q(x)\| \leq \delta\{\|x\|^p + \frac{1}{2 - 2^p} \|x - \sigma(x)\|^p\}\ for\ all\ x \in G.$$

Corollary 2.3. *Let $\delta > 0$ and p, q such that $0 \leq p + q < 1$, let G be a normed space, let $K = \{I, \sigma\}$ where σ is an involution of G. If a function $f : G \longrightarrow B$, where B is a Banach space, satisfies the inequality*

$$\|f(x + y) + f(x + \sigma(y)) - 2f(x)\| \leq \delta\|x\|^p \|y\|^q\ for\ all\ x, y \in G. \tag{2.16}$$

Then, there exists a unique function $J : G \longrightarrow B$ solution of (2.15) such that $J(e) = f(e)$ and

$$\|f(x) - J(x)\| \leq \delta \frac{\|x - \sigma(x)\|^{p+q}}{2 - 2^{p+q}}\ for\ all\ x \in G.$$

Corollary 2.4. [6] *Let G be an abelian group, let $(B, \|\ .\ \|)$ be a Banach space, and let φ: $G \times G \longrightarrow [0, \infty)$ be a mapping such that*

$$\psi(x, y) = \sum_{n=0}^{\infty} 2^{-n}\varphi(2^n x, 2^n y) < \infty \tag{2.17}$$

for all $x, y \in G$.
Assume that the map f: $G \longrightarrow B$ satisfies the inequality

$$\| f(x + y) + f(x + \sigma(y)) - 2f(x) \| \leq \varphi(x, y) \tag{2.18}$$

for all $x, y \in G$. Then, there exists a unique mapping J: $G \longrightarrow B$, solution of equation (2.15), such that $J(e) = f(e)$ and

$$\| f(x) - J(x) \| \leq \frac{1}{2}\varphi(x, -x) + \frac{1}{4}\psi(x - \sigma(x), \sigma(x) - x) \tag{2.19}$$

for all $x \in G$.

3. HYERS-ULAM-RASSIAS STABILITY OF THE GENERALIZED PEXIDER FUNCTIONAL EQUATION IN AMENABLE GROUPS

In this section, we obtain the Hyers-Ulam stability of the following functional equations

$$\sum_{k \in K} f(xk \cdot y) = |K|f(x) + |K|f(y), \quad x, y \in G, \tag{3.1}$$

$$\sum_{k \in K} f(xk \cdot y) = |K|f(x) \quad x, y \in G, \tag{3.2}$$

and

$$\sum_{k \in K} f(xk \cdot y) = |K|g(x) + |K|h(y), \quad x, y \in G, \tag{3.3}$$

where G is an amenable topological group and K is a finite subgroup of $Aut(G)$ not necessarily commutative.
We recall that a semigroup G is said to be amenable if there exists an invariant mean on the space of the bounded complex functions defined on G. We refer to [12] for the definition and properties of invariant means.

Theorem 3.1. *Let G be an amenable group, let $\delta > 0$ and let f: $G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$|\Sigma_{k \in K} f(xk \cdot y) - |K|f(x) - |K|f(y)| \leq \delta \ \text{ for all } x, y \in G. \tag{3.4}$$

Then, there exists a K-quadratical mapping $q : G \longrightarrow \mathbb{C}$ such that

$$|f(x) - q(x)| \leq \frac{\delta}{|K|} \tag{3.5}$$

Proof. Assume that f satisfies the inequality (3.4). Hence, for any fixed $y \in G$ the function

$$x \longmapsto \sum_{k \in K} f(xk \cdot y) - |K|f(x)$$

is bounded. Since G is amenable, there exists a left invariant mean m_x on the space of bounded, complex-functions on G. By using m_x we define the following function on G

$$\psi(y) = m_x\{\sum_{k \in K} f_{k \cdot y} - |K|f\} \tag{3.6}$$

for all $y \in G$, where $f_y(z) = f(zy)$, $z \in G$.
Now, in view of (3.6) we get

$$\sum_{k' \in K} \psi(yk' \cdot z) = \sum_{k' \in K} m_x\{\sum_{k \in K} f_{k \cdot y(kk') \cdot z} - |K|f\} = m_x\{\sum_{k' \in K}\sum_{k \in K} f_{k \cdot y(kk') \cdot z} - |K|^2 f\} \tag{3.7}$$

