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HYPERBOLIC BARYCENTRIC COORDINATES

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Special Issue in Honor of the 100th Anniversary of S.M. Ulam

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ABSTRACT. A powerful and novel way to study Einstein's special theory of relativity and its underlying geometry, the hyperbolic geometry of Bolyai and Lobachevsky, by analogies with classical mechanics and its underlying Euclidean geometry is demonstrated. The demonstration sets the stage for the extension of the notion of barycentric coordinates in Euclidean geometry, first conceived by Möbius in 1827, into hyperbolic geometry. As an example for the application of hyperbolic barycentric coordinates, the hyperbolic midpoint of any hyperbolic segment, and the centroid and orthocenter of any hyperbolic triangle are determined.

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1. INTRODUCTION

It was discovered in 1988 [34] that the Einstein velocity addition law encodes a grouplike algebraic object that became known as a *gyrocommutative gyrogroup*. Furthermore, it was later discovered that Einstein addition admits hyperbolic scalar multiplication, giving rise to the so called *gyrovector spaces*. The latter turned out to form a nonassociative algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry just as vector spaces form an associative algebraic setting for the standard model of Euclidean geometry [40]. The discovery of novel analogies that relativistic mechanics and its underlying hyperbolic geometry share with classical mechanics and its underlying Euclidean geometry has culminated in a series of four books [40, 44, 48, 52]. Thus, a powerful way to study Einstein's special theory of relativity and its underlying hyperbolic geometry, in which analogies with classical results form useful tools, emerged. Accordingly, mathematical analogies play an important role in this article.

Stanislaw Ulam (1909 – 1984) was interested in the mathematical structure of Einstein's special theory of relativity and its generalizations, as evidenced from [7]. His special predilection towards the study of complex mathematical notions by analogy is well known from his 1986 paper *"On the notion of analogy and complexity in some constructive mathematical schemata"* [32], and from his 1990 book on *"Analogies between analogies"* [33]. Ulam was, accordingly, one of the mathematical heroes of a new generation of mathematicians, as Themistocles M. Rassias testified in a recent interview [6].

Sharing with Ulam the special predilection towards the study of complex mathematical notions by analogy, we dedicate this article to the memory of Stanislaw M. Ulam (1909 – 1984) and his love to mathematical analogies on the occasion of his 100th Anniversary.

The aim of this article is to introduce and apply hyperbolic barycentric coordinates, which are fully analogous to the Euclidean barycentric coordinates that were first conceived by Möbius in 1827 [25, p. 71].

As evidenced from [40, 44, 48], the hyperbolic geometry of Bolyai and Lobachevsky is becoming increasingly important owing to the role it plays in Einstein's special theory of relativity. A brief historical background of hyperbolic geometry is found, for instance, in [19, Chap. I]. The first part of this paper is an expository, setting the stage for the presentation of hyperbolic barycentric coordinates and their applications in the second part.

Accordingly, we motivate and present the definition of gyrogroups and gyrovector spaces, which generalize the notion of groups and vector spaces. It turns out that gyrogroups and gyrovector spaces lay a fruitful bridge between nonassociative algebra and hyperbolic geometry, just as groups and vector spaces lay a fruitful bridge between associative algebra and Euclidean geometry. Basically, gyrogroups and gyrovector spaces are groups and vector spaces in which associativity and commutativity are replaced by gyroassociativity and gyrocommutativity. Indeed, in this way we employ our analogies that hyperbolic geometry and relativistic mechanics share with Euclidean geometry and classical mechanics. Remarkably, these analogies stem from a single common mechanism represented by the prefix "gyro". Indeed, in order to elaborate the precise language we need for dealing with analytic hyperbolic geometry, which emphasizes analogies with classical notions, we extensively use the prefix "gyro", giving rise to the *gyrolanguage* that we use in this article.

In Sections 2 – 9 we set the stage for the introduction of barycentric coordinates into hyperbolic geometry, where they are called *gyrobarycentric coordinates*. We study here the hyperbolic geometry of Bolyai and Lobachevsky by its two ball models, (i) the Beltrami-Klein ball model, regulated by the Einstein velocity addition law of special relativity extended to the ball of any real inner product space, and (ii) the Poincaré ball model, regulated by a Möbius transformation of the complex open disc extended to the ball of any real inner product space. The

Einstein velocity addition law was introduced by Einstein in 1905 [3], and its importance in special relativity is well-known [47]. Möbius transformations, in turn, are important in geometry as we see, for instance in [13, 14, 15, 16] and [35, 38, 39].

In Sections 2 – 5 we present Einstein addition as a binary operation in the ball of any real inner product space, and uncover the gyrogroup and gyrovector space structures that it encodes and, finally, link it to the Beltrami-Klein ball model of hyperbolic geometry by elementary methods of differential geometry. Similarly, in Sections 6 – 8 we present Möbius addition as a binary operation in the ball of any real inner product space, and uncover the gyrogroup and gyrovector space structures that it encodes and, finally, link it to the Poincaré ball model of hyperbolic geometry by elementary differential geometry methods. In Section 9 we show that the algebraic structures that Einstein addition and Möbius addition encode are isomorphic.

In order to set the stage for the presentation of hyperbolic barycentric coordinates, in Section 10 we present the well known Euclidean barycentric coordinates. In full analogy we, then, present the Einsteinian hyperbolic barycentric coordinates in Section 11. In Sections 12 and 13 we employ the Einsteinian hyperbolic barycentric coordinates for the study of hyperbolic triangle altitudes and orthocenters. Finally, in Section 14 we determine the hyperbolic triangle orthocenter in the Poincaré ball model of hyperbolic geometry by employing the isomorphism between Einstein addition and Möbius addition studied in Section 9.

2. EINSTEIN ADDITION VS. VECTOR ADDITION

The Einstein addition $\oplus$ of relativistically admissible velocities is a binary operation in the ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$,

$$\mathbb{R}^3_c = \{\mathbf{v}\in\mathbb{R}^3 : \|\mathbf{v}\| < c\} \tag{2.1}$$

of all relativistically admissible velocities, where c is the speed of light in empty space. It takes the form

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{2.2}$$

$\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, where $\gamma_\mathbf{u}$ is the gamma factor

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}} \tag{2.3}$$

in $\mathbb{R}^3_c$, and where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{R}^3_c$ inherits from its space $\mathbb{R}^3$. Einstein velocity addition is seemingly structureless since, counterintuitively, it is neither commutative nor associative.

Einstein gyrations $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\in Aut(\mathbb{R}^3_c,\oplus)$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})) \tag{2.4}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3_c$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}^3_c,\oplus)$, as shown in [40]. We recall that a groupoid $(S,\oplus)$ is a non-empty set S with a binary operation $\oplus$, and that an automorphism of a groupoid $(S,\oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a,b\in S$. The set of all automorphisms of a groupoid $(S,\oplus)$ forms a group, denoted by $Aut(S,\oplus)$. To emphasize that the gyrations of the Einstein groupoid $(\mathbb{R}^3_c,\oplus)$ are automorphisms of the groupoid, Einstein gyrations are also called gyroautomorphisms.

A gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$, is *trivial* if $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}=\mathbf{w}$ for all $\mathbf{w}\in\mathbb{R}_c^3$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0},\mathbf{0}]$, $\mathrm{gyr}[\mathbf{v},\mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v},\ominus\mathbf{v}]$ are trivial for all $\mathbf{v}\in\mathbb{R}_c^3$.

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and associativity as we see from the gyrocommutative and gyroassociative laws in the following identities [40, 44, 48]:

$$\begin{array}{ll}
\mathbf{u}\oplus\mathbf{v}=\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) & \text{Gyrocommutative Law}\\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})=(\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} & \text{Left Gyroassociative Law}\\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w}=\mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}) & \text{Right Gyroassociative Law}\\
\mathrm{gyr}[\mathbf{u},\mathbf{v}]=\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] & \text{Left Loop Property}\\
\mathrm{gyr}[\mathbf{u},\mathbf{v}]=\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] & \text{Right Loop Property}\\
\mathrm{gyr}[\mathbf{u},\ominus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{v}]=\mathrm{gyr}[\mathbf{v},\mathbf{u}] & \text{A Nested Gyration Identity}
\end{array}$$

Einstein addition is thus regulated by its gyrations so that Einstein addition and its gyrations are inextricably linked. Indeed, the Einstein groupoid $(\mathbb{R}_c^3,\oplus)$ forms a grouplike algebraic object called a *gyrocommutative gyrogroup* [36], which was discovered in 1988 [34] as the algebraic grouplike object that Einstein addition encodes. Interestingly, Einstein gyrations are the mathematical abstraction of the relativistic effect known as *Thomas precession* [48, Sec. 10.3].

Einstein addition (2.2) of relativistically admissible velocities was introduced by Einstein in his 1905 paper [3] [4, p. 141]. The Euclidean three-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. In [3], Einstein calculated the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2.2).

In the Newtonian limit, $c\to\infty$, the ball $\mathbb{R}_c^3$ of all relativistically admissible velocities expands to the whole of its space $\mathbb{R}^3$, as we see from (2.1), and Einstein addition $\oplus$ in $\mathbb{R}_c^3$ reduces to ordinary vector addition $+$ in $\mathbb{R}^3$, as we see from (2.2) and (2.3).

Suggestively, we extend Einstein addition of relativistically admissible velocities by abstraction in the following definition of Einstein addition in the ball of any real inner product space.

Definition 2.1 (Einstein Addition in the Ball)**.** *Let* $\mathbb{V}=(\mathbb{V},+,\cdot)$ *be a real inner product space* [23] *and let* $\mathbb{V}_s$ *be the* s*-ball of* $\mathbb{V}$,

$$\mathbb{V}_s=\{\mathbf{v}\in\mathbb{V}:\|\mathbf{v}\|<s\},\tag{2.5}$$

where $s>0$ *is an arbitrarily fixed constant. Einstein addition* $\oplus_E$ *is a binary operation in* $\mathbb{V}_s$ *given by the equation*

$$\mathbf{u}\oplus_E\mathbf{v}=\frac{1}{1+\dfrac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u}+\frac{1}{\gamma_\mathbf{u}}\mathbf{v}+\frac{1}{s^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\},\tag{2.6}$$

where $\gamma_\mathbf{u}$ *is the gamma factor*

$$\gamma_\mathbf{u}=\frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{s^2}}}\tag{2.7}$$

in $\mathbb{V}_s$, *and where* $\cdot$ *and* $\|\cdot\|$ *are the inner product and norm that the ball* $\mathbb{V}_s$ *inherits from its space* $\mathbb{V}$.

We naturally use the abbreviation $\mathbf{u}\ominus_\mathrm{E}\mathbf{v}=\mathbf{u}\oplus_\mathrm{E}(-\mathbf{v})$ for Einstein subtraction. Thus, for instance, $\mathbf{v}\ominus_\mathrm{E}\mathbf{v}=\mathbf{0}$, $\ominus_\mathrm{E}\mathbf{v}=\mathbf{0}\ominus_\mathrm{E}\mathbf{v}=-\mathbf{v}$ and, in particular,

$$\ominus_\mathrm{E}(\mathbf{u}\oplus_\mathrm{E}\mathbf{v})=\ominus_\mathrm{E}\mathbf{u}\ominus_\mathrm{E}\mathbf{v}\tag{2.8}$$

and

$$\ominus_{\mathrm{E}}\mathbf{u}\oplus_{\mathrm{E}}(\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}) = \mathbf{v} \tag{2.9}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball, in full analogy with vector addition and subtraction. Identity (2.8) is known as the *automorphic inverse property*, and Identity (2.9) is known as the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the immediate right counterpart of the left cancellation law (2.9) since, in general,

$$(\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v})\ominus_{\mathrm{E}}\mathbf{v} \neq \mathbf{u}. \tag{2.10}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is repaired in (3.7) below, and in [44, Table 2.1, p. 33].

The gamma factor is related to Einstein addition by the identity

$$\gamma_{\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right) \tag{2.11}$$

that historically provided the first link between Einstein's special theory of relativity and the hyperbolic geometry of Bolyai and Lobachevsky, as described in [47].

The gamma factor possesses useful identities as, for instance,

$$\gamma_{2\otimes\mathbf{a}} = 2\gamma_{\mathbf{a}}^2 - 1 \tag{2.12}$$

[48, Eq. 6.304] and

$$\gamma_{2\otimes\mathbf{a}}(2\otimes\mathbf{a}) = 2\gamma_{\mathbf{a}}^2\mathbf{a} \tag{2.13}$$

[48, Eq. 6.301], where $2\otimes\mathbf{a} = \mathbf{a}\oplus\mathbf{a}$, and where $\oplus$ represents one of Einstein addition, $\oplus = \oplus_{\mathrm{E}}$, and Möbius addition, $\oplus = \oplus_{\mathrm{M}}$. Möbius addition will be studied in Sec. 6, and its relationship with Einstein addition will be studied in Sec. 9.

When the vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{V}_s$ of $\mathbb{V}$ are parallel in $\mathbb{V}$, $\mathbf{u}\|\mathbf{v}$, the Einstein gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is trivial, and Einstein addition $\oplus_{\mathrm{E}}$ reduces to the binary operation between parallel velocities,

$$\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}, \qquad \mathbf{u}\|\mathbf{v} \tag{2.14}$$

which is both commutative and associative. In general, however, owing to the presence of nontrivial gyrations, Einstein addition is neither commutative nor associative. The non-associativity of general Einstein velocity addition (2.2) is hardly known in the literature [46]. Among outstanding exceptions is [31].

In the Newtonian limit of large s, $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole of its space $\mathbb{V}$, and Einstein addition, $\oplus_{\mathrm{E}}$, reduces to common vector addition, $+$, in $\mathbb{V}$, as we see from (2.5) – (2.7).

3. FROM EINSTEIN VELOCITY ADDITION TO GYROGROUPS

Taking the key features of the Einstein groupoid $(\mathbb{V}_s, \oplus_{\mathrm{E}})$ as axioms, and guided by analogies with groups, we are led to the following formal gyrogroup definition.

Definition 3.1 (Gyrogroups). *A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $\quad 0\oplus a = a$

for all $a\in G$. There is an element $0\in G$ satisfying axiom $(G1)$ such that for each $a\in G$ there is an element $\ominus a\in G$, called a left inverse of a, satisfying

(G2) $\quad \ominus a\oplus a = 0\,.$

Moreover, for any $a, b, c\in G$ there exists a unique element $\mathrm{gyr}[a, b]c\in G$ such that the binary operation obeys the left gyroassociative law

(G3) $a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a, b]c$.

The map $\mathrm{gyr}[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is,*

(G4) $\mathrm{gyr}[a, b] \in Aut(G, \oplus)$,

and the automorphism $\mathrm{gyr}[a, b]$ *of* G *is called the gyroautomorphism, or the gyration, of* G *generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to Aut(G, \oplus)$ *is called the gyrator of* G*. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *possesses the left loop property*

(G5) $\mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]$.

