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APPLICATIONS OF RELATIONS AND RELATORS IN THE EXTENSIONS OF STABILITY THEOREMS FOR HOMOGENEOUS AND ADDITIVE FUNCTIONS

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ABSTRACT. By working out an appropriate technique of relations and relators and extending the ideas of the direct methods of Z. Gajda and R. Ger, we prove some generalizations of the stability theorems of D. H. Hyers, T. Aoki, Th. M. Rassias and P. Gavruţă in terms of the existence and unicity of 2–homogeneous and additive approximate selections of generalized subadditive relations of semigroups to vector relator spaces. Thus, we obtain generalizations not only of the selection theorems of Z. Gajda and R. Ger, but also those of the present author.

Key words and phrases: Relations and relators, Homogeneity and additivity properties, Selection and stability theorems.

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1. INTRODUCTION

The first result on approximately additive functions was obtained by Pólya and Szegő [148, p. 17] who proved a somewhat different form of the following:

Theorem 1. *If* $(a_n)_{n=1}^{\infty}$ *is a sequence of real numbers such that*

$$\|a_{n+m} - a_n - a_m\| \leq 1$$

for all $n, m \in \mathbb{N}$*, then there exists a real number* ω *such that*

$$\|a_n - \omega n\| \leq 1$$

for all $n \in \mathbb{N}$*. Moreover,* $\omega = \lim_{n\to\infty} a_n/n$*.*

Remark 1. This theorem has been overlooked by several authors, from Hyers [90] to Maligranda [120]. It was first mentioned by Kuczma [110, p. 424] at the suggestion of R. Ger.

By Ger [77, p. 4], his attention to Theorem 1 was first drawn by M. Laczkovich who indicated that the real-valued particular case of Hyers's stability theorem can be easily derived from Theorem 1. D. H. Hyers, giving a partial answer to a general question of S. M. Ulam, proved a different form of the following stability theorem. The proof given by Hyers is more simple than the one given by Pólya and Szegő.

Theorem 2. *If* f *is an* ε*–approximately additive function of one Banach space* X *to another* Y*, for some* $\varepsilon \geq 0$*, in the sense that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$*, then there exists an additive function* g *of* X *to* Y *such that*

$$\|f(x) - g(x)\| \leq \varepsilon$$

for all $x \in X$*. Moreover,* $g(x) = \lim_{n\to\infty} f_n(x)$ *for all* $x \in X$*, where* $f_n(x) = 2^{-n} f\big(2^n x\big)$*.*

Remark 2. In this case, because $g(nx) = ng(x)$, we also have

$$\Big\|\frac{1}{n} f(nx) - g(x)\Big\| = \frac{1}{n}\big\|f(nx) - g(nx)\big\| \leq \frac{1}{n}\varepsilon$$

for all $x \in X$ and $n \in \mathbb{N}$. Therefore, in accordance with the result of Pólya and Szegő, we can also state that $g(x) = \lim_{n\to\infty} f(nx)/n$ for all $x \in X$.

However, the sequence $\big(f_n\big)_{n=1}^{\infty}$ applied by Hyers is usually more convenient. For instance, it can also be used to prove a similar theorem for a function f of X to Y which is only ε–approximately 2–homogeneous in the sense that $\|f(2x) - 2f(x)\| \leq \varepsilon$ for all $x \in X$. Moreover, it can also be shown that the Hyers sequence is actually rapidly uniformly convergent (see [212]).

Hyers's stability theorem has been generalized by several authors in various ways (see Hyers, Isac and Rassias [92]). For instance, Forti [53] remarked that for the majority of Hyers' theorem the domain X of f may be an arbitrary semigroup; only the additivity of the function g requires X to be commutative. Weaker sufficient conditions were also considered by Rätz [173] and Páles [136] (see also [223] and [233]). Moreover, Székelyhidi [216] noticed that the existence of an invariant mean is also sufficient.

Other general stability theorems, for additive and linear mappings, were also proved independently by Aoki [2] and Th. M. Rassias [157], respectively. They allowed the Cauchy difference $f(x+y) - f(x) - f(y)$ to be unbounded to the extent that

$$\|f(x+y) - f(x) - f(y)\| \leq M\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$ and some $M \geq 0$ and $0 \leq p < 1$. The paper of Aoki was overlooked from Bourgin [31] in 1951 to Maligranda [119] in 2006.

The results and problems of Th. M. Rassias motivated a number mathematicians to investigate the stability of various functional equations and inequalities. The interested reader can obtain a quick overview on the subject by consulting the surveys of Hyers and Rassias [94], Ger [77], Forti [54], Th. M. Rassias [161] and Székelyhidi [222], and the books of Hyers, Isac and Rassias [92], Jung [102] and Czerwik [44].

Bourgin [30, p. 224] already remarked that a direct generalization of Hyers's theorem can be obtained by replacing ε with a more general quantity $\psi(x, y)$. However, such a generalization of Hyers' theorem was only proved by Găvruţă [65] (for more general results, see [51] and [26]).

Following the approach of Th. M. Rassias, P. Găvruţă proved a different form of the following:

Theorem 3. *If f is a ψ–approximately additive function of a commutative group U to a Banach space X, for some function ψ of U^2 to X, in the sense that*

$$\|f(u+v) - f(u) - f(v)\| \leq \psi(u, v)$$

for all $u, v \in X$, and

$$S(u, v) = \sum_{n=0}^{\infty} \psi_n(u, v) < +\infty$$

for all $u, v \in U$, then there exists an additive function g of U to X such that

$$\|f(u) - g(u)\| \leq \frac{1}{2} S(u, u)$$

for all $u \in U$. Moreover, g is given by the same conditions as in Theorem 2.

Remark 3. Here, U can again be a commutative semigroup, and only the additivity of the function g requires the commutativity of U. Moreover, according to Bourgin [30, p. 224] and Ger [77, p. 19], who attributes the result to G. L. Forti and Z. Kominek, the above condition on ψ can also be weakened. However, the existence of the limit considered in Remark 2 and the uniform convergence of the Hyers sequence cannot yet be stated. Hyers' stability theorem has also been generalized in terms of selections of set-valued functions. First of all, if f and g are as in Theorem 2, then by taking

$$A = \{u \in X : \ \|u\| \leq \varepsilon\},$$

W. Smajdor [187] and Gajda and Ger [61] noticed that

$$g(x) - f(x) \in A \qquad \text{and} \qquad f(x+y) - f(x) - f(y) \in A.$$

Hence

$$g(x) \in f(x) + A \qquad \text{and} \qquad f(x+y) \in f(x) + f(y) + A$$

for all $x, y \in X$. Therefore, by defining

$$F(x) = f(x) + A$$

for all $x \in X$, we can obtain a multifunction F of X to Y such that g is a selection of F and F is subadditive. That is,

$$g(x) \in F(x) \qquad \text{and} \qquad F(x+y) \subset F(x) + F(y)$$

for all $x, y \in X$. Therefore, Theorem 2 is essentially a statement of the existence of a unique additive selection of a certain subadditive multifunction (the importance of the above observation was also recognized by Th. M. Rassias [160]).

Z. Gajda and R. Ger proved a different form of the following.

Theorem 4. *If F is a convex and closed valued subadditive multifunction of a commutative semigroup U to a Banach space X such that*

$$\sup_{u\in U} \operatorname{diam}\big(F(u)\big) < +\infty,$$

then F has an additive selection f. Moreover, $\{f(u)\} = \bigcap_{n=1}^{\infty} F_n(u)$ for all $u \in U$, where $F_n(u) = 2^{-n} F\big(2^n u\big)$.

Remark 4. In the same paper, they also proved an extension of the above theorem to a sequentially complete topological vector space X by introducing the notion of the diameter of a subset of X relative to a balanced neighborhood of the origin in X (see also [59]).

Analogous to Hyers' stability theorem, the Hahn–Banach extension theorems can also be generalized in terms of selections of set-valued functions. This also yields stability theorems for additive functions. For some ideas, see [174], [95], [63], [3], [189], and [198].

Moreover, Páles [135] and Badora [8] showed that generalizations of Hyers' stability theorem can also be proved with the help of separation and extension theorems (for the same purposes, fixed point theorems were also used by V. Radu [153] and L. Cădariu and V. Radu [38, 39]). However, the existence of additive selections of set-valued functions has mainly been investigated with the help of generalized invariant means. See, for instance, [60], [6], and [9] (generalizations to monomial and multimonomial selections were obtained by J. J. Tabor [226], and Badora, Páles and Székelyhidi [10]).