$$= m_x\{\sum_{k \in K}(\sum_{k' \in K}(f_{k' \cdot z})_{k \cdot y} - |K| f_{k \cdot y})\} + |K| m_x\{\sum_{k \in K} f_{k \cdot y} - |K|f\}$$

$$= \sum_{k \in K} m_x\{(\sum_{k' \in K}(f_{k' \cdot z}) - |K|f)_{k \cdot y}\} + |K|\|psi(y)$$

$$= |K|\psi(z) + |K|\psi(y).$$

Consequently the function $\frac{\psi}{|K|}$ is a K-quadratical function and we have

$$|\frac{\psi(y)}{|K|} - f(y)| = \frac{1}{|K|}|\psi(y) - |K|f(y)| \tag{3.8}$$

$$= \frac{1}{|K|}|m_x\{\sum_{k \in K} f_{k \cdot y} - |K|f - |K|f(y)\}|$$

$$\leq \frac{1}{|K|} Sup_{x \in G}|\sum_{k \in K} f(xk \cdot y) - |K|f(x) - |K|f(y)| \leq \frac{\delta}{|K|}.$$

This completes the proof. ∎

Theorem 3.2. *Let G be an amenable group, let $\delta > 0$ and let $f\colon G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$|\Sigma_{k \in K} f(xk \cdot y) - |K|f(x)| \leq \delta \text{ for all } x, y \in G. \tag{3.9}$$

Then, there exists a function $J : G \longrightarrow \mathbb{C}$ solution of equation (3.2) such that

$$|f(x) - J(x)| \leq \frac{\delta}{|K|} \tag{3.10}$$

for all $x \in G$.

Proof. Let $f\colon G \longrightarrow \mathbb{C}$ be a solution of (3.9), then we get

$$|\Sigma_{k \in K} f(yk^{-1} \cdot x) - |K|f(y)| < \delta$$

and consequently

$$|\Sigma_{k \in K} k \cdot f(k \cdot yx) - |K|f(y)| < \delta,$$

where $k \cdot f(x) = f(k^{-1} \cdot x)$, for $x \in G$. Now we consider the function defined by

$$\phi(y) = m_x\{\sum_{k \in K}\{_{k \cdot y}(k \cdot f)\}\}.$$

$$\sum_{k' \in K} \phi(yk' \cdot z) = \sum_{k' \in K} m_x\{\sum_{k \in K}\{_{k \cdot (yk' \cdot z)}(k \cdot f)\}\}$$

$$= m_x\{\sum_{k' \in K}\sum_{k \in K}\{_{k \cdot y(kk') \cdot z}(k \cdot f)\}\}$$

$$= m_x\{\sum_{k' \in K}\sum_{k \in K}\{_{k \cdot yk' \cdot z}(k \cdot f)\}\}$$

$$= \sum_{k' \in K} m_x\{_{k' \cdot z}(\sum_{k \in K}\{_{k \cdot y}(k \cdot f)\})\}$$

$$= \sum_{k' \in K} m_x\{(\sum_{k \in K}\{_{k \cdot y}(k \cdot f)\})\}$$

$$= |K|\phi(y).$$

Now, if we take $J = \frac{\phi}{|K|}$, we get the rest of the proof. ∎

The next two corollaries extends the results obtained in [1] and [6], for the particular case where G is supposed to be an abelian group and K a commutative subgroup of $Aut(G)$.

Corollary 3.3. *Let G be an abelian group, let $\delta > 0$ and let $f\colon G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$|\Sigma_{k\in K} f(x + k\cdot y) - |K|f(x) - |K|f(y)| \le \delta \text{ for all } x, y \in G. \tag{3.11}$$

Then, there exists a K-quadratical mapping $q : G \longrightarrow \mathbb{C}$ such that

$$|f(x) - q(x)| \le \frac{\delta}{|K|} \tag{3.12}$$

Corollary 3.4. *Let G be an abelian group, let $\delta > 0$ and let $f\colon G \longrightarrow \mathbb{C}$ satisfies the inequality*

$$|\Sigma_{k\in K} f(x + k\cdot y) - |K|f(x)| \le \delta \text{ for all } x, y \in G. \tag{3.13}$$

Then, there exists a function $J : G \longrightarrow \mathbb{C}$ solution of (3.2) such that

$$|f(x) - J(x)| \le \frac{\delta}{|K|} \tag{3.14}$$

for all $x \in G$.