The gyrogroup axioms $(G1)-(G5)$ in Definition 3.1 are classified into three classes:

(1) The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.
(2) The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
(3) The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a\ominus b = a\oplus(\ominus b)$ for gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into non-gyrocommutative and gyrocommutative gyrogroups.

Definition 3.2 (Gyrocommutative Gyrogroups)**.** *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*

(G6) $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

for all $a, b\in G$*.*

Clearly, a (commutative) group is a degenerate (gyrocommutative) gyrogroup whose gyroautomorphisms are all trivial. The algebraic structure of gyrogroups is, accordingly, richer than that of groups. Thus, without losing the flavor of the group structure we have generalized it into the gyrogroup structure to suit the needs of Einstein addition in the ball. Fortunately, the gyrogroup structure is by no means restricted to Einstein addition in the ball. Rather, it abounds in group theory as demonstrated, for instance, in [10] and [11], where finite and infinite gyrogroups, both gyrocommutative and non-gyrocommutative, are studied. Some initial gyrogroup theorems, some of which are analogous to group theorems, are presented in [44, Chap. 2] and [48, Chap. 2].

In order to capture analogies with groups, we introduce into the abstract gyrogroup $(G, \oplus)$ a second binary operation $\boxplus$ called a *cooperation*, or coaddition, which shares useful duality symmetries with its gyrogroup operation $\oplus$ [40, 44].

Definition 3.3 (The Gyrogroup Cooperation (Coaddition))**.** *Let* $(G, \oplus)$ *be a gyrogroup. The gyrogroup cooperation (or, coaddition)* $\boxplus$ *is a second binary operation in* G *related to the gyrogroup operation (or, addition)* $\oplus$ *by the equation*

$$a \boxplus b = a\oplus\mathrm{gyr}[a, \ominus b]b \tag{3.1}$$

for all $a, b\in G$*.*

Naturally, we use the notation $a \boxminus b = a \boxplus (\ominus b)$ where $\ominus b = -b$, so that

$$a \boxminus b = a\ominus\mathrm{gyr}[a, b]b \tag{3.2}$$

The gyrogroup cooperation is commutative if and only if the gyrogroup operation is gyrocommutative [44, Theorem 3.4]. Hence, in particular, Einstein coaddition $\boxplus$ is commutative since Einstein addition $\oplus$ is gyrocommutative. The commutativity of Einstein coaddition proves useful in the hyperbolic parallelogram (gyroparallelogram) law of relativistic velocities, presented in [48, Sec. 10.8], and illustrated in Figs. 7.3 –7.5 of Sec. 7.

The gyrogroup cooperation $\boxplus$ is expressed in (3.1) in terms of the gyrogroup operation $\oplus$ and the gyrator gyr. It can be shown that, similarly, the gyrogroup operation $\oplus$ can be expressed in terms of the gyrogroup cooperation $\boxplus$ and the gyrator gyr by the identity [44, Theorem 2.10],

$$a\oplus b = a \boxplus \mathrm{gyr}[a, b]b \tag{3.3}$$

for all a, b in a gyrogroup $(G, \oplus)$. Identities (3.1) and (3.3) exhibit one of the duality symmetries that the gyrogroup operation and cooperation share.

Gyrogroup theorems are introduced in [40, 44, 48]. In particular, it is found that any gyrogroup possesses a unique identity (left and right) and each element of any gyrogroup possesses a unique inverse (left and right). Similarly, the left gyroassociative law $(G3)$ and the left loop property $(G5)$ have the following right counterparts:

$$(a\oplus b)\oplus c = a\oplus(b\oplus \mathrm{gyr}[b, a]c) \tag{3.4}$$

and

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b\oplus a] \tag{3.5}$$

respectively.

Furthermore, any gyrogroup obeys the left cancellation law,

$$\ominus a\oplus(a\oplus b) = b \tag{3.6}$$

and the two right cancellation laws,

$$(b\oplus a) \boxminus a = b \tag{3.7}$$

and

$$(b \boxplus a)\ominus a = b \tag{3.8}$$

Like Identities (3.1) and (3.3), Identities (3.7) and (3.8) present a duality symmetry between the gyrogroup operation $\oplus$ and cooperation $\boxplus$.

Applying the left cancellation law (3.6) to the left gyroassociative law $(G3)$ of a gyrogroup we obtain the *gyrator identity*

$$\mathrm{gyr}[a, b]x = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus x)\} \tag{3.9}$$

It demonstrates that the gyrations of a gyrogroup are uniquely determined by the gyrogroup operation.

Furthermore, it is clear from (the gyrocommutative law and) the gyroassociative law that gyrations measure the extent to which the gyrogroup operation deviates from (both commutativity and) associativity. A (commutative) group is accordingly a (gyrocommutative) gyrogroup whose gyrations are trivial. Hence, the gyrogroup structure is richer than the group structure and, in particular, the algebra of Einstein velocity addition is richer than that of Newtonian velocity addition.

4. EINSTEIN GYROVECTOR SPACES

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Einstein addition admits scalar multiplication, giving rise to Einstein gyrovector spaces. The latter, in turn, form the setting for the Beltrami-Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard model of Euclidean geometry, as shown in [48].

Definition 4.1. An Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\rm E}, \otimes_{\rm E})$ is an Einstein gyrogroup $(\mathbb{V}_s, \oplus_{\rm E})$ with scalar multiplication $\otimes_{\rm E}$ given by the equation

$$r\otimes_{\rm E}\mathbf{v} = s\frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} = s\tanh\left(r\,\tanh^{-1}\frac{\|\mathbf{v}\|}{s}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \tag{4.1}$$

where r is any real number, $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{V}_s$, $\mathbf{v}\neq\mathbf{0}$, and $r\otimes_{\rm E}\mathbf{0}=\mathbf{0}$, with which we use the notation $\mathbf{v}\otimes_{\rm E} r = r\otimes_{\rm E}\mathbf{v}$.

Einstein gyrovector spaces are studied in [48, Sec. 6.18]. Einstein scalar multiplication does not distribute with Einstein addition, but it possesses other properties of vector spaces. For any positive integer n, and for all real numbers $r, r_1, r_2\in\mathbb{R}$ and $\mathbf{v}\in\mathbb{V}_s$, we have

$$\begin{aligned} n\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && n \text{ terms}\\ (r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law}\\ (r_1r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law}\end{aligned}$$

in any Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$.

Any Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ inherits an inner product and a norm from its vector space $\mathbb{V}$. These turn out to be invariant under gyrations, that is,

$$\begin{aligned} \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{u}\cdot\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} &= \mathbf{u}\cdot\mathbf{v},\\ \|\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}\| &= \|\mathbf{v}\|, \end{aligned} \tag{4.2}$$

for all $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}\in\mathbb{V}_s$.

Unlike vector spaces, Einstein gyrovector spaces $(\mathbb{V}_s, \oplus, \otimes)$ do not possess the distributive law since, in general,

$$r\otimes(\mathbf{u}\oplus\mathbf{v}) \neq r\otimes\mathbf{u}\oplus r\otimes\mathbf{v} \tag{4.3}$$

for $r\in\mathbb{R}$ and $\mathbf{u},\mathbf{v}\in\mathbb{V}_s$. One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein addition, but this is not the case as demonstrated in [40, 44, 48] and in this article, and as noted by S. Walter in [56].

5. Linking Einstein Addition to Hyperbolic Geometry

In this section we present the link between Einstein addition and the Beltrami-Klein ball model of hyperbolic geometry.

The Einstein distance function, $d(\mathbf{u},\mathbf{v})$, in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is given by the equation

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}\ominus\mathbf{v}\|, \tag{5.1}$$

$\mathbf{u},\mathbf{v}\in\mathbb{V}_s$, where $\oplus=\oplus_{\rm E}$ is Einstein addition, given by (2.6), so that $\ominus$ is Einstein subtraction. We call it a *gyrodistance function* in order to emphasize the analogies it shares with its Euclidean counterpart, the distance function $\|\mathbf{u}-\mathbf{v}\|$ in $\mathbb{V}$. Among these analogies is the gyrotriangle inequality according to which

$$\|\mathbf{u}\oplus\mathbf{v}\| \le \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{5.2}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{V}_s$. For this and other analogies that distance and gyrodistance functions share, see [44, 48, 52].

In a two dimensional Einstein gyrovector space $(\mathbb{R}_s^2, \oplus, \otimes)$ the squared gyrodistance between a point $\mathbf{x}\in\mathbb{R}_s^2$ and an infinitesimally nearby point $\mathbf{x}+d\mathbf{x}\in\mathbb{R}_s^2$, $\mathbf{x}=(x_1,\ x_2)$, $d\mathbf{x}=(dx_1,\ dx_2)$, is given by the equation [48, Sec. 7.5], [44, Sec. 7.5]

$$ds^2 = \|\mathbf{x}\ominus(\mathbf{x}+d\mathbf{x})\|^2 = Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2 + \cdots\ , \tag{5.3}$$

where, if we use the notation $r^2 = x_1^2 + x_2^2$, we have

$$\begin{aligned} E &= s^2\frac{s^2 - x_2^2}{(s^2-r^2)^2}, \\ F &= s^2\frac{x_1x_2}{(s^2-r^2)^2}, \\ G &= s^2\frac{s^2 - x_1^2}{(s^2-r^2)^2}. \end{aligned} \tag{5.4}$$

The triple $(g_{11}, g_{12}, g_{22}) := (E, F, G)$ along with $g_{21} = g_{12}$ is known in differential geometry as the metric tensor g_{ij} [21]. It turns out to be the metric tensor of the Beltrami-Klein disc model of hyperbolic geometry [24, p. 220]. Hence, ds^2 in (5.3) – (5.4) is the Riemannian line element of the Beltrami-Klein disc model of hyperbolic geometry, linked to Einstein velocity addition (2.2) and to Einstein gyrodistance function (5.1) [45].

The Gaussian curvature K of an Einstein gyrovector plane with the triple (E, F, G) of (5.3) – (5.4) turns out to be [24, p. 149], [48, Sec. 7.5], [44, Sec. 7.5],

$$K = -\frac{1}{s^2}. \tag{5.5}$$

The link between Einstein gyrovector spaces and the Beltrami-Klein ball model of hyperbolic geometry, already noted by Fock [9, p. 39], has thus been established in (5.1) – (5.4) in two dimensions. The extension of the link to higher dimensions is presented in [40, Sec. 9, Chap. 3], [48, Sec. 7.5], [44, Sec. 7.5] and [45]. For a brief account of the history of linking Einstein's velocity addition law with hyperbolic geometry, see [29, p. 943].

In full analogy with Euclidean geometry, the graph of the parametric expression

$$A\oplus(\ominus A\oplus B)\otimes t \tag{5.6}$$

in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, for the parameter $t\in\mathbb{R}$, where $A, B\in\mathbb{R}_s^n$, describes a geodesic line in the Beltrami-Klein ball model of hyperbolic geometry. It is a chord of the ball as shown in Fig. 5.1 for the disc. The geodesic (5.6) is the unique geodesic passing through the points A and B. It passes through the point A at "time" $t = 0$ and, owing to the left cancellation law, (3.6), it passes through the point B at "time" $t = 1$. Hence, the geodesic segment that joins the points A and B in Fig. 5.1 is obtained from (5.6) with $0 \le t \le 1$.

6. MÖBIUS ADDITION AND MÖBIUS GYROGROUPS

Definition 6.1 (Möbius Addition in the Ball)**.** Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be a real inner product space [23], and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{\mathbb{V}_s\in\mathbb{V} : \|\mathbf{v}\| < s\}, \tag{6.1}$$

where $s > 0$ is an arbitrarily fixed constant. Möbius addition $\oplus_{\rm M}$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\mathbf{u}\oplus_{\rm M}\mathbf{v} = \frac{\left(1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^2}\|\mathbf{v}\|^2\right)\mathbf{u} + \left(1 - \frac{1}{s^2}\|\mathbf{u}\|^2\right)\mathbf{v}}{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}, \tag{6.2}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

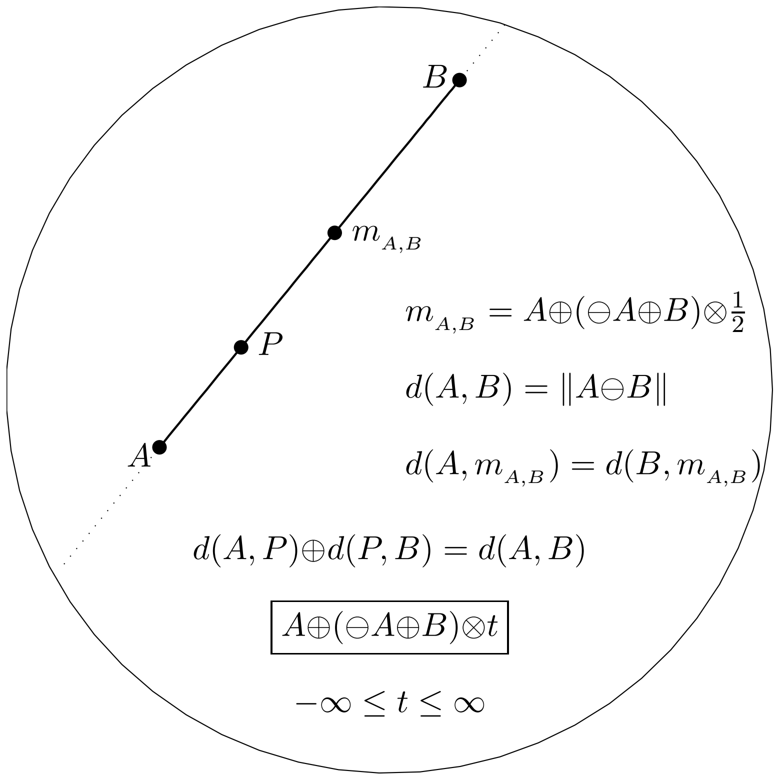

Figure 5.1. The Einstein gyroline $A\oplus(\ominus A\oplus B)\otimes t$, $t \in \mathbb{R}$, in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is a geodesic line in the Beltrami-Klein ball model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in Euclidean geometry. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the gyroline through the points A and B lying between these points. The Einstein sum, $\oplus$, of the Einstein distance (gyrodistance) from A to P and from P to B equals the Einstein distance from A to B. The point $m_{A,B}$ is the hyperbolic midpoint (gyromidpoint) of the points A and B.

Without loss of generality, one may select $s = 1$ in Definition 6.1. We, however, prefer to keep s as a free positive parameter in order to exhibit the result that in the limit as $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole of its real inner product space $\mathbb{V}$, and Möbius addition, $\oplus_{\mathrm{M}}$, reduces to vector addition, +, in $\mathbb{V}$. Like Einstein groupoids $(\mathbb{V}_s, \oplus_{\mathrm{E}})$, Möbius groupoids $(\mathbb{V}_s, \oplus_{\mathrm{M}})$ are gyrocommutative gyrogroups. Interestingly, the right hand side of (6.2) is known as a Möbius translation [28, p. 129]. Owing to the analogies it shares with vector addition we, however, call it Möbius addition. The evolution from Möbius to gyrogroups is unfolded in [49].