On the other hand, by extending the direct methods, the results of Gajda and Ger [61] have also been generalized by Popa [149, 151] and Száz [208]. By using relations and relators instead of set-valued functions and topologies, we have proved the following extension of Theorem 4.

Theorem 5. *If F is a closed-valued, 2–subhomogeneous, subadditive relation of a commutative semigroup U to a separated, sequentially complete vector relator space $X(\mathcal{R})$ such that the sequence $\big(F_n(u)\big)_{n=0}^{\infty}$ is infinitesimal for all $u \in U$, then F has an additive selection f. Moreover, f is given by the same formula as in Theorem 4.*

Remark 5. Here, $\mathcal{R}$ is a nonvoid family of relations on the vector space X which is, to some extent, compatible with the linear operations in X. And the infinitesimality of the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ of subsets of $X(\mathcal{R})$ means only that for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $F_n(u) \subset R(x)$.

In the present paper, by working out a more delicate technique using relations and relators, we shall prove a similar generalization of Găvruţă's stability theorem. Actually, we will transform the results of Gselmann and Száz [88] to relations and relators.

Here, because of the technical difficulties, we can only note that if f and ψ are as in Theorem 3, then by defining

$$\Psi(u,v) = \big\{x \in X : \quad \|x\| \leq \psi(u,v)\big\}$$

for all $u, v \in U$, we have $f(u+v) - f(u) - f(v) \in \Psi(u,v)$, and hence

$$f(u+v) \in f(u) + f(v) + \Psi(u,v)$$

for all $u, v \in U$.

Therefore, a relation F on one semigroup U to another V is called Ψ–approximately subadditive, for some relation Ψ of U^2 to V, if

$$F(u+v) \subset F(u) + F(v) + \Psi(u,v)$$

for all $u, v \in U$. This definition is a natural generalization of that of the subadditive relations. Therefore, it will allow us to prove a common generalization of Theorems 3 and 5.

The existence and unicity theorems of additive selections can again be proved with the help of those of 2–homogeneous selections. Namely, if F is a midconvex-valued Ψ–approximately subadditive relation of a semigroup U to a vector space X, then

$$\begin{aligned} F(2u) &= F(u+u) \subset F(u)+F(u)+\Psi(u,u) \\ &= 2\left(\frac{1}{2}F(u)+\frac{1}{2}F(u)\right)+\Psi(u,u) \subset 2F(u)+\Psi(u,u) \end{aligned}$$

for all $u \in U$. Therefore, by defining

$$\Phi(u) = \Psi(u,u)$$

for all $u \in U$, we can observe that F is Φ–approximately 2-subhomogeneous in the sense that

$$F(2u) \subset 2F(u) + \Phi(u)$$

for all $u \in U$.

The only prerequisites for understanding our forthcoming selection theorems is a familiarity with several old and new results on relations and relators. These will be listed in a rather systematic way in the sections below. However, the proofs are frequently omitted or only outlined. The interested reader can find some of them in our former papers. However, the present terminology and notation may differ from the earlier ones. Moreover, the classification of relators offered here is still not completely satisfactory.

2. Relations and Functions

A subset F of a product set $U \times V$ is called a relation on U to V. If $F \subset U^2$, then we may simply say that F is a relation on U. Thus, $\Delta_U = \{(u,u): \ u \in U\}$ is a relation on U.

If F is a relation on U to V, then for any $u \in U$ and $A \subset U$ the sets $F(u) = \{v \in V: (u,v) \in F\}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the images of u and A under F, respectively.

Moreover, the sets $D_F = \{u \in U: \ F(u) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If $D_F = U$ ($R_F = V$), then we say that F is a relation of U to V (on U onto V).

If F is a relation on U to V, then the values $F(u)$, where $u \in U$, uniquely determine F since we have $F = \bigcup_{u \in U} \{u\} \times F(u)$. Therefore, the inverse relation F^{-1} can be defined such that $F^{-1}(v) = \{u \in U: \ v \in F(u)\}$ for all $v \in V$.

Moreover, if G is a relation on V to W, then the composition relation $G \circ F$ can be defined such that $(G \circ F)(u) = G[F(u)]$ for all $u \in U$. Thus, we also have $(G \circ F)[A] = G[F[A]]$ for all $A \subset U$.

Additionally, if G is a relation on W to Ω, then the box product relation $F \boxtimes G$ can be defined such that $(F \boxtimes G)(u,w) = F(u) \times G(w)$ for all $u \in U$ and $w \in W$. Thus, we have $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subset U \times W$.

A relation R on U is called reflexive, symmetric, and transitive if $\Delta_U \subset R$, $R^{-1} \subset R$, and $R \circ R \subset R$, respectively. Moreover, a reflexive relation is called a tolerance (preorder) relation if it is symmetric (transitive).

If R is a relation on U, then we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_U$. Moreover, we also write $R^\infty = \bigcup_{n=0}^{\infty} R^n$. Thus, R^∞ is the smallest preorder relation on U such that $R \subset R^\infty$.

A relation f on U to V is called a function if for each $u \in D_f$ there exists $v \in V$ such that $f(u) = \{v\}$. In this case, by identifying singletons with their elements, we may write $f(u) = v$ in place of $f(u) = \{v\}$.

If F is a relation on U to V, then a function f of D_F to V is called a selection of F if $f \subset F$, i. e., $f(u) \in F(u)$ for all $u \in D_F$. Thus, the axiom of choice can be briefly expressed by saying that every relation has a selection.

If F is a relation on U to V and $A_i \subset U$ for all $i \in I$, then in general we only have $F\left[\bigcup_{i\in I} A_i\right] = \bigcup_{i\in I} F[A_i]$. However, if f is a function, then all set-theoretic operations are preserved under the relation f^{-1}.

A function a of the set $\mathbb{N}$ of all natural numbers to X is called a sequence in X. In this case, we usually write a_n, $(a_n)_{n=1}^{\infty}$, and $\{a_n\}_{n=1}^{\infty}$ in place of $a(n)$, a, and R_a, respectively.

If $(a_n)_{n=1}^{\infty}$ is a sequence in the set $\overline{\mathbb{R}}$ of all extended real numbers, then the extended real numbers

$$\underline{\lim}_{n\to\infty} a_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} a_k \quad \text{and} \quad \overline{\lim}_{n\to\infty} a_n = \inf_{n\in\mathbb{N}} \sup_{k\geq n} a_k$$

are called the lower and upper limits of the sequence $(a_n)_{n=1}^{\infty}$.

Similarly, if $(A_n)_{n=1}^{\infty}$ is a sequence in the family $\mathcal{P}(X)$ of all subsets of X, then the sets

$$\underline{\lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \quad \text{and} \quad \overline{\lim}_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

are called the set-theoretic lower and upper limits of the sequence $(A_n)_{n=1}^{\infty}$.

A function d of X^2 to $[0, +\infty]$ is called a distance function on X. The distance function d is said to be quasi-semimetric if $d(x,x) = 0$, $d(x,y) < +\infty$, and $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

If d is a distance function on X, then for any $r > 0$ the relations $B_r^d = \{(x,y) \in X^2 : d(x,y) < r\}$ and $\bar{B}_r^d = \{(x,y) \in X^2 : \ d(x,y) \leq r\}$ are called the r-sized open and closed d-surroundings in X.

The surroundings B_r^d and $\bar{B}_r^d$ are, in general, only tolerance relations on X even if d is a metric, while the Pervin relations $R_A = A^2 \cup A^c \times X$, where $A \subset X$ and $A^c = X \setminus A$, are only preorder relations on X.

If d is a distance function on X, then for any $A, B \subset X$ we write $\underline{d}(A,B) = \inf\{d(a,b) : a \in A, \ b \in B\}$ and $\overline{d}(A,B) = \sup\{d(a,b) : \ a \in A, \ b \in B\}$. Moreover, in particular, we also write $d(A) = \overline{d}(A,A)$.

The surroundings B_r^d and $\bar{B}_r^d$ are usually more convenient means than the distance function d. However, the distances $\underline{d}(A,B)$ and $\overline{d}(A,B)$, and the diameter $d(A)$, cannot be directly expressed in terms of the surroundings.