Corollary 3.5. *Let G be an amenable group, let $\delta > 0$ and let $g, h : G \longrightarrow \mathbb{C}$ satisfy the inequality*

$$|\Sigma_{k\in K} g(xk\cdot y) - |K|g(x) - |K|h(y)| \le \delta,\ x, y \in G. \tag{3.15}$$

Then, there exists a K-quadratical function $q : G \longrightarrow \mathbb{C}$, there exists a function $J : G \longrightarrow \mathbb{C}$ solution of equation (3.2) such that

$$|h(x) - q(x)| \le \frac{3\delta}{|K|} \tag{3.16}$$

and

$$|g(x) - q(x) - J(x)| \le \frac{7\delta}{|K|} \tag{3.17}$$

for all $x \in G$.

Proof. By using the following computations,

$$\begin{aligned}
&|K|^2h(x) + |K|^2h(y) - |K|\Sigma_{k\in K} h(xk\cdot y)| \\
&= |[|K|^2g(z) + |K|^2h(x) - |K|\Sigma_{k\in K} g(zk\cdot x)] \\
&+[|K|^2h(y) + |K|\Sigma_{k\in K} g(zk\cdot x) - \Sigma_{k\in K}\Sigma_{t\in K} g(zt\cdot(xk\cdot y))] \\
&+[\Sigma_{k\in K}\Sigma_{t\in K} g(zt\cdot(xk\cdot y) - |K|^2g(z) - |K|\Sigma_{k\in K} h(xk\cdot y))]| \\
&= |[|K|^2g(z) + |K|^2h(x) - |K|\Sigma_{k\in K} g(zk\cdot x)] \\
&+[|K|^2h(y) + |K|\Sigma_{k\in K} g(zk\cdot x) - \Sigma_{k\in K}\Sigma_{t\in K} g((zk\cdot x)t\cdot y))] \\
&+[\Sigma_{k\in K}\Sigma_{t\in K} g(zt\cdot(xk\cdot y) - |K|^2g(z) - |K|\Sigma_{k\in K} h(xk\cdot y))]| \\
&\le |K||\Sigma_{k\in K} g(zk\cdot x) - |K|g(z) - |K|h(x)| \\
&+\Sigma_{k\in K}|\Sigma_{t\in K} g((zk\cdot x)t\cdot y)) - |K|g(zk\cdot x) - |K|h(y)| \\
&+\Sigma_{k\in K}|\Sigma_{t\in K} g(zt\cdot(xk\cdot y) - |K|g(z) - |K|h(xk\cdot y))| \\
&\le 3|K|\delta,
\end{aligned}$$

we obtain

$$|\Sigma_{k\in K}h(xk\cdot y)-|K|h(x)-|K|h(y)|\leq 3\delta. \tag{3.18}$$

Now, in view of Theorem 3.1, there exists a function q: $G\longrightarrow\mathbb{C}$ which is solution of the K-quadratical functional equation (3.1) which satisfies the inequality $|h(x)-q(x)\leq\frac{3\delta}{|K|}$ for all $x\in G$.
By using the inequalities (3.18)-(3.15) the new function $f=g-h$ satisfies the following inequality

$$\begin{aligned}&|\Sigma_{k\in K}f(xk\cdot y)-|K|f(x)|\\ \leq\ &|\Sigma_{k\in K}g(xk\cdot y)-|K|g(x)-|K|h(y)|+|\Sigma_{k\in K}h(xk\cdot y)-|K|h(x)-|K|h(y)|\\ \leq\ &4\delta\ \text{ for all }\ x,y\in G.\end{aligned}$$

So, by Theorem 3.2, there exists a function J: $G\longrightarrow\mathbb{C}$ solution of the functional equation (3.2) such that $|g(x)-q(x)-J(x)|\leq\frac{7\delta}{|K|}$, for all $x\in G$. This completes the proof. ∎

Corollary 3.6. *Let G be an amenable group, let $\delta>0$ and let $f,g,h:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$|\Sigma_{k\in K}f(xk\cdot y)-|K|g(x)-|K|h(y)|\leq\delta\ \textit{ for all } x,y\in G. \tag{3.19}$$

Then, there exists a K-quadratical mapping $q:G\longrightarrow\mathbb{C}$, there exists a function $J:G\longrightarrow\mathbb{C}$ solution of equation (3.2) such that

$$|f(x)-q(x)-J(x)|\leq\frac{14\delta}{|K|} \tag{3.20}$$

$$|g(x)-q(x)-J(x)+h(e)|\leq\frac{15\delta}{|K|} \tag{3.21}$$

and

$$|h(x)-q(x)-h(e)|\leq\frac{6\delta}{|K|} \tag{3.22}$$

for all $x\in G$.