7. MÖBIUS GYROVECTOR SPACES

Definition 7.1 (Möbius Scalar Multiplication)**.** *A Möbius gyrovector space $(\mathbb{V}_s, \oplus_M, \otimes_M)$ is a Möbius gyrogroup $(\mathbb{V}_s, \oplus_M)$ with scalar multiplication $\otimes_M$ given by the equation*

$$r\otimes_M \mathbf{v} = s\frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} = s\tanh\left(r\ \tanh^{-1}\frac{\|\mathbf{v}\|}{s}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|}, \tag{7.1}$$

where $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{V}_s$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes_M\mathbf{0} = \mathbf{0}$.

Interestingly, Einstein and Möbius scalar multiplication, $\otimes_{\mathrm{E}}$ and $\otimes_{\mathrm{M}}$, are identical. Möbius gyrovector spaces are studied in [48, Sec. 6.14].

Any two points $A, B \in \mathbb{R}^n$ of the Euclidean n-space $\mathbb{R}^n$, $n \in \mathbb{N}$, form the vector $-A + B$. The algebraic value of this vector is $-A+B$, its length is $\|-A+B\|$, and for $n = 2, 3$ is represented graphically by a straight arrow from A to B, as shown in Fig. 7.1 for $n = 2$. Two vectors are equivalent if they have the same algebraic value, so that vectors are equivalence classes. Two equivalent vectors, $-A + B$ and $-A' + B'$ in $\mathbb{R}^2$ are shown in Fig. 7.1.

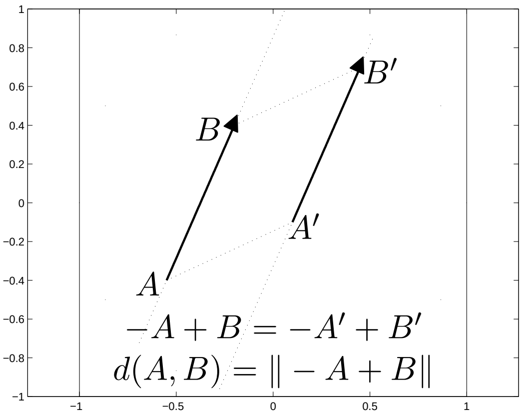

Figure 7.1. Points P of the Euclidean space $\mathbb{R}^n$ are given by their orthogonal Cartesian coordinates $P = (x_1, \ldots, x_n)$, $x_1^2 + \ldots + x_n^2 < \infty$. The vector from point A to point B in the Euclidean space $\mathbb{R}^n$ has the algebraic value $-A + B$ and length $\| - A + B\|$, and is represented graphically by a straight arrow from A to B. Two vectors are equivalent if they have the same algebraic value. Vectors are, thus, equivalence classes. Equivalent vectors have equal lengths and, moreover, they are parallel. Vectors add according to the parallelogram rule. Any point $A \in \mathbb{R}^n$ is identified with the vector $-O + A$, where O is the arbitrarily selected origin of $\mathbb{R}^n$. The Euclidity of $\mathbb{R}^n$ is determined by the Euclidean metric in which the distance between any two points $A, B \in \mathbb{R}^n$ is $d(A, B) = \| - A + B\|$. Like vectors, also gyrovectors are equivalence classes, shown in Fig. 7.2

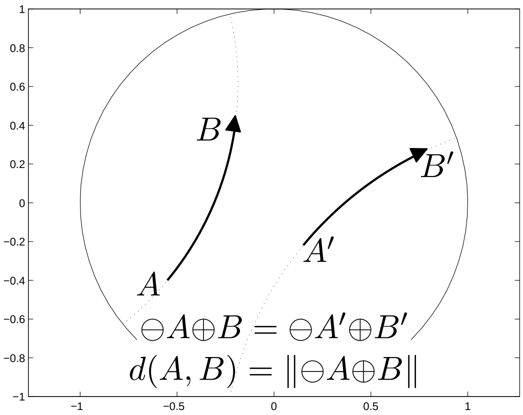

Figure 7.2. Points P of the Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ are given by their orthogonal Cartesian coordinates $P = (x_1, \ldots, x_n)$, $x_1^2 + \ldots + x_n^2 < s^2$. The gyrovector from point A to point B in the Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ has the algebraic value $\ominus A \oplus B$ and gyrolength $\|\ominus A \oplus B\|$, and is represented graphically by a curved arrow from A to B. Two gyrovectors are equivalent if they have the same algebraic value. Gyrovectors are, thus, equivalence classes. Equivalent gyrovectors have equal gyrolengths. Gyrovectors add according to the gyroparallelogram rule, shown in Figs. 7.4−7.5. Any point $A \in \mathbb{R}^n_s$ is identified with the gyrovector $\ominus O \oplus A$, where O is the arbitrarily selected origin of $\mathbb{R}^n_s$. The hyperbolicity of $\mathbb{R}^n_s$ is determined by the hyperbolic metric, gyrometric, in which the gyrodistance between any two points $A, B \in \mathbb{R}^n_s$ is $d(A, B) = \|\ominus A \oplus B\|$.

Similarly, any two points $A, B \in G$ of a gyrovector space $(G, \oplus, \otimes)$ form the gyrovector $\ominus A \oplus B$. The algebraic value of this gyrovector is $\ominus A \oplus B$, its gyrolength is $\|\ominus A \oplus B\|$, and for $G = \mathbb{R}^n_s$, $n = 2, 3$, is represented graphically by a curved arrow from A to B, as shown in Fig. 7.2 for the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$.

In full analogy with vectors, gyrovectors are equivalence classes, defined in [48, Def. 5.4, p. 133], and illustrated in Fig. 7.2, where two equivalent gyrovectors, $\ominus A \oplus B$ and $\ominus A' \oplus B'$ in a Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are shown.

The vector $-A' + B'$ is obtained in Fig. 7.1 from the vector $-A + B$ by a vector translation of the latter by some $\mathbf{t} \in \mathbb{R}^2$ according to the equations

$$\begin{aligned} A' &= \mathbf{t} + A, \\ B' &= \mathbf{t} + B. \end{aligned} \tag{7.2}$$

It follows from (7.2) that $-A' + B' = -A + B$ so that the two vectors $-A' + B'$ and $-A + B$ are equivalent. Owing to their equivalence, these two vectors are parallel and have equal lengths.

In full analogy, the gyrovector $\ominus A' \oplus B'$ is obtained in Fig. 7.2 from the gyrovector $\ominus A \oplus B$ by a gyrovector translation of the latter by some $\mathbf{t} \in \mathbb{R}^2_s$ according to the equations

$$\begin{aligned} A' &= \mathrm{gyr}[A, \mathbf{t}](\mathbf{t} \oplus A), \\ B' &= \mathrm{gyr}[A, \mathbf{t}](\mathbf{t} \oplus B), \end{aligned} \tag{7.3}$$

as we see from the gyrovector translation definition [48, Def. 5.6, p. 135]. Hence, by [48, Def. 5.6 and Theorem 5.7, p. 135], we have $\ominus A' \oplus B' = \ominus A \oplus B$ so that the two gyrovectors $\ominus A' \oplus B'$ and $\ominus A \oplus B$ are equivalent. Owing to their equivalence, these two gyrovectors have equal gyrolengths.

Thus, gyrovectors are equivalence classes just like vectors. To extend the analogies between gyrovectors and vectors, we must introduce Euclidean parallelograms into hyperbolic geometry. Indeed, in what seemingly sounds like a contradiction in terms we have introduced in [48, Def. 6.41] the Euclidean parallelogram into hyperbolic geometry where the parallel postulate is denied. In the same way that a Euclidean parallelogram is a Euclidean quadrilateral whose diagonals intersect each other at their midpoints, a hyperbolic parallelogram, called a gyroparallelogram, is defined to be a gyroquadrilateral whose gyrodiagonals intersect each other at their gyromidpoints. The resulting gyroparallelogram, shown in Fig. 7.3, shares remarkable analogies with its Euclidean counterpart, giving rise to the gyroparallelogram gyrovector addition law, shown in Figs. 7.4 and 7.5, which is fully analogous to the common parallelogram vector addition law in Euclidean geometry.

We find in [48, Sec. 8.14], and illustrate in Figs. 7.4 and 7.5, that gyrovectors add according to the gyroparallelogram rule just like vectors, which add according to the parallelogram rule.

In the years 1908 – 1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865 – 1942), professor and rector of Zagreb University, showed that this theory has a natural interpretation in hyperbolic geometry [53]. However, much to his chagrin, he had to admit in 1924 [54, p. 80] that the adaption of vector algebra for use in hyperbolic geometry was just not feasible, as Scott Walter notes in [55, p. 121]. Vladimir Varičak's hyperbolic geometry program, cited by Pauli [27, p. 74], is described by Walter in [55, p. 112–115].

Following Varičak's 1924 realization that, unlike Euclidean geometry, the hyperbolic geometry of Bolyai and Lobachevsky does not admit vectors, explorers of hyperbolic geometry could not treat the geometry vectorially till the appearance of [40, 44, 48]. Owing to the analogies that Euclidean vectors share with gyrovectors, illustrated in Figs. 7.1 and 7.2, gyrovectors became the vectors of hyperbolic geometry enabling us to develop a gyrovector space approach to hyperbolic geometry [52] that is fully analogous to the common vector space approach to Euclidean geometry. Moreover, it was found in [41, 42] and [50] that the *Bloch vector* of quantum information and computation is, in fact, a gyrovector rather than a vector.

8. LINKING MÖBIUS ADDITION TO HYPERBOLIC GEOMETRY

In this section we present the link between Möbius addition and the Poincaré ball model of hyperbolic geometry.

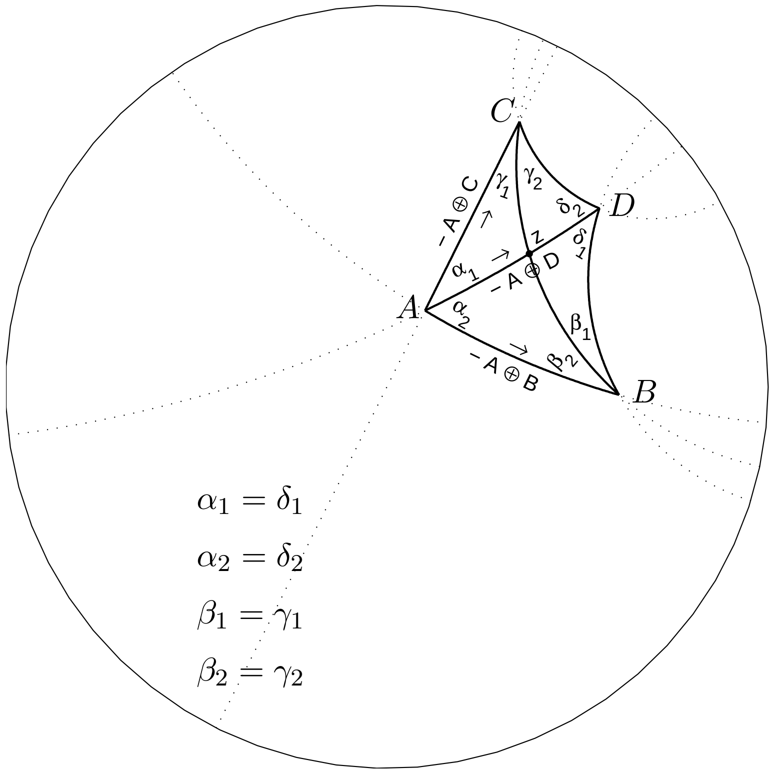

Figure 7.3. The Möbius Gyroparallelogram. Seemingly, a hyperbolic parallelogram sounds like a contradiction in terms. A Möbius gyroparallelogram in a Möbius gyrovector space is a gyroquadrilateral the two gyrodiagonals of which intersect at their gyromidpoints.

The Möbius distance function, $d(\mathbf{u},\mathbf{v})$ in a Möbius gyrovector space $(\mathbb{V}_s,\oplus,\otimes)$ is given by the equation

$$d(\mathbf{u},\mathbf{v}) = \|\mathbf{u}\ominus\mathbf{v}\|, \tag{8.1}$$

$\mathbf{u},\mathbf{v}\in\mathbb{V}_s$, where $\oplus = \oplus_{\mathrm{M}}$ is Möbius addition, given by (6.2), so that $\ominus$ is Möbius subtraction. We call it a *gyrodistance function* in order to emphasize the analogies it shares with its Euclidean counterpart, the distance function $\|\mathbf{u}-\mathbf{v}\|$ in $\mathbb{V}$. Among these analogies is the gyrotriangle inequality according to which

$$\|\mathbf{u}\oplus\mathbf{v}\| \le \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{8.2}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{V}_s$. For this and other analogies that distance and gyrodistance functions share, see [44, 48, 52].

In a two dimensional Möbius gyrovector space $(\mathbb{R}^2_s,\oplus,\otimes)$ the squared gyrodistance between a point $\mathbf{x}\in\mathbb{R}^2_s$ and an infinitesimally nearby point $\mathbf{x}+d\mathbf{x}\in\mathbb{R}^2_s$, $\mathbf{x}=(x_1,\ x_2)$, $d\mathbf{x}=(dx_1,\ dx_2)$, is given by the equation [48, Sec. 7.3], [44, Sec. 7.3]

$$ds^2 = \|\mathbf{x}\ominus(\mathbf{x}+d\mathbf{x})\|^2 = Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2 + \cdots\,, \tag{8.3}$$

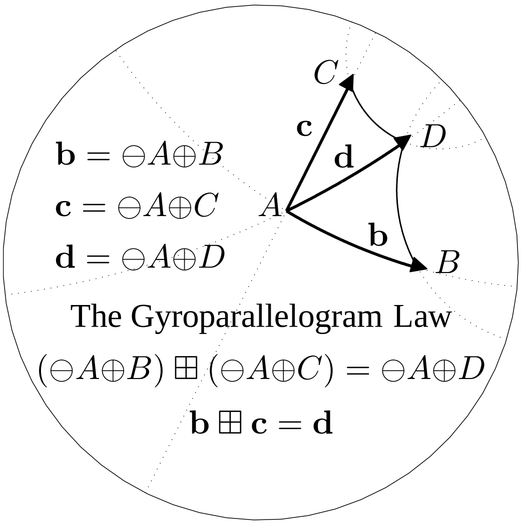

Figure 7.4. The gyroparallelogram law and the Möbius gyroparallelogram $ABDC$ in Fig. 7.3 give rise to the commutative gyroparallelogram addition law of gyrovectors, shown here as a first example. Remarkably, the gyroparallelogram addition coincides with the Möbius gyrogroup coaddition $\boxplus$, as verified in Theorem 6.43, p. 180 of [48].