3. SEMIGROUPS AND VECTOR SPACES

Definition 3.1. If U is a nonvoid set, then a function $+$ of U^2 to U is called an operation in U. And the ordered pair $U(+) = (U,+)$ is called a groupoid.

Remark 3.1. In this case, we may simply write $u + v$ in place of $+(u,v)$ for all $u, v \in U$. Moreover, we may also simply write U in place of $U(+)$.

Instead of groupoids, it is usually sufficient to consider only semigroups. However, several definitions on semigroups can be naturally extended to groupoids.

Definition 3.2. If U is a groupoid and $u \in U$, then we define $1u = u$. Moreover, if $n \in \mathbb{N}$ such that nu is already defined, then

$$(n+1)u = nu + u.$$

By induction, we can easily prove the following theorems.

Theorem 3.1. *If U is a semigroup, then for any $u \in U$ and $m, n \in \mathbb{N}$ we have*

(1) $(m+n)u = mu + nu$,
(2) $(nm)u = n(mu)$.

Proof. To prove (2), note that $(1m)u = mu = m(1u)$. Moreover, if (2) holds, then by (1) we also have

$$\big((n+1)m\big)u = (nm+m)u = (nm)u + mu = n(mu) + mu = (n+1)(mu).$$

∎

Theorem 3.2. *If U is a commutative semigroup, then for any $u, v \in X$ and $n \in \mathbb{N}$ we have*

$$n(u+v) = nu + nv.$$

Remark 3.2. A commutative group U can be made a module over the ring $\mathbb{Z}$ of integers by using the definitions $0u = 0$ and $(-n)u = -(nu)$ for all $u \in U$ and $n \in \mathbb{N}$.

Moreover, U can be naturally extended to a vector space V over the field $\mathbb{Q}$ of rationals by using the quotients $u/k = \{(l, v) \in \mathbb{Z} \times U : \quad lu = kv\}$ with $u \in U$ and $0 \neq k \in \mathbb{Z}$.

Remark 3.3. In the sequel, $\mathbb{K}$ will denote any one of the number fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Moreover, we shall only consider vector spaces over $\mathbb{K}$.

Note that if X is a vector space then $1x = x$ and $(n+1)x = nx + x$. Moreover, $0x = 0$ and $(-n)x = -nx$ for all $x \in X$ and $n \in \mathbb{N}$. Therefore, the two possible definitions for kx, with $k \in \mathbb{Z}$ and $x \in X$, coincide.

Definition 3.3. If U is a groupoid, then for any $A, B \subset U$ and $n \in \mathbb{N}$ we define

$$A + B = \big\{a + b : \quad a \in A, \ b \in B\big\} \qquad \text{and} \qquad nA = \big\{na : \quad a \in A\big\}.$$

Remark 3.4. If U is a group, then for any $A \subset U$ and $k \in \mathbb{Z}$ we also define $kA = \big\{ka : a \in A\big\}$.

Also, for any $A, B \subset U$, we write $-A = (-1)A$ and $A - B = A + (-B)$ despite the fact that $\mathcal{P}(U)$ is only a semigroup with zero element.

Remark 3.5. Moreover, if X is a vector space over $\mathbb{K}$, then for any $A \subset X$ and $\lambda \in \mathbb{K}$ we also define $\lambda A = \big\{\lambda x : \quad x \in A\big\}$.

Thus, only two axioms of a vector space fail to hold for $\mathcal{P}(X)$. Namely, only the one point subsets of X can have additive inverses. Moreover, in general we only have $(\lambda + \mu)A \subset \lambda A + \mu A$.

4. Additivity Properties of Relations

Definition 4.1. A relation R on a groupoid U is called a translation relation if

$$u + R(v) \subset R(u+v)$$

for all $u, v \in U$ (actually, it should be called a left translation relation).

Remark 4.1. By using the global addition of relations, the above condition can be briefly expressed by writing $\Delta_U \oplus R \subset R$.

Moreover, the latter inclusion can be expressed by saying that vRw implies $(u+v)R(u+w)$ for all $u \in U$. Thus, the usual order relation of $\mathbb{R}$ is a translation relation.

By using the corresponding definitions, we can easily prove the following

Theorem 4.1. *If R is a relation on a group U, then the following assertions are equivalent:*

(1) *R is a translation relation;*
(2) *$R(u) = u + R(0)$ for all $u \in U$;*

(3) $R(u+v) = u + R(v)$ *for all* $u, v \in U$;
(4) $R(u+v) \subset u + R(v)$ *for all* $u, v \in U$.

Proof. For instance, if (4) holds, then

$$R(u) = R(u+0) \subset u + R(0) = u + R(-u+u) \subset u - u + R(u) = R(u)$$

for all $u \in U$. Therefore, (2) also holds. ∎

From the above theorem, it is clear that we have:

Corollary 4.2. *If R is a nonvoid translation relation on a group U, then* $U = D_R$.

Definition 4.2. A relation F on one groupoid U to another V is called
(1) subadditive if $F(u+v) \subset F(u) + F(v)$ for all $u, v \in U$;
(2) superadditive if $F(u) + F(v) \subset F(u+v)$ for all $u, v \in U$.

Remark 4.2. By using the global addition of relations, the superadditivity of F can be briefly expressed by $F \oplus F \subset F$. That is, F is a subgroupoid of $U \times V$.

Moreover, the latter inclusion can be expressed by saying that uFw and vFz imply $(u+v)F(w+z)$. Thus, the usual order relation of $\mathbb{R}$ is a superadditive relation.

Remark 4.3. It is also worth noting that if R is a reflexive superadditive relation on a groupoid U, then R is a translation relation on U.

Definition 4.3. A relation F on one group U to another V is called:
(1) odd if $-F(u) \subset F(-u)$ for all $u \in D_F$;
(2) quasi-odd if $-F(u) \cap F(-u) \neq \emptyset$ for all $u \in D_F$.

Remark 4.4. Note that if F is an odd relation on U to V, then for any $u \in U$ we have $-F(-u) \subset F(u)$, and hence $F(-u) \subset -F(u)$. Therefore, the corresponding equality is also true.

Remark 4.5. By using the global negative of relations, the oddness of F can be briefly expressed by writing $\ominus F \subset F$, or equivalently $F = \ominus F$.

Moreover, the above inclusion can be expressed by saying that uFv implies $(-u)F(-v)$. Thus, the usual order relation of $\mathbb{R}$ is not odd. However, from the following theorem it is clear that it is quasi-odd.

Theorem 4.3. *If F is a relation on one group X to another Y, then the following assertions are equivalent:*
(1) *F is quasi-odd;*
(2) $0 \in F(u) + F(-u)$ *for all* $u \in D_F$.

Proof. For instance, if (1) holds, then for each $u \in D_F$ there exists $v \in F(-u)$ such that $v \in -F(u)$, and thus $-v \in F(u)$. Hence, it is clear that $0 = -v + v \in F(u) + F(-u)$. Therefore, (2) also holds. ∎

Hence, we also have:

Corollary 4.4. *If R is a reflexive relation on a group X, then R is quasi-odd.*

Moreover, by using Theorem 4.3, we can prove the following theorems.

Theorem 4.5. *If F is a subadditive relation on one group U to another V such that* $0 \in F(0)$, *then F is quasi-odd and* $U = D_F$.

Proof. Namely, for any $u \in U$, we have

$$0 \in F(0) = F(u - u) \subset F(u) + F(-u).$$

Thus, $F(u) \neq \emptyset$. Moreover, Theorem 4.3 can be applied. ∎

Theorem 4.6. *If F is a quasi-odd superadditive relation of one group U to another V, then F is subadditive and $0 \in F(0)$.*

Proof. If $u, v \in U$, then by Theorem 4.3 and the superadditivity of F we have

$$0 \in F(u) + F(-u) \subset F(u - u) = F(0)$$

and

$$F(u+v) \subset F(u) + F(-u) + F(u+v) \subset F(u) + F(-u+u+v) = F(u) + F(v).$$

∎

Remark 4.6. If $U = D_F$ is not assumed, then instead of the subadditivity of F we can only prove that F is quasi-subadditive in the sense that $F(u+v) \subset F(u) + F(v)$ for all $u, v \in U$ with either $u \in D_F$ or $v \in D_F$.

Definition 4.4. A selection f of a relation F on one group U to another V is called odd-like if $-f(u) \in F(-u)$ for all $u \in D_F$.