Proof. Setting $y=e$ in (3.19), yields

$$||K|f(x)-|K|g(x)-|K|h(e)|\leq\delta, \tag{3.23}$$

so inequality (3.19) implies that

$$|\Sigma_{k\in K}f(xk\cdot y)-|K|f(x)-|K|(h(y)-h(e))|\leq 2\delta\ \text{ for all } x,y\in G. \tag{3.24}$$

Now, from Corollary 3.5, we get the rest of the proof. ∎

Corollary 3.7. *Let $K=\{I,\sigma\}$, let G be an amenable group, let $\delta>0$ and let $f,g,h:G\longrightarrow\mathbb{C}$ satisfy the inequality*

$$|f(xy)+f(x\sigma(y))-2g(x)-2h(y)|\leq\delta\ \textit{ for all } x,y\in G. \tag{3.25}$$

Then, there exists a quadratical mapping $q:G\longrightarrow\mathbb{C}$,

$$q(xy)+q(x\sigma(y))=2q(x)+2q(y),\ \ x,y\in G \tag{3.26}$$

there exists a function $J:G\longrightarrow\mathbb{C}$ solution of Jensen functional equation

$$J(xy)+J(x\sigma(y))=2J(x),\ \ x,y\in G \tag{3.27}$$

such that

$$|f(x)-q(x)-J(x)|\leq 7\delta \tag{3.28}$$

$$|g(x) - q(x) - J(x) + h(e)| \leq \frac{15\delta}{2} \tag{3.29}$$

and

$$|h(x) - q(x) - h(e)| \leq 3\delta \tag{3.30}$$

for all $x \in G$.

Corollary 3.8. *Let* G *be an abelian group, let* $\delta > 0$ *and let* $f, g, h : G \longrightarrow \mathbb{C}$ *satisfy the inequality*

$$|f(x+y) + f(x-y) - 2g(x) - 2h(y)| \leq \delta \text{ for all } x, y \in G. \tag{3.31}$$

Then, there exists a unique quadratical mapping, there exists a unique function J *solution of Jensen functional equation such that*

$$|f(x) - q(x) - J(x)| \leq 7\delta \tag{3.32}$$

$$|g(x) - q(x) - J(x) + h(e)| \leq \frac{15\delta}{2} \tag{3.33}$$

and

$$|h(x) - q(x) - h(e)| \leq 3\delta \tag{3.34}$$

for all $x \in G$.

Corollary 3.9. *Let* G *be an amenable group, let* $\delta > 0$ *and let* $f, g, h : G \longrightarrow \mathbb{C}$ *satisfy the inequality*

$$|f(xy) - g(x) - h(y)| \leq \delta \text{ for all } x, y \in G. \tag{3.35}$$

Then, there exists $\lambda \in \mathbb{C}$*, there exits an additive function* a*, such that*

$$|f(x) - a(x) - \lambda| \leq 14\delta \tag{3.36}$$

$$|g(x) - a(x) - \lambda + h(e)| \leq 15\delta \tag{3.37}$$

and

$$|h(x) - a(x) - h(e)| \leq 6\delta \tag{3.38}$$

for all $x \in G$.

4. THE STABILITY OF K-SPHERICAL MATRIX FUNCTIONAL EQUATION

In this section we investigate the Hyers-Ulam stability of the K-spherical matrix functional equation

$$\int_K \phi(x + k \cdot y)dk = \phi(x)\phi(y), \quad x, y \in G, \tag{4.1}$$

where K is a compact subgroup of automorphisms of G.
In addition to the terminology introduced in the introduction, we shall need the following notations:
The norm in $\mathrm{M}_n(\mathbb{C})$ is given by $\| (a_{i,j})_{1\leq i,j\leq n} \| = max_{1\leq i,j\leq n} \mid a_{i,j} \mid$. We set $U = \{f(x), x \in G\}$ and $V = \{\int_K f(x + k \cdot y)dk - f(x)f(y)\ x, y \in G\}$.