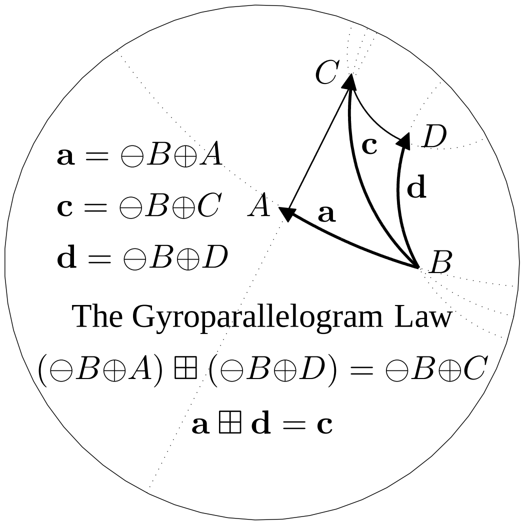

Figure 7.5. The gyroparallelogram law and the Möbius gyroparallelogram $ABDC$ in Figs. 7.3 – 7.4 give rise to the additional commutative gyroparallelogram addition of gyrovectors, which is shown here as a second example. Remarkably, as in Fig. 7.4, the gyroparallelogram addition coincides with the Möbius gyrogroup coaddition $\boxplus$.

where, if we use the notation $r^2 = x_1^2 + x_2^2$, we have

$$
\begin{aligned}
E &= \frac{s^4}{(s^2-r^2)^2}, \\
F &= 0, \\
G &= \frac{s^2}{(s^2-r^2)^2}.
\end{aligned}
\tag{8.4}
$$

The triple $(g_{11}, g_{12}, g_{22}) := (E, F, G)$ along with $g_{21} = g_{12}$ is known in differential geometry as the metric tensor g_{ij} [21]. It turns out to be the metric tensor of the Poincaré disc model of hyperbolic geometry [24, p. 226]. Hence, ds^2 in (8.3) – (8.4) is the Riemannian line element of the Poincaré disc model of hyperbolic geometry, linked to Möbius addition (6.2) and to the Möbius gyrodistance function (8.1) [45].

The Gaussian curvature K of a Möbius gyrovector plane with the triple (E, F, G) of (8.3) – (8.4) turns out to be [24, p. 149], [48, Sec. 7.5], [44, Sec. 7.5],

$$K = -\frac{4}{s^2}. \tag{8.5}$$

The link between Möbius gyrovector spaces and the Poincaré ball model of hyperbolic geometry has thus been established in (8.1) – (8.4) in two dimensions. The extension of the link to higher dimensions is presented in [48, Sec. 7.3], [44, Sec. 7.3] and [45].

In full analogy with Euclidean geometry, the graph of the parametric expression

$$A \oplus (\ominus A \oplus B) \otimes t \tag{8.6}$$

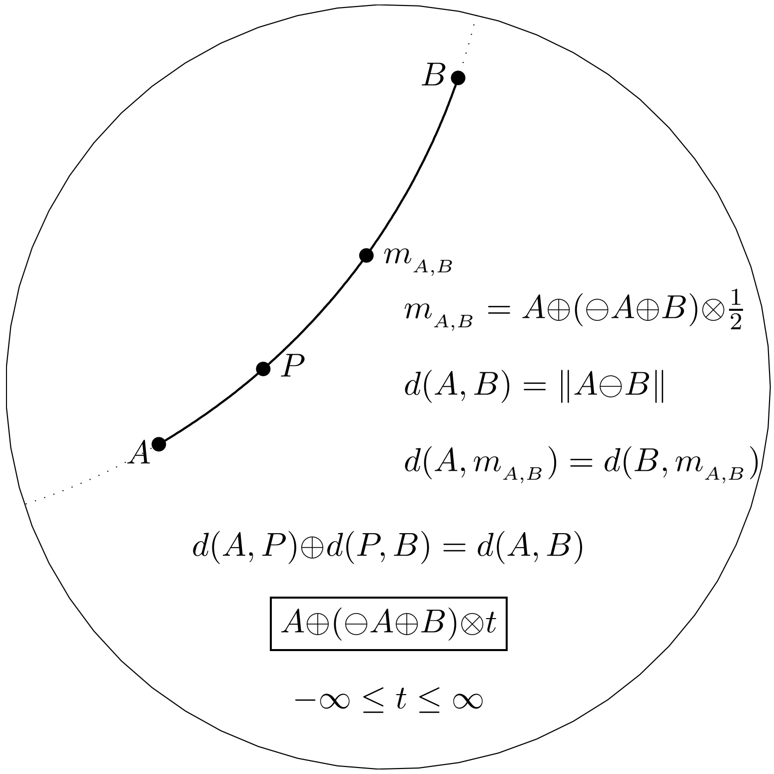

Figure 8.1. The Möbius gyroline $A\oplus(\ominus A\oplus B)\otimes t$, $t \in \mathbb{R}$, in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is a geodesic line in the Poincaré ball model of hyperbolic geometry, fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in Euclidean geometry. The points A and B correspond to $t = 0$ and $t = 1$, respectively. The point P is a generic point on the gyroline through the points A and B lying between these points. The Möbius sum, $\oplus$, of the Möbius distance (gyrodistance) from A to P and from P to B equals the Möbius distance from A to B. The point $m_{A,B}$ is the hyperbolic midpoint (gyromidpoint) of the points A and B.

in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, for the parameter $t\in\mathbb{R}$, where $A, B\in\mathbb{R}^n_s$, describes a geodesic line in the Poincaré ball model of hyperbolic geometry. It is a circular arch in the ball that intersects the boundary of the ball orthogonally, as shown in Fig. 8.1 for the disc. The geodesic (8.6) is the unique geodesic passing through the points A and B. It passes through the point A at "time" $t = 0$ and, owing to the left cancellation law, (3.6), it passes through the point B at "time" $t = 1$. Hence, the geodesic segment that joins the points A and B in Fig. 8.1 is obtained from (8.6) with $0 \le t \le 1$.

9. EINSTEIN AND MÖBIUS GYROVECTOR SPACES ARE ISOMORPHIC

Isomorphisms between gyrovector spaces are studied in [48, Sec. 6.21]. In particular, it is shown there that the isomorphism between Einstein and Möbius Gyrovector Spaces is given by each of the two identities

$$\begin{aligned} \tfrac{1}{2}\otimes_{\mathrm{E}}(\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}) &= \tfrac{1}{2}\otimes_{\mathrm{M}}\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes_{\mathrm{M}}\mathbf{v} \\ 2\otimes_{\mathrm{M}}(\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v}) &= 2\otimes_{\mathrm{E}}\mathbf{u}\oplus_{\mathrm{E}}2\otimes_{\mathrm{E}}\mathbf{v} \end{aligned} \tag{9.1}$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{V}_s$.

The operations $\otimes_{\mathrm{E}}$ and $\otimes_{\mathrm{M}}$, which represent scalar gyromultiplication in Einstein and Möbius gyrovector spaces respectively, are identical to each other, $\otimes_{\mathrm{E}} = \otimes_{\mathrm{M}} =: \otimes$. Hence, Identities

(9.1) can be written equivalently as

$$\begin{aligned}\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes\mathbf{v}),\\ \mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v})\end{aligned} \tag{9.2}$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{V}_s$. The isomorphism in (9.2) is not trivial owing to the result that scalar gyromultiplication, $\otimes$, is non-distributive, that is, it does not distribute with gyroaddition, $\oplus$.

Möbius addition in the ball is obtained from Möbius transformation of the complex open unit disc of a complex plane, as demonstrated in [20, 37, 43, 49]. The related connection between Möbius transformation of the disc and Lorentz transformation of Einstein's special theory of relativity was recognized by H. Liebmann in 1905 [26, pp. 122–123]. The isomorphism (9.2) thus generalizes this well-known connection.

10. EUCLIDEAN BARYCENTRIC COORDINATES

In order to set the stage for the study of hyperbolic barycentric coordinates that we naturally call *gyrobarycentric coordinates*, we present in this section the Euclidean barycentric coordinates [30, pp. 7,12] that were introduced by Möbius [8] in 1827.

A barycenter in astronomy is the point between two objects where they balance each other. It is the center of gravity where two or more celestial bodies orbit each other. In 1827 Möbius published a book whose title, *Der Barycentrische Calcul*, translates as *The Barycentric Calculus*. The word *barycentric* means center of gravity, but the book is entirely geometrical and, hence, called by Jeremy Gray [12], *Möbius's Geometrical Mechanics*. The 1827 Möbius book is best remembered for introducing a new system of coordinates, the *barycentric coordinates*. The historical contribution of Möbius' barycentric coordinates to vector analysis is described in [2, pp. 48–50].

The Möbius idea, for a triangle as an illustrative example, is to attach masses, $m_{\mathbf{a}}$, $m_{\mathbf{b}}$, $m_{\mathbf{c}}$, respectively, to three non-collinear points, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, in the Euclidean plane $\mathbb{R}^2$, and consider their center of mass, or momentum, CM, called *barycenter*, given by the equation

$$CM = \frac{m_{\mathbf{a}}\mathbf{a} + m_{\mathbf{b}}\mathbf{b} + m_{\mathbf{c}}\mathbf{c}}{m_{\mathbf{a}} + m_{\mathbf{b}} + m_{\mathbf{c}}}. \tag{10.1}$$

Following Hocking and Young [18, pp. 195 – 200], a set of $h+1$ vectors $\{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_h\}$ in $\mathbb{R}^n$ is *pointwise independent* if the h vectors $-\mathbf{a}_0 + \mathbf{a}_k$, $k = 1, \dots, h$, are linearly independent.

Definition 10.1 (Euclidean Barycentric Coordinates)**.** Let $A=\{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_h\}$ be a pointwise independent set of $h+1$ vectors in $\mathbb{R}^n$. Then, the real numbers $m_0, m_1, \dots, m_h$ normalized by the condition

$$\sum_{k=0}^{h} m_k = 1 \tag{10.2}$$

are the *barycentric coordinates* of a vector $\mathbf{a}\in\mathbb{R}^n$ with respect to the set A if

$$\mathbf{a} = \frac{\sum_{k=0}^{h} m_k\mathbf{a}_k}{\sum_{k=0}^{h} m_k}. \tag{10.3}$$

When the normalization condition (10.2) is relaxed,

$$\sum_{k=0}^{h} m_k \neq 0 \tag{10.4}$$

the barycentric coordinates become the so called *homogeneous barycentric coordinates*. They are homogeneous in the sense that the homogeneous barycentric coordinates $(m_0, m_1, \dots, m_h)$

of $\mathbf{a}$ in (10.3) are equivalent to the homogeneous barycentric coordinates $(\lambda m_0, \lambda m_1, \ldots, \lambda m_h)$ for any $\lambda \neq 0$. Since in homogeneous barycentric coordinates only ratios of coordinates are relevant, the homogeneous barycentric coordinates $(m_0, m_1, \ldots, m_h)$ are also written as $(m_0 : m_1 : \cdots : m_h)$.

It is easy to see from (10.3) that the barycentric coordinates are independent of the choice of the origin of their vector space, that is,

$$-\mathbf{x} + \mathbf{a} = \frac{\sum_{k=0}^{h} m_k(-\mathbf{x} + \mathbf{a}_k)}{\sum_{k=0}^{h} m_k} \tag{10.5}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

It follows from (10.5) that the vector $\mathbf{a}$ is *covariant* with respect to translations of $\mathbb{R}^n$ since the vector $\mathbf{a}$ and the vectors $\mathbf{a}_k$, $k = 0, 1, \ldots, h$, of its generating set A vary together under translations.

Let $R \in SO(n)$ be an element of the group $SO(n)$ of all rotations of the space $\mathbb{R}^n$ about its origin. Since R is linear, it follows from (10.3) that

$$R\mathbf{a} = \frac{\sum_{k=0}^{h} m_k R\mathbf{a}_k}{\sum_{k=0}^{h} m_k} \tag{10.6}$$

for all $R \in SO(n)$. It follows from (10.6) that the vector $\mathbf{a}$ is covariant with respect to rotations of $\mathbb{R}^n$ since the vector $\mathbf{a}$ and the vectors $\mathbf{a}_k$, $k = 0, 1, \ldots, h$, of its generating set A vary together under rotations.

It is owing to the transformation rules of $\mathbf{a}$ in (10.5) and (10.6) that $\mathbf{a}$ of (10.3) is qualified for the title of vector.

The group of all translations and all rotations of $\mathbb{R}^n$ forms the group of rigid motions of $\mathbb{R}^n$, which is the group of all direct isometries of $\mathbb{R}^n$ (that is, isometries preserving orientation) for the Euclidean distance function $d(\mathbf{a}, \mathbf{b}) = \|-\mathbf{a} + \mathbf{b}\|$. Hence, the vector $\mathbf{a}$ in (10.3) is said to be covariant since the vector $\mathbf{a}$ and the vectors of its generating set A vary together in $\mathbb{R}^n$ under the rigid motions of $\mathbb{R}^n$. The motion group of a Euclidean geometry, which is a semidirect product of an orthogonal group and the group of translations, is studied, for instance, in [1], as Ellers mentions in an article dedicated to the memory of Friedrich Bachmann [5].

The set of all points in $\mathbb{R}^n$ for which the barycentric coordinates with respect to A are all positive form an open convex subset of $\mathbb{R}^n$, called the open *h-simplex* with the $h + 1$ vertices $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h$. Following Hocking and Young [18, p. 199], the h-simplex with vertices $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h$ may be denoted by the symbol $\langle \mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h \rangle$. If the positive number m_k is viewed as the mass of a massive object situated at the point $\mathbf{a}_k$, $0 \leq k \leq h$, the point $\mathbf{a}$ in (10.3) turns out to be the center of mass of the $h + 1$ masses m_k, $0 \leq k \leq h$. If, furthermore, all the masses are equal, the center of mass turns out to be the *centroid* of the h-simplex. Three illustrative examples follow [48, Sec. 11.12].

Example 10.1. *The 2-simplex* $\langle \mathbf{u}, \mathbf{v} \rangle$ *in* $\mathbb{R}^3$ *is the Euclidean segment* $\mathbf{uv}$ *with endpoints* $\mathbf{u}$ *and* $\mathbf{v}$ *and midpoint*

$$\mathbf{m_{uv}} = \frac{\mathbf{u} + \mathbf{v}}{2}. \tag{10.7}$$

The barycentric coordinates of the endpoints $\mathbf{u}$ *and* $\mathbf{v}$ *of the segment* $\mathbf{uv}$ *with respect to* $A = \{\mathbf{u}, \mathbf{v}\}$ *are, respectively,* $(1, 0)$ *and* $(0, 1)$. *As we see from* (10.7)*, the barycentric coordinates of the midpoint* $\mathbf{m_{uv}}$ *of the segment with respect to* A *are* $(1/2, 1/2)$.