Remark 4.7. Note that if f is an odd selection of F, then $-f(u) = f(-u) \in F(-u)$ for all $u \in D_F$. Therefore, f is odd-like.

Moreover, if f is a selection of an odd relation F, then $-f(u) \in -F(u) = F(-u)$ for all $u \in D_F$. Therefore, f is again odd-like.

Now, in addition to Theorem 4.3, we can establish the following.

Theorem 4.7. *If F is a relation on one group U to another V, then the following assertions are equivalent:*

(1) *F is quasi-odd;*
(2) *F has an odd-like selection.*

Proof. For instance, if (1) holds, then $-F(u) \cap F(-u) \neq \emptyset$, and hence $F(u) \cap \big(-F(-u)\big) \neq \emptyset$ for all $u \in D_F$. Thus, by the axiom of choice, there exists a function f of D_F to V such that $f(u) \in F(u) \cap \big(-F(-u)\big)$, and hence $f(u) \in F(u)$ and $-f(u) \in F(-u)$ for all $u \in D_F$. Therefore, f is an odd-like selection of F, and thus (2) also holds. ∎

Definition 4.5. A selection f of a relation F on one groupoid U with zero to an arbitrary groupoid V is called a representing selection of F if $F(u) = f(u) + F(0)$ for all $u \in D_F$.

Moreover, a representing selection f of F is called normal if $F(0) + f(u) = f(u) + F(0)$ for all $u \in D_F$.

Remark 4.8. If R is a translation relation on a group U such that $0 \in R(0)$, then by Theorem 4.1 the identity function Δ_U is a representing selection of R.

Moreover, if the translation relation R is normal in the sense that $u + R(0) = R(0) + u$ for all $u \in U$, then Δ_U is already a normal representing selection of R.

Now, analogously to Theorem 4.1, we can prove the following.

Theorem 4.8. *If F is a superadditive relation on one group U to another V and f is an odd-like selection of F, then f is a normal representing selection of F.*

Proof. For any $u \in D_F$, we have

$$\begin{aligned} f(u) + F(0) \subset F(u) + F(0) \subset F(u) \\ = f(u) - f(u) + F(u) \subset f(u) + F(-u) + F(u) \subset f(u) + F(0). \end{aligned}$$

Therefore, $F(u) = f(u) + F(0)$. Moreover, we can similarly see that $F(u) = F(0) + f(u)$ is also true. ∎

Remark 4.9. Note that if f is a representing selection of a superadditive relation F on a groupoid U with zero to an arbitrary one V, then

$$F(0) + f(u) \subset F(0) + F(u) \subset F(u) = f(u) + F(0)$$

for all $u \in D_F$.

Hence, if V is a group and for each $u \in D_F$ there exists $v \in D_F$ such that $f(v) = -f(u)$, then we can infer that

$$\begin{aligned} f(u) + F(0) &= f(u) + F(0) + f(v) + f(u) \subset f(u) + f(v) + F(0) + f(u) \\ &= F(0) + f(u). \end{aligned}$$

Therefore, the representing selection f is normal.

5. FURTHER RESULTS ON TRANSLATION RELATIONS

Theorem 5.1. *If R is a translation relation on U, then R^{-1} is also a translation relation on U.*

Proof. If $u, v \in U$ and $z \in R^{-1}(v)$, then $v \in R(z)$ and $u + v \in u + R(z) \subset R(u + z)$, and hence $u + z \in R^{-1}(u + v)$. Therefore, $u + R^{-1}(v) \subset R^{-1}(u + v)$. ∎

Theorem 5.2. *If R and S are translation relations on U, then $S \circ R$ is also a translation relation on U.*

Proof. It can be easily seen that, for any $u, v \in U$, we have

$$\begin{aligned} u + (S \circ R)(v) &= u + S[R(v)] \subset S[u + R(v)] \subset S[R(u + v)] \\ &= (S \circ R)(u + v). \end{aligned}$$

∎

Remark 5.1. Unfortunately, the pointwise linear operations lead out from the family of all translation relations on a vector space X.

In addition to the above theorems, we can prove the following theorems.

Theorem 5.3. *If R and S are translation relations on U, then $R \setminus S$ is also a translation relation on U.*

Corollary 5.4. *If R is a translation relation on U, then its complement R^c is also a translation relation on U.*

Theorem 5.5. *If R_i is a translation relation on U for all $i \in I$, then $\bigcap_{i \in I} R_i$ and $\bigcup_{i \in I} R_i$ are also translation relations on U.*

Hence, we also have:

Corollary 5.6. *If R is a relation on a groupoid U, then there exists a smallest translation relation $R^\top$ on U such that $R \subset R^\top$.*

Moreover, by using Theorems 5.2 and 5.5, we can also easily establish:

Theorem 5.7. *If R is translation relation on U, then the generated preorder R^{∞} is also a translation relation on U.*

By using Theorem 4.1, in addition to Theorems 5.1 and 5.2, we can prove the following theorems.

Theorem 5.8. *If R is a translation relation on a group U, then*

$$R^{-1}(0) = -R(0).$$

Proof. By Theorems 5.1 and 4.1, we have

$$u \in R^{-1}(0) \iff 0 \in R(u) \iff 0 \in u + R(0) \iff -u \in R(0) \iff u \in -R(0)$$

for any $u \in U$. Therefore, the required assertion is also true. ∎

Corollary 5.9. *If R is a normal translation relation on a group U, then for any $u \in U$ we have*

$$R^{-1}(u) = -R(-u).$$

Proof. By Theorems 5.1, 4.1 and 5.8 and the normality of $R(0)$, we have

$$\begin{aligned} R^{-1}(u) &= u + R^{-1}(0) = u - R(0) = -\big(R(0) - u\big) \\ &= -\big(-u + R(0)\big) = -R(-u). \end{aligned}$$

∎

Theorem 5.10. *If R is an arbitrary translation relation and S is a translation relation on a group U, then*

$$(S \circ R)(0) = R(0) + S(0).$$

Proof. By using Theorem 4.1, we can easily see that

$$(S \circ R)(0) = S[R(0)] = S[R(0) + 0] = R(0) + S(0).$$

∎

Corollary 5.11. *If R and S are translation relations on a group U, then for any $u \in U$ we have*

$$(S \circ R)(u) = R(u) + S(0).$$

Theorem 5.12. *If R is a normal translation relation and S is an arbitrary translation relation on a group U, then for any $u, v \in U$ we have*

$$(S \circ R)(u + v) = R(u) + S(v).$$

Proof. By Theorems 5.2, 4.1 and 5.10 and the normality of $R(0)$, we have

$$\begin{aligned} (S \circ R)(u + v) &= u + v + (S \circ R)(0) \\ &= u + v + R(0) + S(0) \\ &= u + R(0) + v + S(0) = R(u) + S(v). \end{aligned}$$

∎

Corollary 5.13. *If R and S are as in the above theorem, then $S \circ R = R \circ S$.*

Proof. Namely, by Theorem 5.12 and the normality of $R(0)$ and Corollary 5.11, we have

$$(S \circ R)(u) = (S \circ R)(0 + u) = R(0) + S(u) = S(u) + R(0) = (R \circ S)(u)$$

for all $u \in U$. Therefore, the required assertion is also true. ∎

By using our former results, we can also easily prove the following theorems.

Theorem 5.14. *If R, S and T are translation relations on a group U, then*
(1) $R \subset S \iff R(0) \subset S(0)$;
(2) $R^{-1} \subset S \iff -R(0) \subset S(0)$;
(3) $S \circ R \subset T \iff R(0) + S(0) \subset T(0)$;
(4) $R \subset S \cap T \iff R(0) \subset S(0) \cap T(0)$.

Corollary 5.15. *If R is a translation relation on a group U, then*
(1) *R is reflexive* $\iff 0 \in R(0)$;
(2) *R is symmetric* $\iff -R(0) \subset R(0)$;
(3) *R is transitive* $\iff R(0) + R(0) \subset R(0)$;
(4) *R is antisymmetric* $\iff -R(0) \cap R(0) \subset \{0\}$.

Theorem 5.16. *If R is a normal translation relation on a group U, then*
(1) *R is odd* $\iff -R(0) \subset R(0)$;
(2) *R is quasi-odd* $\iff -R(0) \cap R(0) \neq \emptyset$;
(3) *R is subadditive* $\iff R(0) \subset R(0) + R(0)$;
(4) *R is superadditive* $\iff R(0) + R(0) \subset R(0)$.