In the following we prove some lemmas that we need later

Lemma 4.1. *Assume that the mapping $f: G \longrightarrow M_n(\mathbb{C})$ satisfies the inequality*

$$\| \int_K f(x+k\cdot y)dk - f(x)f(y)\| \leq \delta \tag{4.2}$$

for all $x, y \in G$ and for some $\delta > 0$.
Let $v \in V$, then there exists $\lambda \geq 0$ such that $\|uv\| \leq \lambda$ and $\|vu\| \leq \lambda$ for all $u \in U$.

Proof. Let $v = \int_K f(x+k\cdot y)dk - f(x)f(y)$ and $u = f(z)$ for $x, y, z \in G$. By using the triangle inequality, we get

$$\begin{aligned} &\|f(z)[\int_K f(x+k\cdot y)dk - f(x)f(y)]\| \\ &\leq \| \int_K\int_K f(z+k'\cdot(x+k\cdot y))dkdk' - f(z)\int_K f(x+k\cdot y)dk\| \\ &+\| \int_K\int_K f(z+k'\cdot(x+k\cdot y))dkdk' - \int_K f(z+k'\cdot x)dk'f(y)\| \\ &+\| \int_K f(z+k\cdot x))dkf(y) - f(z)f(x)f(y)\|. \end{aligned}$$

According to (4.2), we obtain

$$\begin{aligned} &\| \int_K\int_K f(z+k'\cdot(x+k\cdot y))dkdk' - f(z)\int_K f(x+k\cdot y)dk\| \\ &= \| \int_K [\int_K f(z+k'\cdot(x+k\cdot y))dk' - f(z)f(xk\cdot y)]dk\| \leq \delta \end{aligned}$$

and

$$\| \int_K f(z+k\cdot x)dkf(y) - f(z)f(x)f(y)\| = \|[\int_K f(z+k\cdot x)dk - f(z)f(x)]f(y)\| \leq \delta\|f(y)\|.$$

By using the invariance of the normalized Haar measure dk, we deduce that

$$\begin{aligned} &\| \int_K\int_K f(z+k'\cdot(x+k\cdot y))dkdk' - \int_K f(z+k'\cdot x)dk'f(y)\| \\ &= \| \int_K [\int_K f(z+k'\cdot(x+k\cdot y))dk - f(z+k'\cdot x)f(y)]dk'\| \\ &= \| \int_K [\int_K f(z+k'\cdot x+k'k\cdot y))dk - f(z+k'\cdot x)f(y)]dk'\| \\ &= \| \int_K [\int_K f(z+k'\cdot x+k\cdot y))dk - f(z+k'\cdot x)f(y)]dk'\| \leq \delta. \end{aligned}$$

Hence

$$\|f(z)[\int_K f(x+k\cdot y)dk - f(x)f(y)]\| \leq \delta(2+\|f(y)\|),$$

which proves the desired result. ∎

The following lemma extends the one obtained in [24].

Lemma 4.2. *Let $f : G \longrightarrow \mathrm{M}_n(\mathbb{C})$ be a solution of the inequality (4.2) and α an inner automorphism of $\mathrm{M}_n(\mathbb{C})$. Then the function g given by $g = \alpha \circ f$ satisfies the inequality (4.2). Moreover, f can be approximated by a solution of the functional equation*

$$[\int_K f(x+k\cdot y)dk - f(x)f(y)]^2 = 0, \quad x, y \in G \tag{4.3}$$

if and only if g is approximated by a solution of (4.3).

Now, by using the above lemmas, the superstability of the K-spherical mapping [2] and by following the Lawrence's proof [24], we get the following theorem.

Theorem 4.3. *Let* $f : G \longrightarrow \mathrm{M}_n(\mathbb{C})$ *satisfying (4.2). Then there exits* $\psi : G \longrightarrow \mathrm{M}_n(\mathbb{C})$, *and* $\gamma \geq 0$ *such that*

$$\|f(x) - \psi(x)\| \leq \gamma \tag{4.4}$$

and

$$[\int_K \psi(x + k \cdot y)dk - \psi(x)\psi(y)]^2 = 0 \quad \textit{for all} \;\; x, y \in G. \tag{4.5}$$

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