Example 10.2. *The 3-simplex* $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ *in* $\mathbb{R}^3$ *is the Euclidean triangle* $\mathbf{uvw}$ *with vertices* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *and centroid*

$$\mathbb{C}_{\mathbf{uvw}} = \frac{\mathbf{u}+\mathbf{v}+\mathbf{w}}{3}. \tag{10.8}$$

The barycentric coordinates of the vertices $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *of triangle* $\mathbf{uvw}$ *with respect to* $A = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *are, respectively,* $(1,0,0)$, $(0,1,0)$ *and* $(0,0,1)$. *As we see from* (10.8)*, the barycentric coordinates of the centroid with respect to* A *are* $(1/3, 1/3, 1/3)$.

Example 10.3. *The 4-simplex* $\langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p} \rangle$ *in* $\mathbb{R}^3$ *is a Euclidean tetrahedron* $\mathbf{uvwp}$ *with vertices* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{p}$*, and centroid at the point*

$$\mathbb{C}_{\mathbf{uvwp}} = \frac{\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{p}}{4} \tag{10.9}$$

of the tetrahedron. The barycentric coordinates of the vertices $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *and* $\mathbf{p}$ *of the tetrahedron with respect to the set* $A = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}\}$ *are, respectively,* $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$ *and* $(0,0,0,1)$. *As we see from* (10.9)*, the barycentric coordinates of the centroid of the tetrahedron with respect to* A *are* $(1/4, 1/4, 1/4, 1/4)$.

11. EINSTEINIAN GYROBARYCENTRIC COORDINATES

Definition 11.1 (Einsteinian Gyrobarycentric Coordinates)**.** Let

$$\mathbb{R}^n_s = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\} \tag{11.1}$$

be the s-ball of the Euclidean n-space, and let the set $A = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_h\}$ of points in the Einstein gyrovector space $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus, \otimes)$ be a pointwise independent set of $h+1$ vectors in $\mathbb{R}^n$. Then, the real numbers $m_0, m_1, \dots, m_h$ normalized by the condition

$$\sum_{k=0}^{h} m_k = 1 \tag{11.2}$$

are the *gyrobarycentric coordinates* of a gyrovector $\mathbf{a} \in \mathbb{R}^n_s$ with respect to A if

$$\mathbf{a} = \frac{\sum_{k=0}^{h} m_k \gamma_{\mathbf{a}_k} \mathbf{a}_k}{\sum_{k=0}^{h} m_k \gamma_{\mathbf{a}_k}}. \tag{11.3}$$

When the normalization condition (11.2) is relaxed,

$$\sum_{k=0}^{h} m_k \neq 0 \tag{11.4}$$

the gyrobarycentric coordinates become the so called *homogeneous gyrobarycentric coordinates*. They are homogeneous in the sense that the homogeneous gyrobarycentric coordinates $(m_0, m_1, \dots, m_h)$ of $\mathbf{a}$ in (11.3) are equivalent to the homogeneous gyrobarycentric coordinates $(\lambda m_0, \lambda m_1, \dots, \lambda m_h)$ for any $\lambda \neq 0$. Since in homogeneous gyrobarycentric coordinates only ratios of coordinates are relevant, the homogeneous gyrobarycentric coordinates $(m_0, m_1, \dots, m_h)$ are also written as $(m_0 : m_1 : \cdots : m_h)$.

Interestingly, in full analogy with (10.5), which follows from (10.3), it follows from (11.3) that

$$\ominus\mathbf{x} \oplus \mathbf{a} = \frac{\sum_{k=0}^{h} m_k \gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_k} (\ominus\mathbf{x}\oplus\mathbf{a}_k)}{\sum_{k=0}^{h} m_k \gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_k}} \tag{11.5}$$

for all $\mathbf{x}, \mathbf{a}_k \in \mathbb{R}_s^n$ in the Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, as we see from [48, Eq. (11.112), p. 473]. Noting that $\sum$ in (11.5) denotes ordinary vector addition, the interplay between ordinary vector addition, $+$, and Einstein addition, $\oplus$, in (11.5) is remarkable.

It follows from (11.5) that the vector $\mathbf{a}$ is *covariant* with respect to *left gyrotranslations* of $\mathbb{R}_s^n$ since the vector $\mathbf{a}$ and the vectors $\mathbf{a}_k$, $k = 0, 1, \ldots, h$, of its generating set A vary together under left gyrotranslations.

Let $R \in SO(n)$ be an element of the group $SO(n)$ of all rotations of the ball $\mathbb{R}_s^n$ about its origin. Since R is linear, and since by (2.3) we have $\gamma_{\mathbf{a}_k} = \gamma_{R\mathbf{a}_k}$, it follows from (11.3) that

$$R\mathbf{a} = \frac{\sum_{k=0}^{h} m_k \gamma_{R\mathbf{a}_k} R\mathbf{a}_k}{\sum_{k=0}^{h} m_k \gamma_{R\mathbf{a}_k}} \tag{11.6}$$

for all $R \in SO(n)$. It follows from (11.6) that the vector $\mathbf{a}$ is covariant with respect to rotations of $\mathbb{R}_s^n$ about its origin since the vector $\mathbf{a}$ and the vectors $\mathbf{a}_k$, $k = 0, 1, \ldots, h$, of its generating set A vary together in $\mathbb{R}_s^n$ under rotations of $\mathbb{R}_s^n$.

It is owing to the transformation rules of $\mathbf{a}$ in (11.5) and (11.6) that $\mathbf{a}$ of (11.3) is qualified for the title of gyrovector.

The group of all left gyrotranslations and all rotations of $\mathbb{R}_s^n$ forms the group of hyperbolic rigid motions of $\mathbb{R}_s^n$, which is the group of all direct isometries of $\mathbb{R}_s^n$ (that is, isometries preserving orientation) for the hyperbolic distance, or gyrodistance, function $d(\mathbf{a}, \mathbf{b}) = \|\ominus\mathbf{a}\oplus\mathbf{b}\|$. Hence, the hyperbolic vector, gyrovector, $\mathbf{a}$ in (11.3) is said to be hyperbolically covariant, or gyrocovariant, since the hyperbolic vector $\mathbf{a}$ and the hyperbolic vectors of its generating set A vary together in $\mathbb{R}_s^n$ under the hyperbolic rigid motions of $\mathbb{R}_s^n$. The motion group of a non Euclidean geometry is studied, for instance, in [1], as Ellers mentions in [5].

Three illustrative examples 11.1 – 11.3, which are respectively analogous to examples 10.1 – 10.3, follow.

Example 11.1. *The 2-simplex $\langle \mathbf{u}, \mathbf{v} \rangle$ in the Einstein 3-gyrovector space $\mathbb{R}_s^3 = (\mathbb{R}_s^3, \oplus, \otimes)$ is the hyperbolic segment, gyrosegment, $\mathbf{uv}$ with endpoints $\mathbf{u}$ and $\mathbf{v}$ and gyromidpoint, Figs. 5.1 and 11.1,*

$$\mathbf{m}_{\mathbf{uv}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}. \tag{11.7}$$

The gyrobarycentric coordinates of the endpoints $\mathbf{u}$ and $\mathbf{v}$ of the gyrosegment $\mathbf{uv}$ with respect to $A = \{\mathbf{u}, \mathbf{v}\}$ are, respectively, $(1, 0)$ and $(0, 1)$. As we see from (11.7), the gyrobarycentric coordinates of the gyromidpoint $\mathbf{m}_{\mathbf{uv}}$ of the gyrosegment with respect to A are $(1/2, 1/2)$.

It can be shown [40, 44, 48] that, in full analogy with vector space midpoints, the gyrovector space gyromidpoint $\mathbf{m}_{\mathbf{uv}}$ in (11.7) can be written as

$$\begin{aligned} \mathbf{m}_{\mathbf{uv}} &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \\ &= \tfrac{1}{2}\otimes(\mathbf{u} \boxplus \mathbf{v}) \\ &= \mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes\tfrac{1}{2} \\ &= \mathbf{v}\oplus(\ominus\mathbf{v}\oplus\mathbf{u})\otimes\tfrac{1}{2} \end{aligned} \tag{11.8}$$

in any Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$.

Example 11.2. *The 3-simplex $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ in $\mathbb{R}_s^3$ is the hyperbolic triangle, gyrotriangle, $\mathbf{uvw}$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and gyrocentroid at the point*

$$\mathbb{C}_{\mathbf{uvw}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}} \tag{11.9}$$

of the gyrotriangle. The gyrobarycentric coordinates of the vertices $\mathbf{u}$*,* $\mathbf{v}$ *and* $\mathbf{w}$ *of gyrotriangle* $\mathbf{uvw}$ *with respect to* $A = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *are, respectively,* $(1, 0, 0)$*,* $(0, 1, 0)$ *and* $(0, 0, 1)$*. As we see from* (11.9)*, the gyrobarycentric coordinates of the gyrocentroid with respect to* A *are* $(1/3, 1/3, 1/3)$.

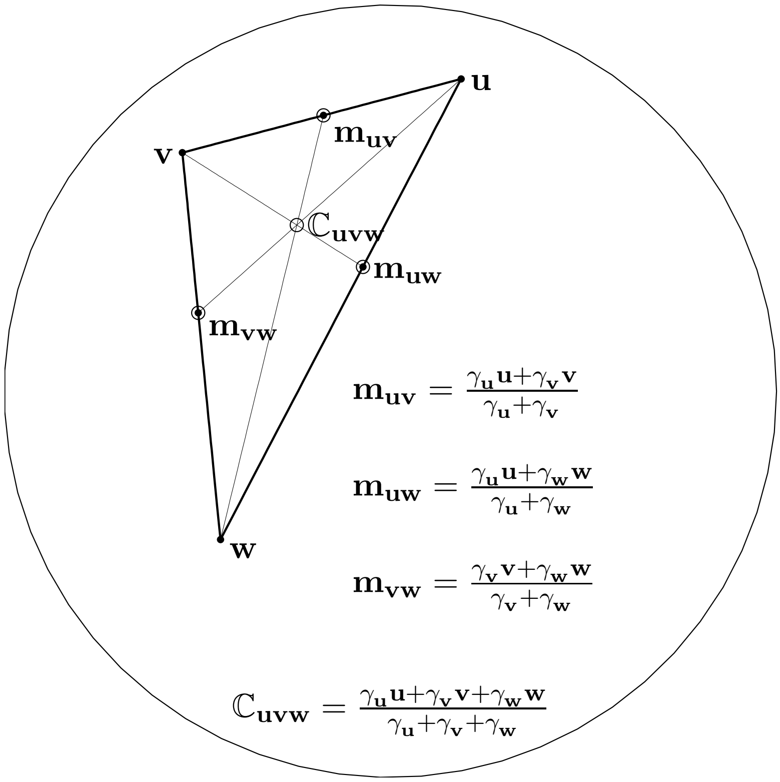

Figure 11.1. The gyromidpoints of the three sides of a gyrotriangle $\mathbf{uvw}$ in the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus_{\rm E}, \otimes_{\rm E})$ are shown along with its gyromedians and gyrocentroid. Interestingly, Einsteinian gyromidpoints and gyrocentroids have interpretation in relativistic mechanics, fully analogous to the interpretation of Euclidean midpoints and centroids in classical mechanics that one encounters in the vector space approach to Euclidean geometry [17]; see Example 11.2.

The gyrotriangle $\mathbf{uvw}$ of this example along with its side gyromidpoints and gyrocentroid is shown in Fig. 11.1. The gyrotriangle gyrocentroid is the point of concurrency of the gyrotriangle gyromedians, in full analogy with its Euclidean counterpart.

Example 11.3. *The 4-simplex* $\langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p} \rangle$ *in* $\mathbb{R}^3_s$ *is a hyperbolic tetrahedron, gyrotetrahedron,* $\mathbf{uvwp}$ *with vertices* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *and* $\mathbf{p}$*, and gyrocentroid at the point*

$$\mathbb{C}_{\mathbf{uvwp}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w} + \gamma_{\mathbf{p}}\mathbf{p}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}} + \gamma_{\mathbf{p}}} \tag{11.10}$$

of the gyrotetrahedron. The gyrobarycentric coordinates of the vertices $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *and* $\mathbf{p}$ *of the gyrotetrahedron with respect to the set* $A = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}\}$ *are, respectively,* $(1, 0, 0, 0)$*,* $(0, 1, 0, 0)$*,* $(0, 0, 1, 0)$ *and* $(0, 0, 0, 1)$*. As we see from* (11.10)*, the gyrobarycentric coordinates of the gyrocentroid of the gyrotetrahedron with respect to* A *are* $(1/4, 1/4, 1/4, 1/4)$.

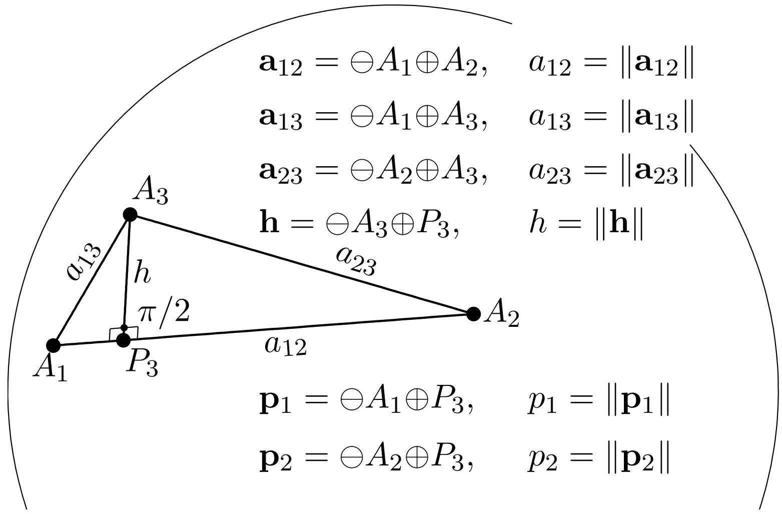

Figure 12.1. Orthogonal projection, P_3, of a point, A_3, onto a gyrosegment, A_1A_2, in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. The gyrosegment A_3P_3 is the gyroaltitude of gyrotriangle $A_1A_2A_3$ dropped from vertex A_3 to side A_1A_2. Ambigiously, both the gyrovector $\mathbf{h}$ and its gyrolength h are called a gyroaltitude of the gyrotriangle. The gyrobarycentric coordinates of the point P_3 with respect to the set of points $\{A_1, A_2, A_3\}$ are determined in Sec. 12.