Remark 5.2. Corollary 5.15 and Theorem 5.16 show that a normal translation relation on a group is odd (superadditive) if and only if it is symmetric (transitive).

From Theorem 5.10, we obtain the following.

Theorem 5.17. *If R is a translation relation on a group U, then*

$$R^{\infty}(0) = \{0\} \cup \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} R(0).$$

Proof. From Theorem 5.10, by induction, we obtain $R^n(0) = \sum_{k=1}^{n} R(0)$ for all $n \in \mathbb{N}$. Hence, by the corresponding definitions, it is clear that

$$R^{\infty}(0) = \left(\bigcup_{n=0}^{\infty} R^n \right)(0) = \bigcup_{n=0}^{\infty} R^n(0)$$
$$= R^0(0) \cup \bigcup_{n=1}^{\infty} R^n(0) = \{0\} \cup \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} R(0).$$

∎

Now, by calling a translation relation R on a group U absorbing if $R(0)$ is an absorbing subset of U in the sense $U = \bigcup_{n=1}^{\infty} nR(0)$, we can easily establish that:

Corollary 5.18. *If R is an absorbing translation relation on a group U, then R is well-chained in the sense that $R^{\infty} = U^2$.*

Proof. By using Theorem 5.17, we can see that

$$U = \bigcup_{n=1}^{\infty} nR(0) \subset \bigcup_{n=1}^{\infty} \sum_{k=1}^{n} R(0) \subset R^{\infty}(0),$$

and thus $R^{\infty}(0) = U$. Hence, by Theorems 5.7 and 4.1, it follows that

$$R^{\infty}(u) = u + R^{\infty}(0) = u + U = U$$

for all $u \in U$. Therefore, the required assertion is also true. ∎

Moreover, as direct consequences of the latter corollary, we can also state:

Corollary 5.19. *If R is an absorbing translation relation on a group U, then $R[A] \subset A$ implies $A \in \{\emptyset, U\}$ for all $A \subset U$.*

Corollary 5.20. *If R is a transitive, absorbing translation relation on a group U, then $R = X^2$.*

6. Homogeneity Properties of Relations

Definition 6.1. A relation F on one vector space X to another Y over $\mathbb{K}$ is called:

(1) homogeneous if $\lambda F(x) \subset F(\lambda x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$;
(2) balanced if $\lambda F(x) \subset F(\lambda x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;
(3) convex if $\lambda F(x) + (1-\lambda)F(y) \subset F\big(\lambda x + (1-\lambda)y\big)$ for all $x, y \in X$ and $\lambda \in [0,1]$.

Remark 6.1. Note that thus a subset A of X is homogeneous, balanced, and convex, respectively, if and only if the relation $F = X \times A$ has the corresponding property.

Moreover, it can be easily seen that a relation F on X to Y is homogeneous, balanced, and convex if and only if it is a homogeneous, balanced, and convex subset of the product space $X \times Y$, respectively.

By using the corresponding definitions, we can prove the following theorems.

Theorem 6.1. *If F is a nonvoid, homogeneous relation on one vector space X to another Y over $\mathbb{K}$, then $0 \in F(0)$, and for any $x \in X$ and $\lambda \in \mathbb{K}$, with $\lambda \neq 0$, we have*

$$F(\lambda x) = \lambda F(x).$$

Proof. Note that $0 \in 0F(u) \subset F(0u) = F(0)$ for all $u \in D_F$. Moreover, $\lambda^{-1}F(\lambda x) \subset F\big(\lambda^{-1}\lambda x\big) = F(x)$, and thus $F(\lambda x) \subset \lambda F(x)$ also holds. ∎

Theorem 6.2. *If F is a balanced relation on one vector space X to another Y over $\mathbb{K}$, then for any $x \in X$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda| \leq |\mu|$, we have*

$$\lambda F(\mu x) \subset \mu F(\lambda x).$$

Proof. If in addition to the above conditions we have $\mu \neq 0$, then

$$\lambda F(\mu x) = \mu \lambda \mu^{-1} F(\mu x) \subset \mu F\big(\lambda \mu^{-1} \mu A\big) = \mu F(\lambda A),$$

while if $\mu = 0$, then we also have $\lambda = 0$. Therefore, the required inclusion is again true. ∎

From this theorem, by using $|\lambda| \leq ||\lambda|| \leq |\lambda|$, we can immediately obtain:

Corollary 6.3. *If F is a balanced relation on one vector space X to another Y over $\mathbb{K}$, then for any $x \in X$ and $\lambda \in \mathbb{K}$,*

$$\lambda F(|\lambda| x) = |\lambda| F(\lambda x).$$

Theorem 6.4. *If F is a convex relation on one vector space X to another Y over $\mathbb{K}$, then for any $x, y \in X$ and $\lambda, \mu \in \mathbb{K}$, with $\lambda, \mu \geq 0$ and $\lambda + \mu \neq 0$, we obtain*

$$\lambda F(x) + \mu F(y) \subset (\lambda + \mu) F\big(\lambda(\lambda+\mu)^{-1} x + \mu(\lambda+\mu)^{-1} y\big).$$

Hence, we also have:

Corollary 6.5. *If F is a convex relation on one vector space X to another Y over $\mathbb{K}$, then for any $x \in X$ and $\lambda, \mu \in \mathbb{K}$, with $\lambda, \mu \geq 0$,*

$$(\lambda + \mu) F(x) = \lambda F(x) + \mu F(x).$$

Remark 6.2. Note that, by using Remark 6.1, several useful properties of balanced and convex sets can be immediately derived from the above results on balanced and convex relations.

By using Theorem 4.1 and the corresponding definitions, we can easily prove the following counterpart of Corollary 5.15 and Theorem 5.16.

Theorem 6.6. *If R is a translation relation on a vector space X over $\mathbb{K}$, then*

(1) *R is convex $\iff$ $R(0)$ is convex;*
(2) *R is balanced $\iff$ $R(0)$ is balanced;*
(3) *R is homogeneous $\iff$ $R(0)$ is homogeneous.*

To provide a natural example for balanced translation relations, we shall use an important generalization of the notion of seminorms.

Definition 6.2. A function p of a vector space X over $\mathbb{K}$ to $\mathbb{R}$ is called a preseminorm if

(1) $\lim\limits_{\lambda \to 0} p(\lambda x) = 0$ for all $x \in X$;
(2) $p(\lambda x) \leq p(x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;
(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$.

Remark 6.3. Thus, we have $p(0) = \lim\limits_{\lambda \to 0} p(0) = \lim\limits_{\lambda \to 0} p(\lambda 0) = 0$. Moreover,

$$0 = p(0) = p(x-x) \leq p(x) + p(-x) \leq p(x) + p(x) = 2p(x),$$

and hence $0 \leq p(x)$ for all $x \in X$.

Now, by Definition 6.2, we also have the following:

Theorem 6.7. *If p is a preseminorm on a vector space X over $\mathbb{K}$, then the function $d = d_p$, defined by*

$$d_p(x, y) = p(x-y)$$

for all $x, y \in X$, is a semimetric on X such that

(1) $\lim\limits_{\lambda \to 0} d(\lambda x, \lambda y) = 0$ *for all* $x, y \in X$;
(2) $d(\lambda x, \lambda y) \leq d(x, y)$ *for all* $x, y \in X$ *and* $\lambda \in \mathbb{K}$ *with* $|\lambda| \leq 1$;
(3) $d(x+y, z+w) \leq d(x, z) + d(y, w)$ *for all* $x, y, z, w \in X$.

Remark 6.4. If p is a seminorm on X, then for any $x \in X$ and $\lambda \in \mathbb{K}$ we simply have

$$d(\lambda x, \lambda y) = |\lambda| d(x, y).$$

Definition 6.3. If p a preseminom on X, then by the above theorem we may write

$$B_r^p = B_r^{d_p} \qquad \text{and} \qquad \bar{B}_r^p = \bar{B}_r^{d_p}$$

for all $r > 0$.

The open and closed surroundings have almost the same properties. Therefore, in the following two simple theorems, we shall only list those of the open ones.