12. GYROTRIANGLE GYROALTITUDES

Let $A_1A_2A_3$ be a gyrotriangle with vertices A_1, A_2, and A_3, in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, and let the point P_3 be the orthogonal projection of vertex A_3 onto its opposite side, A_1A_2 (or its extension), as shown in Fig. 12.1 and in Figs. 12.2 – 12.3 for $\mathbb{V}_s = \mathbb{R}^2_s$. Furthermore, let (m_1, m_2) be the gyrobarycentric coordinates of P_3 with respect to the set $\{A_1, A_2\}$ in the Einstein gyrovector space, as presented in Def. 11.1. Then, P_3 is given in terms of its gyrobarycentric coordinates (m_1, m_2) with respect to the set $\{A_1, A_2\}$ by the equation

$$P_3 = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2}{m_1\gamma_{A_1} + m_2\gamma_{A_2}}, \tag{12.1}$$

where the gyrobarycentric coordinates m_1 and m_2 of P_3 are to be determined in (12.16) below in terms of the side gyrolengths of gyrotriangle $A_1A_2A_3$.

Furthermore, by the gyrocovariance of P_3 in (12.1), which follows from (11.5), we have

$$\ominus X \oplus P_3 = \frac{m_1\gamma_{\ominus X \oplus A_1}(\ominus X \oplus A_1) + m_2\gamma_{\ominus X \oplus A_2}(\ominus X \oplus A_2)}{m_1\gamma_{\ominus X \oplus A_1} + m_2\gamma_{\ominus X \oplus A_2}} \tag{12.2}$$

so that, in particular, for $X = A_1$ and for $X = A_2$ in (12.2) we have, respectively,

$$\begin{aligned} \ominus A_1 \oplus P_3 &= \frac{m_2\gamma_{\ominus A_1 \oplus A_2}(\ominus A_1 \oplus A_2)}{m_1 + m_2\gamma_{\ominus A_1 \oplus A_2}}, \\ \ominus A_2 \oplus P_3 &= \frac{m_1\gamma_{\ominus A_1 \oplus A_2}(\ominus A_2 \oplus A_1)}{m_1\gamma_{\ominus A_1 \oplus A_2} + m_2}. \end{aligned} \tag{12.3}$$

Along with the notation in Fig. 12.1 we use the notation

$$\begin{aligned} &\mathbf{a}_{12} = \ominus A_1 \oplus A_2, && a_{12} = \|\mathbf{a}_{12}\|, && \gamma_{21} = \gamma_{12} = \gamma_{a_{12}}, \\ &\mathbf{a}_{13} = \ominus A_1 \oplus A_3, && a_{13} = \|\mathbf{a}_{13}\|, && \gamma_{31} = \gamma_{13} = \gamma_{a_{13}}, \\ &\mathbf{a}_{23} = \ominus A_2 \oplus A_3, && a_{23} = \|\mathbf{a}_{23}\|, && \gamma_{32} = \gamma_{23} = \gamma_{a_{23}} \end{aligned} \tag{12.4}$$

and

$$\begin{aligned} &\mathbf{p}_1 = \ominus A_1 \oplus P_3, && p_1 = \|\mathbf{p}_1\|, \\ &\mathbf{p}_2 = \ominus A_2 \oplus P_3, && p_2 = \|\mathbf{p}_2\| \end{aligned} \tag{12.5}$$

and

$$\mathbf{h} = \ominus A_3 \oplus P_3, \qquad h = \|\mathbf{h}\|. \tag{12.6}$$

The Einstein-Pythagoras Identity says that if the gyrolength of the two legs and hypotenuse of a right-gyroangled gyrotriangle in an Einstein gyrovector space are $a,\ b$ and c, respectively, then [48, Eq. (12.61), p. 553]

$$\gamma_a \gamma_b = \gamma_c\,. \tag{12.7}$$

Following the notation in Fig. 12.1 and in (12.4) – (12.6), the application of the Einstein-Pythagoras Identity, (12.7), to the two right gyrotriangles $A_1P_3A_3$ and $P_3A_2A_3$ in Fig. 12.1 gives rise, respectively, to the equations

$$\begin{aligned} \gamma_{p_1}\gamma_h &= \gamma_{13}, \\ \gamma_{p_2}\gamma_h &= \gamma_{23}. \end{aligned} \tag{12.8}$$

By [48, Eq. (11.148) of Theorem 11.6, p. 481], it follows from (12.1) that

$$\gamma_{\ominus X \oplus P_3} = \frac{m_1\gamma_{\ominus X \oplus A_1} + m_2\gamma_{\ominus X \oplus A_2}}{m_0} \tag{12.9}$$

for any $X \in \mathbb{V}_s$, where $m_0 > 0$ is given by the equation

$$m_0^2 = (m_1 + m_2)^2 + 2m_1m_2(\gamma_{12} - 1). \tag{12.10}$$

In its relativistic mechanical interpretation, m_0 of (12.10) is the resultant invariant mass of the invariant masses m_1 and m_2 [51]. The relativistic invariant mass resultant m_0 depends on the speed of m_1 and m_2 relative to each other, as we see from the presence of the gamma factor γ_{12} in (12.10). When this relative speed vanishes, we have $\gamma_{12} = 1$ so that (12.10) reduces in this case to its classical counterpart, $m_0 = m_1 + m_2$.

For $X = A_1$ and for $X = A_2$, respectively, (12.9) with the notation in (12.5) gives the equations

$$\begin{aligned} \gamma_{p_1} &= \frac{m_1 + m_2\gamma_{12}}{m_0}, \\ \gamma_{p_2} &= \frac{m_1\gamma_{12} + m_2}{m_0} \end{aligned} \tag{12.11}$$

and Einstein-Pythagoras Identities (12.8) give

$$\frac{m_0\gamma_{p_1}}{\gamma_{13}} = \frac{m_0\gamma_{p_2}}{\gamma_{23}}. \tag{12.12}$$

Eliminating $m_0\gamma_{p_1}$ and $m_0\gamma_{p_2}$ between (12.11) and (12.12), we have

$$\frac{m_1 + m_2\gamma_{12}}{\gamma_{13}} = \frac{m_1\gamma_{12} + m_2}{\gamma_{23}}. \tag{12.13}$$

Equation (12.13) and the gyrobarycentric coordinates normalization condition, $m_1 + m_2 = 1$, form a system of two equations for the two unknowns m_1 and m_2, which can be written as the matrix equation

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{\gamma_{13}} - \frac{\gamma_{12}}{\gamma_{23}} & \frac{\gamma_{12}}{\gamma_{13}} - \frac{1}{\gamma_{23}} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{12.14}$$

The 2×2 matrix M in (12.14) is invertible, having the determinant

$$det(M) = \frac{\gamma_{13}+\gamma_{23}}{\gamma_{13}\gamma_{23}}(\gamma_{12}-1) > 0, \tag{12.15}$$

implying

$$\begin{aligned} m_1 &= \frac{\gamma_{12}\gamma_{23}-\gamma_{13}}{(\gamma_{13}+\gamma_{23})(\gamma_{12}-1)}, \\ m_2 &= \frac{\gamma_{12}\gamma_{13}-\gamma_{23}}{(\gamma_{13}+\gamma_{23})(\gamma_{12}-1)}, \\ m_1 + m_2 &= 1, \end{aligned} \tag{12.16}$$

thus determining the gyrobarycentric coordinates (m_1, m_2) of P_3 with respect to the set $\{A_1, A_2\}$ in (12.1).

By (12.9) with $X = A_3$ and the notation in (12.4) and (12.6) we have

$$\gamma_{\mathbf{h}} = \gamma_{\ominus A_3 \oplus P_3} = \frac{m_1\gamma_{13} + m_2\gamma_{23}}{m_0}, \tag{12.17}$$

where m_1 and m_2 are given by (12.16) and m_o is given by (12.10).

We use the notation $\mathbf{h}^2 = \mathbf{h}\cdot\mathbf{h} = \|\mathbf{h}\|^2 = h^2$, so that $\gamma_{\mathbf{h}}^2 = \gamma_h^2$. With this notation, it follows from (2.7) that

$$\mathbf{h}^2 = s^2\frac{\gamma_{\mathbf{h}}^2 - 1}{\gamma_{\mathbf{h}}^2}. \tag{12.18}$$

It follows from (12.18) and from (12.17), (12.16) and (12.10) that

$$\begin{aligned} \gamma_{\mathbf{h}}^2 &= \frac{2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{13}^2 - \gamma_{23}^2}{\gamma_{12}^2 - 1}, \\ \gamma_{\mathbf{h}}^2\mathbf{h}^2 &= s^2(\gamma_{\mathbf{h}}^2 - 1) = s^2\frac{1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{\gamma_{12}^2 - 1}, \\ \mathbf{h}^2 &= \frac{\gamma_{\mathbf{h}}^2\mathbf{h}^2}{\gamma_{\mathbf{h}}^2} = s^2\frac{1 + 2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{13}^2 - \gamma_{23}^2}, \end{aligned} \tag{12.19}$$

where $\mathbf{h}$ is the gyrotriangle gyroaltitude drawn from vertex A_3, $\mathbf{h} = \ominus A_3 \oplus P_3$, as shown in Fig. 12.1.

Substituting the gyrobarycentric coordinates m_1 and m_2 from (12.16) into (12.1) we have

$$P_3 = \frac{(\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{A_1}A_1 + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{A_2}A_2}{(\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{A_1} + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{A_2}} =: \frac{N_{312}}{D_{312}}, \tag{12.20}$$

where N_{312} and D_{312} are the numerator and the denominator, respectively, of the right-hand side of (12.20). The index notation in (12.20) will prove useful when index permutations are needed.

In (12.20) we obtain a homogeneous gyrobarycentric coordinate representation of the orthogonal projection P_3 of vertex A_3 onto its opposite side A_1A_2 (or its extension) of gyrotriangle $A_1A_2A_3$ of Fig. 12.1 with respect to the set $\{A_1, A_2\}$.

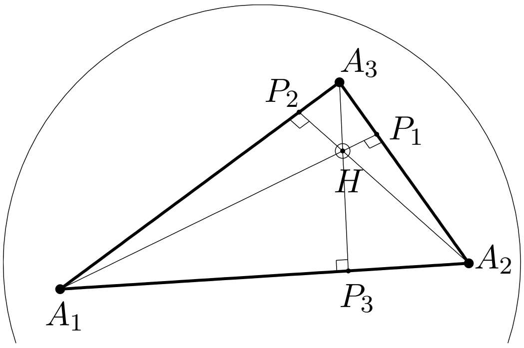

Figure 12.2. The gyroaltitudes, and the gyroorthocenter H, of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space. Case I: The gyroorthocenter H lies inside the acute gyrotriangle. The homogeneous gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of the gyrobarycenter H relative to the set $\{A_1, A_2, A_3\}$ of the gyrotriangle vertices, given by Theorem 13.1, are all positive.

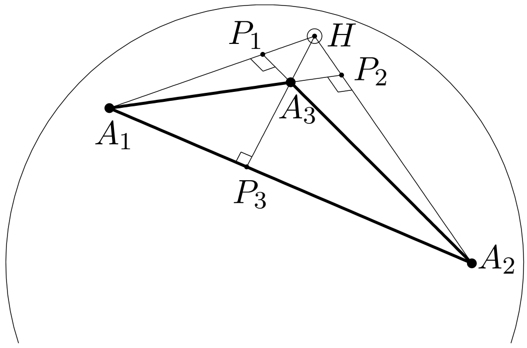

Figure 12.3. The gyroaltitudes, and the gyroorthocenter H, of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space. Case II: The gyroorthocenter H lies outside the obtuse gyrotriangle. One of the homogeneous gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ of the gyrobarycenter H relative to the set $\{A_1, A_2, A_3\}$ of the gyrotriangle vertices, given by Theorem 13.1, is positive and the other two are negative.

Formalizing the result in (12.20) and previous related results, we have the following theorem.

Theorem 12.1. *Let $A_1A_2A_3$ be a gyrotriangle in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, and let P_k, $k = 1, 2, 3$, be the orthogonal projection of vertex A_k onto its opposite side, as shown in Figs. 12.2 – 12.3. Then*

$$\begin{aligned}
P_1 &= \frac{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{A_2}A_2 + (\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{A_3}A_3}{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{A_2} + (\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{A_3}} = \frac{N_{123}}{D_{123}}, \\
P_2 &= \frac{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{A_1}A_1 + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{A_3}A_3}{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{A_1} + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{A_3}} = \frac{N_{231}}{D_{231}}, \\
P_3 &= \frac{(\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{A_1}A_1 + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{A_2}A_2}{(\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{A_1} + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{A_2}} = \frac{N_{312}}{D_{312}}
\end{aligned} \tag{12.21}$$

and

$$\begin{aligned}
\ominus A_1\oplus P_1 &= \frac{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{12}(\ominus A_1\oplus A_2) + (\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{13}(\ominus A_1\oplus A_3)}{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{12} + (\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{13}}, \\
\ominus A_2\oplus P_2 &= \frac{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{12}(\ominus A_2\oplus A_1) + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{23}(\ominus A_2\oplus A_3)}{(\gamma_{13}\gamma_{23}-\gamma_{12})\gamma_{12} + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{23}}, \\
\ominus A_3\oplus P_3 &= \frac{(\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{13}(\ominus A_3\oplus A_1 + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{23}(\ominus A_3\oplus A_2)}{(\gamma_{12}\gamma_{23}-\gamma_{13})\gamma_{13} + (\gamma_{12}\gamma_{13}-\gamma_{23})\gamma_{23}}.
\end{aligned} \tag{12.22}$$

Furthermore, the squares of the gyrotriangle gyroaltitudes are

$$\|\ominus A_1 \oplus P_1\|^2 = s^2 \frac{1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2},$$

$$\|\ominus A_2 \oplus P_2\|^2 = s^2 \frac{1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{23}^2}, \tag{12.23}$$

$$\|\ominus A_3 \oplus P_3\|^2 = s^2 \frac{1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{13}^2 - \gamma_{23}^2},$$

their relativistically corrected versions are

$$\gamma_{\ominus A_1 \oplus P_1}^2 \|\ominus A_1 \oplus P_1\|^2 = s^2 \frac{1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{\gamma_{23}^2 - 1},$$

$$\gamma_{\ominus A_2 \oplus P_2}^2 \|\ominus A_2 \oplus P_2\|^2 = s^2 \frac{1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{\gamma_{13}^2 - 1}, \tag{12.24}$$

$$\gamma_{\ominus A_3 \oplus P_3}^2 \|\ominus A_3 \oplus P_3\|^2 = s^2 \frac{1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2}{\gamma_{12}^2 - 1},$$

and the gyrotriangle constant, $S_{A_1A_2A_3}$, is given by each of the following three equations,

$$\gamma_{\ominus A_2 \oplus A_3}^2 \|\ominus A_2 \oplus A_3\|^2 \gamma_{\ominus A_1 \oplus P_1}^2 \|\ominus A_1 \oplus P_1\|^2 = S_{A_1A_2A_3}^2,$$

$$\gamma_{\ominus A_1 \oplus A_3}^2 \|\ominus A_1 \oplus A_3\|^2 \gamma_{\ominus A_2 \oplus P_2}^2 \|\ominus A_2 \oplus P_2\|^2 = S_{A_1A_2A_3}^2, \tag{12.25}$$

$$\gamma_{\ominus A_1 \oplus A_2}^2 \|\ominus A_1 \oplus A_2\|^2 \gamma_{\ominus A_3 \oplus P_3}^2 \|\ominus A_3 \oplus P_3\|^2 = S_{A_1A_2A_3}^2,$$

where we define

$$S_{A_1A_2A_3}^2 := s^4(1+2\gamma_{12}\gamma_{13}\gamma_{23} - \gamma_{12}^2 - \gamma_{13}^2 - \gamma_{23}^2). \tag{12.26}$$

Proof. Note that the three equations in each of the five groups in (12.21) – (12.25) follow from each other by an index cyclic permutation since, consistent with the notation in (12.4), we have $\gamma_{21} = \gamma_{12}$, etc., where $\gamma_{21} = \gamma_{\|\ominus A_2 \oplus A_1\|}$.