Theorem 6.8. *If p is a preseminorm on X, then $B_r = B_r^p$ is a tolerance relation on X such that, for any $r, s > 0$, we have*

(1) $B_{r \wedge s} = B_r \cap B_s$;
(2) $B_r \circ B_s \subset B_{r+s}$.

Remark 6.5. Moreover, it is worth noting that if $x \in X$ and $y \in B_r(x)$ then $s = r - d(x, y) > 0$ and $B_s(y) \subset B_r(x)$.

Theorem 6.9. *If p is a preseminorm on X, then $B_r = B_r^p$ is an absorbing, balanced translation relation on X such that, for any $x, y \in X$ and $r, s > 0$, we have*

$$B_r(x) + B_s(y) \subset B_{r+s}(x+y).$$

Remark 6.6. If p is a seminorm on a vector space X over $\mathbb{K}$, then we can also state that $B_r = B_r^p$ is a convex relation on X such that, for any $x \in X$ and $\lambda \in \mathbb{K}$, with $\lambda \neq 0$, we have

$$\lambda B_r(x) = B_{|\lambda| r}(\lambda x).$$

7. A Few Basic Facts on Relators

Definition 7.1. If $\mathcal{R}$ is a family of relations on a set X, then we say that the family $\mathcal{R}$ is a relator on X and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is a relator space.

Remark 7.1. Thus, relator spaces are natural generalizations of ordered sets and uniform spaces (see [190]). Moreover, all reasonable generalizations of the usual topological structures can be easily derived from relators (see [194]).

However, to include the theory of Galois connections and formal contexts (see [64, p. 17]) and to briefly express the continuity properties of relations, relators on one set to another also have to be considered (see [201, 207, 208]).

Example 7.1. If $\mathcal{A}$ is a family of subsets of X, then the family of $\mathcal{R}_{\mathcal{A}}$ of all Pervin relations R_{A}, where $A \in \mathcal{A}$, is an important relator on X.

Namely, all minimal structures, generalized topologies and ascending systems on X can be derived from $\mathcal{R}_{\mathcal{A}}$ according to [210].

Example 7.2. If $\mathcal{D}$ is a family of distance functions on X, then the family $\mathcal{R}_{\mathcal{D}}$ of all surroundings B_r^d, where $d \in \mathcal{D}$ and $r > 0$, is also an important relator on X.

Namely, each topology can be derived from a family of quasi-semimetrics according to Pervin [147, Theorem 11.1.2 and an analogue of Theorem 11.3.4]. Moreover, the relator $\mathcal{R}_{\mathcal{D}}$ is usually a more convenient tool than the family $\mathcal{D}$.

Remark 7.2. Apart from preorder relators, tolerance relators are also important particular cases of reflexive relators. Note that a relator may be called reflexive if each of its members is reflexive.

Among the several basic algebraic and topological structures derivable from relators, we shall only need here the induced closures and interiors, and sequential convergences and adherences.

Definition 7.2. If $\mathcal{R}$ is a relator on X, then for any $x \in X$ and $A \subset X$ we write

(1) $x \in \operatorname{int}_{\mathcal{R}}(A)$ if $R(x) \subset A$ for some $R \in \mathcal{R}$;
(2) $x \in \operatorname{cl}_{\mathcal{R}}(A)$ if $R(x) \cap A \neq \emptyset$ for all $R \in \mathcal{R}$.

Remark 7.3. More generally, for any $A, B \subset X$, we may also write $B \in \operatorname{Int}_{\mathcal{R}}(A)$ if $R[B] \subset A$ for some $R \in \mathcal{R}$.

On the other hand, if R is a relation on X, then by identifying singletons with their elements we may write $\operatorname{int}_{R} = \operatorname{int}_{\{R\}}$.

A simple application of the corresponding definitions immediately yields:

Example 7.3. If $\mathcal{D}$ is family of distance functions on X, then for any $x \in X$ and $A \subset X$, we have $x \in \operatorname{cl}_{\mathcal{R}_{\mathcal{D}}}(A)$ if and only if $\underline{d}(x, A) = 0$ for all $d \in \mathcal{D}$.

By using the corresponding definitions, we can establish the following theorems.

Theorem 7.1. *If $\mathcal{R}$ is a relator on X, then for any $A \subset X$ we have*

(1) $\operatorname{cl}_{\mathcal{R}}(A) = \operatorname{int}_{\mathcal{R}}\big(A^c\big)^c$;
(2) $\operatorname{int}_{\mathcal{R}}(A) = \operatorname{cl}_{\mathcal{R}}\big(A^c\big)^c$.

Theorem 7.2. *If $\mathcal{R}$ is a relator on X, then*
(1) $\operatorname{cl}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \operatorname{cl}_R$;
(2) $\operatorname{int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \operatorname{int}_R$.

Theorem 7.3. *If R is a relation on X, then for any $A \subset X$ we have*

$$\operatorname{cl}_R(A) = R^{-1}[A].$$

From the above theorems, we can immediately derive the following

Theorem 7.4. *If $\mathcal{R}$ is a relator on X, then*
(1) $\operatorname{cl}_{\mathcal{R}}(\emptyset) = \emptyset \quad \big(\operatorname{int}_{\mathcal{R}}(X) = X \big)$ *if* $\mathcal{R} \neq \emptyset$,
(2) $\operatorname{cl}_{\mathcal{R}}(A) \subset \operatorname{cl}_{\mathcal{R}}(B) \quad \big(\operatorname{int}_{\mathcal{R}}(A) \subset \operatorname{int}_{\mathcal{R}}(B) \big)$ *if* $A \subset B \subset X$.

Definition 7.3. If $\mathcal{R}$ is a relator on X, then the members of the families

$$\mathcal{T}_{\mathcal{R}} = \big\{ A \subset X : \ A \subset \operatorname{int}_{\mathcal{R}}(A) \big\} \qquad \text{and} \qquad \mathcal{F}_{\mathcal{R}} = \big\{ A \subset X : \ \operatorname{cl}_{\mathcal{R}}(A) \subset A \big\}$$

are called the open and closed subsets of the relator space $X(\mathcal{R})$, respectively.

By Theorem 7.1, we have the following

Theorem 7.5. *If $\mathcal{R}$ is a relator on X, then*
(1) $\mathcal{T}_{\mathcal{R}} = \big\{ A^c : \ A \in \mathcal{F}_{\mathcal{R}} \big\}$;
(2) $\mathcal{F}_{\mathcal{R}} = \big\{ A^c : \ A \in \mathcal{T}_{\mathcal{R}} \big\}$.

Remark 7.4. Note that if $\mathcal{R}$ is a relator on X and

$$\tau_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathcal{T}_R \qquad \text{and} \qquad \digamma_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathcal{F}_R,$$

then in contrast to Theorem 7.2 we only have $\tau_{\mathcal{R}} \subset \mathcal{T}_{\mathcal{R}}$ and $\digamma_{\mathcal{R}} \subset \mathcal{F}_{\mathcal{R}}$.

By Theorem 7.4, we also have the following:

Theorem 7.6. *If $\mathcal{R}$ is a relator on X, then*
(1) $\emptyset \in \mathcal{F}_{\mathcal{R}} \quad \big(X \in \mathcal{T}_{\mathcal{R}} \big)$ *if* $\mathcal{R} \neq \emptyset$,
(2) $\bigcap \mathcal{A} \in \mathcal{F}_{\mathcal{R}}$ *if* $\mathcal{A} \subset \mathcal{F}_{\mathcal{R}} \quad \big(\bigcup \mathcal{A} \in \mathcal{T}_{\mathcal{R}}$ *if* $\mathcal{A} \subset \mathcal{T}_{\mathcal{R}} \big)$.

Remark 7.5. From (2), by taking $\mathcal{A} = \emptyset$, we can see that $\emptyset \in \mathcal{T}_{\mathcal{R}}$ and $X \in \mathcal{F}_{\mathcal{R}}$ are always true. Thus, in general, $\mathcal{T}_{\mathcal{R}}$ is only a generalized topology and $\mathcal{F}_{\mathcal{R}}$ is only a convexity structure on X (see [210] and [89]).

Definition 7.4. If $\mathcal{R}$ is a relator on X, then the members of the families

$$\mathcal{E}_{\mathcal{R}} = \big\{ A \subset X : \ \operatorname{int}_{\mathcal{R}}(A) \neq \emptyset \big\} \qquad \text{and} \qquad \mathcal{D}_{\mathcal{R}} = \big\{ A \subset X : \ \operatorname{cl}_{\mathcal{R}}(A) = X \big\}$$

are called the fat and dense subsets of the relator space $X(\mathcal{R})$, respectively.

Remark 7.6. In a relator space, the fat and dense sets are usually more important tools than the open and closed ones.

For instance, if $\leq$ is a certain order relation on X, then $\mathcal{T}_{\leq}$ and $\mathcal{E}_{\leq}$ are just the families of all ascending and residual subsets of the ordered set $X(\leq)$, respectively. Moreover, it may occur that $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R \neq \{X\}$ for some relation R on X.

Furthermore, in contrast to the open and closed sets, we now have the following theorems.

Theorem 7.7. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \{A \subset X : \ A^c \notin \mathcal{D}_{\mathcal{R}}\}$;
(2) $\mathcal{D}_{\mathcal{R}} = \{A \subset X : \ A^c \notin \mathcal{E}_{\mathcal{R}}\}$.

Theorem 7.8. *If $\mathcal{R}$ is a relator on X, then*
(1) $\mathcal{E}_{\mathcal{R}} = \{A \subset X : \ \forall \ B \in \mathcal{D}_{\mathcal{R}} : \ A \cap B \neq \emptyset\}$;
(2) $\mathcal{D}_{\mathcal{R}} = \{A \subset X : \ \forall \ B \in \mathcal{E}_{\mathcal{R}} : \ A \cap B \neq \emptyset\}$.

Remark 7.7. In this respect, note that $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{E}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{R}} \cap \mathcal{D}_{\mathcal{R}} \subset \{X\}$.

Theorem 7.9. *If $\mathcal{R}$ is a relator on X, then*
(1) $\mathcal{E}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathcal{E}_{R}$;
(2) $\mathcal{D}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathcal{D}_{R}$.

Theorem 7.10. *If R is a relation on X, then*

$$\mathcal{D}_{R} = \{A \subset X : \quad X = R^{-1}[A]\}.$$

Theorem 7.11. *If $\mathcal{R}$ is a relator on X, then $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are ascending systems in $\mathcal{P}(X)$ such that:*
(1) $\mathcal{E}_{\mathcal{R}} \neq \emptyset$ $(\emptyset \notin \mathcal{D}_{\mathcal{R}})$ *if and only if* $X \neq \emptyset$ *and* $\mathcal{R} \neq \emptyset$;
(2) $\mathcal{D}_{\mathcal{R}} \neq \emptyset$ $(\emptyset \notin \mathcal{E}_{\mathcal{R}})$ *if and only if* $X = D_{R}$ *for all* $R \in \mathcal{R}$.

Remark 7.8. A relator $\mathcal{R}$ on X is called total if each member R of $\mathcal{R}$ is total on X in the sense that X is the domain of R.

8. Convergent and Adherent Sequences

Our following definitions on sequences can be naturally extended to nets. However, in the sequel, we shall only be interested in sequences. Therefore, for the sake of simplicity, we shall restrict ourselves to sequences here.

Definition 8.1. A subset A of a relator space $X(\mathcal{R})$ is called infinitesimal if for each $R \in \mathcal{R}$ there exists $x \in X$ such that $A \subset R(x)$.

Moreover, a sequence $(A_n)_{n=1}^{\infty}$ of subsets of $X(\mathcal{R})$ is called infinitesimal if for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that $A_n \subset R(x)$.

Remark 8.1. A subset A of $X(\mathcal{R})$ is infinitesimal if and only if the constant sequence $(A)_{n=1}^{\infty}$ is infinitesimal.

Moreover, if $(A_n)_{n=1}^{\infty}$ is an infinitesimal sequence of subsets of $X(\mathcal{R})$, then $A = \bigcap_{n=1}^{\infty} A_n$ is an infinitesimal subset of $X(\mathcal{R})$.

By using the corresponding definitions, we can easily establish the following characterizations of infinitesimal sets.

Example 8.1. If $\mathcal{D}$ is a family of distance functions on X and A is a subset of the relator space $X(\mathcal{R}_{\mathcal{D}})$, then the following assertions are equivalent:
(1) A is infinitesimal;
(2) $\inf_{x \in X} \overline{d}(x, A) = 0$ for all $d \in \mathcal{D}$.

Theorem 8.1. *If A is a subset of a relator space $X(\mathcal{R})$, then the following assertions are equivalent:*
(1) *A is infinitesimal;*
(2) $\bigcap_{a \in A} R^{-1}(a) \neq \emptyset$ *for all* $R \in \mathcal{R}$.

Proof. Namely, for any $x \in X$ and $R \in \mathcal{R}$, we have

$$\begin{aligned} A \subset R(x) &\iff \forall\, a \in A: \ \ a \in R(x) \\ &\iff \forall\, a \in A: \ \ x \in R^{-1}(a) \iff x \in \bigcap_{a \in A} R^{-1}(a). \end{aligned}$$

∎

Remark 8.2. It is also worth noting that

$$\begin{aligned} A \subset R(x) &\iff A \cap R(x)^c = \emptyset \\ &\iff A \cap R^c(x) = \emptyset \iff x \notin \big(R^c\big)^{-1}[A] \iff x \in \big(R^c\big)^{-1}[A]^c. \end{aligned}$$

Therefore, the equality $\bigcap_{a \in A} R^{-1}(a) = \big(R^c\big)^{-1}[A]^c$ is also true.

We may also introduce the following definition concerning sequences of sets.

Definition 8.2. If $\mathcal{R}$ is a relator on X and $A = \big(A_n\big)_{n=1}^{\infty}$ is a sequence of subsets of X, then for any $x \in X$ we write

(1) $x \in \lim_{\mathcal{R}}\big(A\big)$ if for any $R \in \mathcal{R}$ there exists $n \in \mathbb{N}$ such that $A_k \subset R(x)$ for all $k \in \mathbb{N}$ with $k \geq n$;

(2) $x \in \operatorname{adh}_{\mathcal{R}}(A)$ if for any $R \in \mathcal{R}$ and $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$, with $k \geq n$, such that $A_k \subset R(x)$.

Remark 8.3. Thus, $\lim_{\mathcal{R}}$ and $\operatorname{adh}_{\mathcal{R}}$ are relations between sequences of subsets of X and points of X such that $\lim_{\mathcal{R}} \subset \operatorname{adh}_{\mathcal{R}}$.

More generally, for any pair $A = \big(A_n\big)_{n=1}^{\infty}$ and $B = \big(B_n\big)_{n=1}^{\infty}$ of sequences of subsets of X, we may write $B \in \operatorname{Lim}_{\mathcal{R}}(A)$ if for any $R \in \mathcal{R}$ there exists $n \in \mathbb{N}$ such that $B_k \times A_k \subset R$ for all $k \in \mathbb{N}$ with $k \geq n$.

Note that, by identifying singletons with their elements, the latter definition can be immediately applied to sequences of points of X as well.

Now, analogous to Example 8.1 and Theorem 8.1, we can establish the following descriptions of convergences and adherences.

Example 8.2. If $\mathcal{D}$ is family of distance functions on X, then for any $x \in X$ and sequence $A = \big(A_n\big)_{n=1}^{\infty}$ of subsets of X we have

(1) $x \in \lim_{\mathcal{R}_{\mathcal{D}}}(A)$ if and only if $\overline{\lim\limits_{n\to\infty}}\, \overline{d}(x, A_n) = 0$ for all $d \in \mathcal{D}$;

(2) $x \in \operatorname{adh}_{\mathcal{R}_{\mathcal{D}}}(A)$ if and only if $\underline{\lim\limits_{n\to\infty}}\, \overline{d}(x, A_n) = 0$ for all $d \in \mathcal{D}$.

Theorem 8.2. *If $\mathcal{R}$ is a relator on X, then*

(1) $\lim_{\mathcal{R}} = \bigcap\limits_{R \in \mathcal{R}} \lim_{R}$*;*

(2) $\operatorname{adh}_{\mathcal{R}} = \bigcap\limits_{R \in \mathcal{R}} \operatorname{adh}_{R}$.