Proof of (12.21): The third equation in (12.21) is a copy of (12.20), which has already been proved. The second equation in (12.21) is obtained from the third by the a cyclic permutation of the gyrotriangle vertices in Fig. 12.1 that corresponds to the index cyclic permutation $(1, 2, 3) \to (3, 1, 2)$. The first equation in (12.21) is obtained from the third by the a cyclic permutation of the gyrotriangle vertices in Fig. 12.1 that corresponds to the index cyclic permutation $(1, 2, 3) \to (2, 3, 1)$.

Proof of (12.22): Each of the three equations in (12.22) follows from a corresponding equation in (12.21) and from Identity [48, Eq. (11.112), p. 473], and the notation in (12.4).

Proof of (12.23): The third equation in (12.23) is identical with the third equation in (12.19) with $\mathbf{h} = \ominus A_3 \oplus P_3$, which has already been proved. The second equation in (12.23) is obtained from the third by the cyclic permutation $(1, 2, 3) \to (3, 1, 2)$, and the first equation in (12.23) is obtained from the third by the cyclic permutation $(1, 2, 3) \to (2, 3, 1)$ of indices.

Proof of (12.24): The third equation in (12.24) follows from the second equation in (12.19) with $\mathbf{h} = \ominus A_3 \oplus P_3$, which has already been proved. The second equation in (12.24) is obtained from the third by the cyclic permutation $(1, 2, 3) \to (3, 1, 2)$, and the first equation in (12.24) is obtained from the third by the cyclic permutation $(1, 2, 3) \to (2, 3, 1)$ of indices.

Proof of (12.25): The first equation in (12.25) follows from the first equation in (12.24), noting that by (12.18),

$$s^2(\gamma_{23}^2 - 1) = s^2(\gamma_{\ominus A_2 \oplus A_3}^2 - 1) = \gamma_{\ominus A_2 \oplus A_3}^2 \|\ominus A_2 \oplus A_3\|^2. \tag{12.27}$$

The second and third equations in (12.25) follow from the second and third equations in (12.24) in a similar manner. ∎

13. EINSTEIN GYROTRIANGLE GYROORTHOCENTER

Figs. 12.2 – 12.3 indicate that the three gyroaltitudes of a gyrotriangle are concurrent. To prove analytically that this is indeed the case, we calculate the point of concurrency H which, in gyrolanguage, we naturally call the gyrotriangle *gyroorthocenter*. Since gyrolines in Einstein gyrovector spaces are segments of Euclidean straight lines in Euclidean geometry, we can use linear algebra in order to uncover homogeneous gyrobarycentric coordinates for the point of concurrency H, if it exists, with respect to the set of the gyrotriangle vertices.

Indeed, by methods of linear algebra one may find that the gyroorthocenter H of any gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, shown in Figs. 12.2 – 12.3 for $\mathbb{V}_s = \mathbb{R}_s^2$, exists in $\mathbb{V}$ (but not necessarily in the ball $\mathbb{V}_s \subset \mathbb{V}$), and is given by the gyrotriangle gyroorthocenter equation

$$H = \frac{C_{12}C_{13}\gamma_{A_1}A_1 + C_{12}C_{23}\gamma_{A_2}A_2 + C_{13}C_{23}\gamma_{A_3}A_3}{C_{12}C_{13}\gamma_{A_1} + C_{12}C_{23}\gamma_{A_2} + C_{13}C_{23}\gamma_{A_3}} \in \mathbb{V}, \tag{13.1}$$

where

$$\begin{aligned} C_{12} &= \gamma_{13}\gamma_{23} - \gamma_{12}, \\ C_{13} &= \gamma_{12}\gamma_{23} - \gamma_{13}, \\ C_{23} &= \gamma_{12}\gamma_{13} - \gamma_{23}, \end{aligned} \tag{13.2}$$

and where we use the notation in (12.4). Here, $\mathbb{V}$ is the space of the ball $\mathbb{V}_s$. When $H \in \mathbb{V}$ does not lie in $\mathbb{V}_s$, the gyrotriangle $A_1A_2A_3$ does not have a gyroorthocenter, as shown in Fig. 13.1.

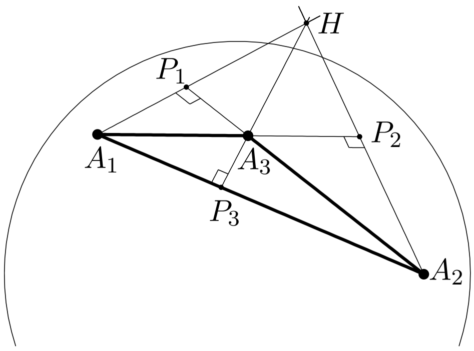

Figure 13.1. A gyrotriangle $A_1A_2A_3$ that does not possess a gyroorthocenter H in an Einstein gyrovector plane $(\mathbb{V}_s, \oplus, \otimes)$. The point $H \in \mathbb{V}$ lies outside the ball $\mathbb{V}_s \subset \mathbb{V}$ and, hence, according to Theorem 13.1, $f_2 < 0$.

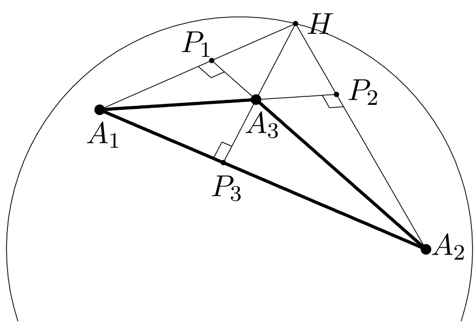

Figure 13.2. A gyrotriangle $A_1A_2A_3$ that does not possess a gyroorthocenter H in an Einstein gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. The point $H \in \mathbb{V}$ lies on the boundary of the ball $\mathbb{V}_s \subset \mathbb{V}$. Hence, by Theorem 13.1, $f_2 = 0$.

Equations (13.1) – (13.2) present homogeneous gyrobarycentric coordinates for the point H with respect to the set $\{A_1, A_2, A_3\}$ of the gyrotriangle vertices, as defined in Def. 11.1. Formalizing the result in (13.1) – (13.2), we have the following theorem.

Theorem 13.1. *Let H be the gyroorthocenter of a gyrotriangle $A_1A_2A_3$ in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Figs. 12.2 – 12.3, and let $\mathbb{V}$ be the space of the ball $\mathbb{V}_s \subset \mathbb{V}$. Then, the homogeneous gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ for H with respect to the set $\{A_1, A_2, A_3\}$ of the gyrotriangle vertices are given by the equations*

$$\begin{aligned} m_1 &= C_{12}C_{13}, \\ m_2 &= C_{12}C_{23}, \\ m_3 &= C_{13}C_{23}, \end{aligned} \tag{13.3}$$

where C_{ij}, $i, j = 1, 2, 3$, $i < j$, are given by (13.2)*. Accordingly, the gyrobarycentric coordinate representation of the gyroorthocenter H of a gyrotriangle $A_1A_2A_3$ with respect to the gyrotriangle vertices in an Einstein gyrovector space is given by the gyroorthocenter equation*

$$H = \frac{m_1\gamma_{A_1}A_1 + m_2\gamma_{A_2}A_2 + m_3\gamma_{A_3}A_3}{m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}} \in \mathbb{V}, \tag{13.4}$$

where the homogeneous gyrobarycentric coordinates m_1, m_2, and m_3 are given by (13.3)*. If $m_1, m_2, m_3 > 0$ then $H \in \mathbb{V}_s \subset \mathbb{V}$.*

The gyrotriangle $A_1A_2A_3$ possesses a gyroorthocenter H, that is, H lies in the ball $\mathbb{V}_s$, if and only if

$$f_2 > 0, \tag{13.5}$$

where f_2 is a gyrotriangle constant, given by the equation

$$f_2 = \gamma_{12}^2\gamma_{13}^2 + \gamma_{12}^2\gamma_{23}^2 + \gamma_{13}^2\gamma_{23}^2 + 3\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2 - 2\gamma_{12}\gamma_{13}\gamma_{23}(\gamma_{12}^2 + \gamma_{13}^2 + \gamma_{23}^2). \tag{13.6}$$

Furthermore, H lies on the boundary of the ball $\mathbb{V}_s$ if and only if

$$f_2 = 0. \tag{13.7}$$

Proof. The point H is the concurrency point of the three Euclidean straight lines that pass, respectively, through the pairs of points (A_1, P_1), (A_2, P_2) and (A_3, P_3), as shown in Figs. 12.2 – 12.3, where A_1,A_2 and A_3 are the gyrotriangle vertices, and where P_1,P_2 and P_3 are the corresponding orthogonal projections given by (12.21). The representation of H as a linear combination of the points $A_1, A_2, A_3 \in \mathbb{V}$ in (13.4) is found by employing common methods of linear algebra.

The point H of $\mathbb{V}$ lies in the ball, $H \in \mathbb{V}_s \subset \mathbb{V}$, if and only if

$$\frac{H^2}{s^2} < 1, \tag{13.8}$$

where $H^2 = H \cdot H = \|H\|^2$ is given by the inner product of H by itself.

Let the numerator and the denominator of H/s in (13.4) be, respectively,

$$\begin{aligned} N &= m_1\gamma_{A_1}\frac{A_1}{s} + m_2\gamma_{A_2}\frac{A_2}{s} + m_3\gamma_{A_3}\frac{A_3}{s}, \\ D &= m_1\gamma_{A_1} + m_2\gamma_{A_2} + m_3\gamma_{A_3}, \end{aligned} \tag{13.9}$$

so that

$$\frac{H^2}{s^2} = \frac{N^2}{D^2}, \tag{13.10}$$

where $N^2 = N \cdot N = \|N\|^2$.

In calculating N^2 of (13.9) – (13.10), one encounters the inner products $(\gamma_{A_i}A_i)\cdot(\gamma_{A_j}A_j)$, $i,j=1,2,3$. These inner products can be manipulated in the way shown in (13.11) and (13.13) below.

By (12.18), we have

$$\gamma_{A_i}\frac{A_i}{s}\cdot\gamma_{A_i}\frac{A_i}{s}=\gamma_{A_i}^2\frac{\|A_i\|^2}{s^2}=\gamma_{A_i}^2-1, \tag{13.11}$$

$i=1,2,3$.

By (2.11), we have

$$\gamma_{ij}:=\gamma_{\ominus A_i\oplus A_j}=\gamma_{A_i}\gamma_{A_j}\left(1-\frac{A_iA_j}{s^2}\right), \tag{13.12}$$

implying

$$\gamma_{A_i}\frac{A_i}{s}\cdot\gamma_{A_j}\frac{A_j}{s}=\gamma_{A_i}\gamma_{A_j}-\gamma_{ij}, \tag{13.13}$$

$i,j=1,2,3$, $i<j$.

The square D^2 of D in (13.10) is clearly a function of $\gamma_{12},\gamma_{13},\gamma_{23}$, and of $\gamma_{A_1},\gamma_{A_2},\gamma_{A_3}$, as one can see from (13.9) and (13.2) – (13.3). Similarly, the square N^2 of N in (13.10) can also be written as a function of $\gamma_{12},\gamma_{13},\gamma_{23}$, and of $\gamma_{A_1},\gamma_{A_2},\gamma_{A_3}$, if we use identities (13.11) and (13.13).

Surprisingly, the difference of squares, D^2-N^2, which remains a function of $\gamma_{12},\gamma_{13},\gamma_{23}$, is independent of $\gamma_{A_1},\gamma_{A_2},\gamma_{A_3}$. Indeed, it turns out that

$$D^2-N^2=f_1f_2, \tag{13.14}$$

where the two factors f_1 and f_2 are given by

$$\begin{aligned} f_1&=1+2\gamma_{12}\gamma_{13}\gamma_{23}-\gamma_{12}^2-\gamma_{13}^2-\gamma_{23}^2=S_{A_1A_2A_3}^2/s^4,\\ f_2&=\gamma_{12}^2\gamma_{13}^2+\gamma_{12}^2\gamma_{23}^2+\gamma_{13}^2\gamma_{23}^2+3\gamma_{12}^2\gamma_{13}^2\gamma_{23}^2-2\gamma_{12}\gamma_{13}\gamma_{23}(\gamma_{12}^2+\gamma_{13}^2+\gamma_{23}^2). \end{aligned} \tag{13.15}$$

Mysteriously, the factor f_1 turns out to be identical with the squared gyroarea constant $S_{A_1A_2A_3}^2$ in (12.26), divided by s^4.

It follows from (13.8), (13.10) and (13.14) that $H\in\mathbb{V}_s$ if and only if $f_1f_2>0$. However, it can be shown that the factor f_1 is positive, $f_1>0$, for any gyrotriangle $A_1A_2A_3$ in $\mathbb{V}_s$ [52]. Hence, $H\in\mathbb{V}_s$ if and only if $f_2>0$, as desired. Furthermore, H lies on the boundary of $\mathbb{V}_s$, that is, $\|H\|^2/s^2=1$, if and only if $f_2=0$, as desired. ∎

The factor f_2 is a gyrotriangle constant, like f_1, in the sense that it is gyrocovariant and independent of gyrotriangle vertex permutations. It is a gyrotriangle constant that determines whether a gyrotriangle possesses a gyroorthocenter.

According to Theorem 13.1, the factor f_2 of the gyrotriangles in Figs. 12.2 – 12.3 is positive, the factor f_2 of the gyrotriangle in Fig. 13.1 is negative, and the factor f_2 of the gyrotriangle in Fig. 13.2 is zero.

Homogeneous gyrobarycentric coordinates of a gyrotriangle gyroorthocenter H with respect to the set of the gyrotriangle vertices need not be positive. Thus, for instance, in Fig. 12.2 these three homogeneous gyrobarycentric coordinates of H, given by (13.3), are all positive, while in Fig. 12.3 only one of these is positive and the other two are negative.

The special case of the right gyroangled gyrotriangle gyroorthocenter equation (13.4) is interesting and instructive. In this case gyrotriangle $A_1A_2A_3$ in Theorem 13.1 and Fig. 12.2 is right with, say, $\angle A_1A_3A_2=\pi/2$, so that the legs and hypotenuse of the gyrotriangle satisfy the

Einstein-Pythagoras Identity (12.7). In the presence of the latter identity, in turn, the gyrotriangle gyroorthocenter equation (13.4) reduces to

$$H = A_3 \tag{13.16}$$

as expected.

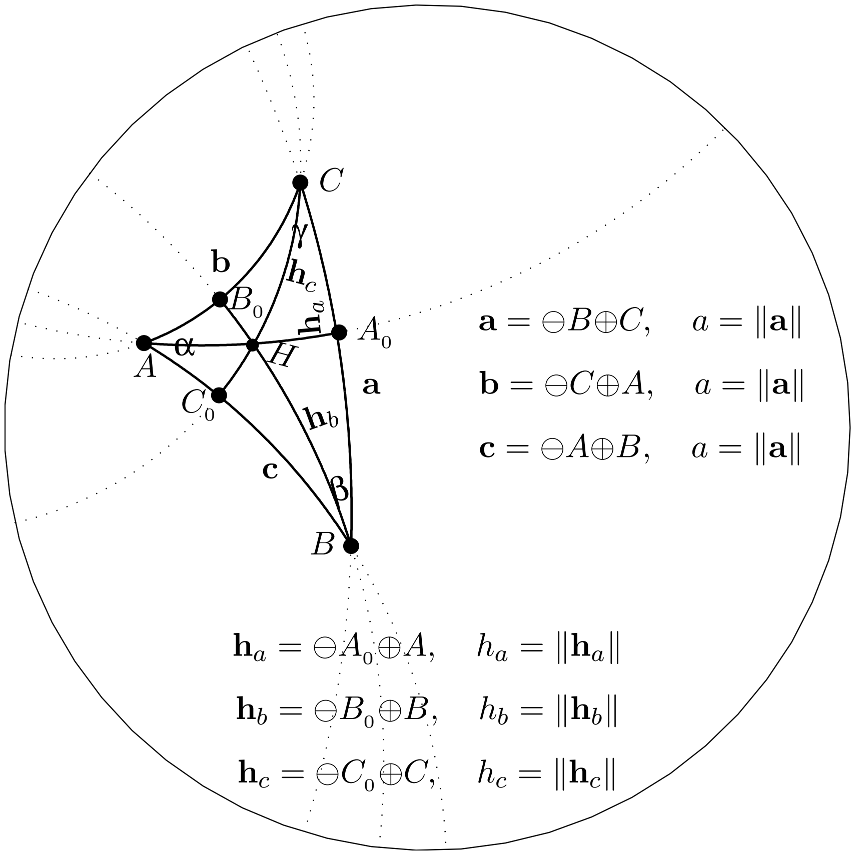

Figure 14.1. A Möbius gyrotriangle ABC in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ and its gyroaltitudes are shown. The gyroaltitude $\mathbf{h}_c$ of ABC is the gyrosegment drawn perpendicularly from vertex C to its opposite side AB. Ambigiously, also the gyrolength $h_c = \|\mathbf{h}_c\|$ of the gyroaltitude $\mathbf{h}_c$ is known as a gyroaltitude of the gyrotriangle ABC. The gyrolines containing the gyrotriangle gyroaltitudes are concurrent at the point H, called the gyroorthocenter of the gyrotriangle ABC. The position of the gyroorthocenter H relative to the gyrotriangle vertices as calculated in Identity (14.15) of Theorem 14.1. is shown here.

14. MÖBIUS GYROTRIANGLE GYROORTHOCENTER

A Möbius gyrotriangle gyroorthocenter is shown in Fig. 14.1. A direct calculation of a point of intersection of two gyrolines in a Möbius gyrovector space is a slightly complicated task. In contrast, a direct calculation of a point of intersection of two gyrolines in an Einstein gyrovector space is a simple task since gyrolines in this space coincide with Euclidean segments of straight lines, so that methods of linear algebra can be employed. Hence, we make no attempt to calculate directly the position of the Möbius gyrotriangle gyroorthocenter with respect to some coordinates, while we did calculate the position of the Einstein gyrotriangle gyroorthocenter in terms of homogeneous gyrobarycentric coordinates in Sec. 13. Rather than calculating the position of a Möbius gyrotriangle gyroorthocenter directly, we therefore calculate it by translating the result from Einstein gyrovector spaces into corresponding Möbius gyrovector spaces.

Accordingly, we wish in this section to translate each side of the Einstein gyrotriangle gyroorthocenter equation (13.4) from an Einstein gyrovector space into a corresponding Möbius gyrovector space. To accomplish the task, let us rewrite (13.4) in a form that emphasizes the governing Einstein gyrovector space by a subscript "e", so that its translation into a corresponding Möbius gyrovector space can be contrasted by emphasizing it by a subscript "m". The resulting Einstein gyroorthocenter equation (13.4) takes the form

$$H_e = \frac{m_{1,e}\gamma_{A_{1,e}}A_{1,e} + m_{2,e}\gamma_{A_{2,e}}A_{2,e} + m_{3,e}\gamma_{A_{3,e}}A_{3,e}}{m_{1,e}\gamma_{A_{1,e}} + m_{2,e}\gamma_{A_{2,e}} + m_{3,e}\gamma_{A_{3,e}}}. \tag{14.1}$$

In (14.1), H_e is the gyroorthocenter of a gyrotriangle $A_{1,e}A_{2,e}A_{3,e}$ in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$, represented by its homogeneous gyrobarycentric coordinates $(m_{1,e} : m_{2,e} : m_{3,e})$ with respect to the set of the gyrotriangle vertices $\{A_{1,e}, A_{2,e}, A_{3,e}\}$.

According to (13.3), the homogeneous gyrobarycentric coordinates $(m_{1,e} : m_{2,e} : m_{3,e})$ in (14.1) are given by the equations

$$\begin{aligned} m_{1,e} &= C_{12,e}C_{13,e}, \\ m_{2,e} &= C_{12,e}C_{23,e}, \\ m_{3,e} &= C_{13,e}C_{23,e}, \end{aligned} \tag{14.2}$$

where, by (13.2),

$$\begin{aligned} C_{12,e} &= \gamma_{13,e}\gamma_{23,e} - \gamma_{12,e}, \\ C_{13,e} &= \gamma_{12,e}\gamma_{23,e} - \gamma_{13,e}, \\ C_{23,e} &= \gamma_{12,e}\gamma_{13,e} - \gamma_{23,e}, \end{aligned} \tag{14.3}$$

and where, by (12.4),

$$\gamma_{ij,e} = \gamma_{\ominus_{\mathrm{E}} A_{i,e} \oplus_{\mathrm{E}} A_{j,e}}, \tag{14.4}$$

$i, j = 1, 2, 3$, and $i < j$.

By the isomorphism studied in Sec. 9 we have the relationships

$$H_e = 2\otimes_{\mathrm{M}} H_m \tag{14.5}$$

and

$$A_{k,e} = 2\otimes_{\mathrm{M}} A_{k,m}, \tag{14.6}$$

$k = 1, 2, 3$. It follows from (14.6) that

$$\gamma_{A_{k,e}} = \gamma_{2\otimes_{\mathrm{M}} A_{k,m}} = 2\gamma^2_{A_{k,m}} - 1 \tag{14.7}$$

by (2.12), and

$$\gamma_{A_{k,e}}A_{k,e} = \gamma_{2\otimes_{\mathrm{M}} A_{k,m}}(2\otimes_{\mathrm{M}} A_{k,m}) = 2\gamma^2_{A_{k,m}}A_{k,m} \tag{14.8}$$

by (2.13), $k = 1, 2, 3$.

In (14.5) – (14.8) we have translated several terms of (14.1) from an Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$ into its corresponding Möbius gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$. It remains to translate the homogeneous gyrobarycentric coordinates $(m_{1,e} : m_{2,e} : m_{3,e})$ as well.

For the translation of the homogeneous gyrobarycentric coordinates $m_{k,e}$ into $m_{k,m}$, $k = 1, 2, 3$, we need the translation that we obtain in the following chain of equations, in which equalities are numbered for subsequent derivation.

$$
\begin{aligned}
\gamma_{12,e} &\overbrace{:=}^{(1)} \gamma_{\ominus_{\mathrm{E}} A_{1,e} \oplus_{\mathrm{E}} A_{2,e}} \\
&\overbrace{=\!=}^{(2)} \gamma_{\ominus_{\mathrm{E}}(2\otimes_{\mathrm{E}} A_{1,m}) \oplus_{\mathrm{E}} (2\otimes_{\mathrm{E}} A_{2,m})} \\
&\overbrace{=\!=}^{(3)} \gamma_{2\otimes_{\mathrm{M}}(\ominus_{\mathrm{M}} A_{1,m} \oplus_{\mathrm{M}} A_{2,m})} \\
&\overbrace{=\!=}^{(4)} 2\gamma^2_{\ominus_{\mathrm{M}} A_{1,m} \oplus_{\mathrm{M}} A_{2,m}} - 1 \\
&\overbrace{=:}^{(5)} 2\gamma^2_{12,m} - 1.
\end{aligned}
\tag{14.9}
$$

Derivation of the numbered equalities in (14.9) follows.

(1) Notation that follows from (12.4) with $\oplus = \oplus_{\mathrm{E}}$ into which the subscript "e" is introduced to emphasize that the equation under (1) is considered in an Einstein gyrovector space.
(2) Follows from (1) by the isomorphism studied in Sec. 9.
(3) Follows from (2) and the isomorphism between $\oplus_{\mathrm{E}}$ and $\oplus_{\mathrm{M}}$ in (9.1).
(4) Follows from (3) and the identity $\gamma_{2\otimes a} = 2\gamma_a^2 - 1$ in (2.12).
(5) Notation that follows from (12.4) with $\oplus = \oplus_{\mathrm{M}}$ into which the subscript "m" is introduced to emphasize that the equation under (5) is considered in a Möbius gyrovector space.

It follows from (13.2) and (14.9) that

$$
\begin{aligned}
C_{12,e} &:= \gamma_{13,e}\gamma_{23,e} - \gamma_{12,e} \\
&= (2\gamma^2_{13,m} - 1)(2\gamma^2_{23,m} - 1) - (2\gamma^2_{12,m} - 1) \\
&=: C_{12,m},
\end{aligned}
\tag{14.10}
$$

etc., so that

$$
\begin{aligned}
C_{12,e} &= C_{12,m} = (2\gamma^2_{13,m} - 1)(2\gamma^2_{23,m} - 1) - (2\gamma^2_{12,m} - 1), \\
C_{13,e} &= C_{13,m} = (2\gamma^2_{12,m} - 1)(2\gamma^2_{23,m} - 1) - (2\gamma^2_{13,m} - 1), \\
C_{23,e} &= C_{23,m} = (2\gamma^2_{12,m} - 1)(2\gamma^2_{13,m} - 1) - (2\gamma^2_{23,m} - 1),
\end{aligned}
\tag{14.11}
$$

where, following the notation introduced in (14.9), we use the notation

$$
\gamma_{ij,m} = \gamma_{\ominus_{\mathrm{M}} A_{i,m} \oplus_{\mathrm{M}} A_{j,m}},
\tag{14.12}
$$

$i, j = 1, 2, 3$, and $i < j$.

Finally, by (14.2) and (14.11),

$$
\begin{aligned}
m_{1,e} &= C_{12,e}C_{13,e} = C_{12,m}C_{13,m} =: m_{1,m}, \\
m_{2,e} &= C_{12,e}C_{23,e} = C_{12,m}C_{23,m} =: m_{2,m}, \\
m_{3,e} &= C_{13,e}C_{23,e} = C_{13,m}C_{23,m} =: m_{3,m}.
\end{aligned}
\tag{14.13}
$$

We are now in the position to rewrite the Einstein gyroorthocenter equation (14.1) by means of corresponding terms that involve the subscript "m" rather than "e". We thus substitute in (14.1):

(1) H_e from (14.5);
(2) $\gamma_{A_{k,e}}$, $k = 1, 2, 3$, from (14.7); and

(3) $\gamma_{A_{k,e}} A_{k,e}$, $k = 1, 2, 3$, from (14.8);
(4) $m_{k,e}$, $k = 1, 2, 3$, from (14.13).

These substitutions result in the Möbius gyroorthocenter equation,

$$2 \otimes_{\mathrm{M}} H_m = \frac{2m_{1,m}\gamma^2_{A_{1,m}} A_{1,m} + 2m_{2,m}\gamma^2_{A_{2,m}} A_{2,m} + 2m_{3,m}\gamma^2_{A_{3,m}} A_{3,m}}{2m_{1,m}\gamma^2_{A_{1,m}} + 2m_{2,m}\gamma^2_{A_{2,m}} + 2m_{3,m}\gamma^2_{A_{3,m}}}, \tag{14.14}$$

where H_m is the gyroorthocenter of a gyrotriangle $A_{1,m}A_{2,m}A_{3,m}$ in a Möbius gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$, and where the homogeneous gyrobarycentric coordinates $(m_{1,m} : m_{2,m} : m_{3,m})$ are given by (14.13) and (14.11).

To formalize the result in (14.14) we slightly rearrange Identity (14.14) and omit the subscript "m", obtaining the following theorem.

Theorem 14.1 (Möbius Gyrotriangle Gyroorthocenter)**.** *Let $A_1A_2A_3$ be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. The gyroorthocenter H of the gyrotriangle, Fig. 14.1, is given by the equation*

$$H = \tfrac{1}{2} \otimes \frac{m_1\gamma^2_{A_1} A_1 + m_2\gamma^2_{A_2} A_2 + m_3\gamma^2_{A_3} A_3}{m_1\gamma^2_{A_1} + m_2\gamma^2_{A_2} + m_3\gamma^2_{A_3} - \frac{1}{2}(m_1 + m_2 + m_3)}, \tag{14.15}$$

where the homogeneous gyrobarycentric coordinates $(m_1 : m_2 : m_3)$ are given by

$$\begin{aligned} m_1 &= C_{12}C_{13}, \\ m_2 &= C_{12}C_{23}, \\ m_3 &= C_{13}C_{23}, \end{aligned} \tag{14.16}$$

where

$$\begin{aligned} C_{12} &= (2\gamma^2_{13} - 1)(2\gamma^2_{23} - 1) - (2\gamma^2_{12} - 1), \\ C_{13} &= (2\gamma^2_{12} - 1)(2\gamma^2_{23} - 1) - (2\gamma^2_{13} - 1), \\ C_{23} &= (2\gamma^2_{12} - 1)(2\gamma^2_{13} - 1) - (2\gamma^2_{23} - 1), \end{aligned} \tag{14.17}$$

and where

$$\gamma_{ij} = \gamma_{\ominus A_i \oplus A_j}, \tag{14.18}$$

$i, j = 1, 2, 3$, *and* $i < j$.

A Möbius gyrotriangle gyroorthocenter, with position calculated by (14.15), is shown in Fig. 14.1.

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