Theorem 8.3. *If R is a relation on X, then for any sequence $A = \big(A_n\big)_{n=1}^{\infty}$ of subsets of X we have*

(1) $\lim_{R}(A) = \underline{\lim\limits_{n\to\infty}} \bigcap\limits_{a \in A_n} R^{-1}(a)$*;*

(2) $\operatorname{adh}_{R}(A) = \overline{\lim\limits_{n\to\infty}} \bigcap\limits_{a \in A_n} R^{-1}(a)$.

From Theorems 8.2 and 8.3, we can derive several properties of convergences and adherences. For instance, by using Theorems 7.2 and 7.3, we obtain:

Theorem 8.4. *If $\mathcal{R}$ is a relator on X, then for any sequence $a=(a_n)_{n=1}^{\infty}$ in X we have*

$$\operatorname{adh}_{\mathcal{R}}(a)=\bigcap_{n=1}^{\infty}\operatorname{cl}_{\mathcal{R}}\Big(\{a_k\}_{k=n}^{\infty}\Big).$$

Hence, by Theorem 7.4, we get:

Corollary 8.5. *If $\mathcal{R}$ is a relator on X and $A\subset X$, then for any sequence a in A we have* $\operatorname{adh}_{\mathcal{R}}(a)\subset\operatorname{cl}_{\mathcal{R}}(A)$.

For the origins of the second part of the following definition, the reader is referred to the historical notes of [195].

Definition 8.3. A sequence A of subsets of a relator space $X(\mathcal{R})$ is called

(1) convergent (adherent) if $\lim_{\mathcal{R}}(A)\neq\emptyset$ $\big(\operatorname{adh}_{\mathcal{R}}(A)\neq\emptyset\big)$;
(2) convergence (adherence) Cauchy if $\lim_{R}(A)\neq\emptyset$ $\big(\operatorname{adh}_{R}(A)\neq\emptyset\big)$ for all $R\in\mathcal{R}$.

Remark 8.4. Additionally, if X is a groupoid with zero, then the sequence A is called convergence (adherence) null if $0\in\lim_{\mathcal{R}}(A)$ $\big(0\in\operatorname{adh}_{\mathcal{R}}(A)\big)$.

By Theorem 8.2 and Definition 8.3, we have the following:

Theorem 8.6. *If A is a convergent (adherent) sequence of subsets of a relator space $X(\mathcal{R})$, then A is convergence (adherence) Cauchy.*

Remark 8.5. By the corresponding definitions, it is also clear that if A is a convergence (adherence) Cauchy sequence of subsets of a relator space $X(\mathcal{R})$, then A is infinitesimal.

It is also worth noting that if A is a decreasing infinitesimal sequence of subsets of a relator space $X(\mathcal{R})$, then A is already convergence (adherence) Cauchy.

By Theorem 8.2 and Definition 8.3, it is obvious that the converse of Theorem 8.6 need not be true. Therefore, we may also introduce the following

Definition 8.4. A relator $\mathcal{R}$ on X, or a relator space $X(\mathcal{R})$, is called

(1) sequentially convergence point-complete (set-complete) if each convergence Cauchy sequence of points (non-void subsets) of $X(\mathcal{R})$ is convergent;
(2) sequentially convergence–adherence point-complete (set-complete) if each convergence Cauchy sequence of points (non-void subsets) of $X(\mathcal{R})$ is adherent.

Remark 8.6. Hence, by Remark 8.3, it is clear that "sequentially convergence complete" implies "sequentially convergence-adherence complete", but the converse implication need not be true.

9. Some Important Operations on Relators

Among the several important refinements of relators, we shall only need here the following ones.

Definition 9.1. If $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{*}=\big\{S\subset X\times Y:\quad \exists\ R\in\mathcal{R}:\quad R\subset S\big\},$$
$$\mathcal{R}^{\#}=\big\{S\subset X\times Y:\quad \forall\ A\subset X:\quad A\in\operatorname{Int}_{\mathcal{R}}\big(S(A)\big)\big\},$$
$$\mathcal{R}^{\wedge}=\big\{S\subset X\times Y:\quad \forall\ x\in X:\quad x\in\operatorname{int}_{\mathcal{R}}\big(S(x)\big)\big\},$$
$$\mathcal{R}^{\triangle}=\big\{S\subset X\times Y:\quad \forall\ x\in X:\quad S(x)\in\mathcal{E}_{\mathcal{R}}\big\}$$

are called the uniform, proximal, topological and paratopological refinements of $\mathcal{R}$, respectively.

Remark 9.1. Thus, we have $\mathcal{R} \subset \mathcal{R}^* \subset \mathcal{R}^\# \subset \mathcal{R}^\wedge \subset \mathcal{R}^\vartriangle$ for any relator $\mathcal{R}$ on X.
Moreover, we have $\{R\}^\wedge = \{R\}^*$ and $\{R\}^\vartriangle = \big(R \circ X^X\big)^*$ for any relation R on X.

By using the corresponding definitions, we can establish the following.

Theorem 9.1. *If* $\Box \in \{*, \#, \wedge, \vartriangle\}$*, then* $\Box$ *is a closure operation on the family of all relators on* X *in the sense that:*

(1) $\mathcal{R} \subset \mathcal{R}^\Box$ *and* $\mathcal{R}^\Box = \mathcal{R}^{\Box\Box}$ *for any relator* $\mathcal{R}$ *on* X*;*
(2) $\mathcal{R}^\Box \subset \mathcal{S}^\Box$ *whenever* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X *such that* $\mathcal{R} \subset \mathcal{S}$.

Remark 9.2. Now, by Remark 9.1 and Theorem 9.1, it is clear that we also have $\mathcal{R}^\vartriangle = \mathcal{R}^{\Box\vartriangle} = \mathcal{R}^{\vartriangle\Box}$. Thus, $\vartriangle$ is $\Box$-absorbing and $\Box$-invariant.

The appropriateness of the topological refinement is apparent from the following.

Theorem 9.2. *If* $\mathcal{R}$ *is a relator on* X*, then* $\mathcal{R}^\wedge$ *is the largest relator on* X *such that*

$$\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^\wedge}, \qquad \textit{resp.} \qquad \operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}^\wedge}.$$

Hint. To prove the maximality of $\mathcal{R}^\wedge$, note that if $\mathcal{S}$ is a relator on X such that $\operatorname{int}_{\mathcal{S}} \subset \operatorname{int}_{\mathcal{R}}$, then $x \in \operatorname{int}_{\mathcal{S}}\big(S(x)\big) \subset \operatorname{int}_{\mathcal{R}}\big(S(x)\big)$ for all $x \in X$ and $S \in \mathcal{S}$. Therefore, $S \in \mathcal{R}^\wedge$ for all $S \in \mathcal{S}$, and thus $\mathcal{S} \subset \mathcal{R}^\wedge$. ∎

Remark 9.3. By this theorem and Remark 7.6, two relators $\mathcal{R}$ and $\mathcal{S}$ on X are called topologically equivalent if $\mathcal{R}^\wedge = \mathcal{S}^\wedge$. Moreover, a relator $\mathcal{R}$ is called topologically simple if it is topologically equivalent to a singleton relator.

From the above theorem, by Definition 7.3, we have

Corollary 9.3. *If* $\mathcal{R}$ *is a relator on* X*, then* $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^\wedge}$ *and* $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^\wedge}$.

Remark 9.4. It is worth mentioning that if $\mathcal{R}$ is a nonvoid relator on X, then we already have $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$ and $\digamma_{\mathcal{R}^\wedge} = \mathcal{F}_{\mathcal{R}}$.

By using the corresponding definitions, we can establish the following.

Theorem 9.4. *If* $\mathcal{R}$ *is a relator on* X*, then*

$$\lim_{\mathcal{R}} = \lim_{\mathcal{R}^\wedge} \qquad \textit{and} \qquad \operatorname{adh}_{\mathcal{R}} = \operatorname{adh}_{\mathcal{R}^\wedge}.$$

Remark 9.5. It is worth mentioning that, by [195, Theorem 2.1], the relator $\mathcal{R}^\wedge$ is always point-complete.

Analogous to Theorem 9.2, we can also prove the following theorem which is again a very particular case of some more general theorems proved by Pataki [144] and Száz [206].

Theorem 9.5. *If* $\mathcal{R}$ *is a relator on* X*, then* $\mathcal{R}^\vartriangle$ *is the largest relator on* X *such that*

$$\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^\vartriangle}, \qquad \textit{resp.} \qquad \mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^\vartriangle}.$$

Remark 9.6. We also note that if $\mathcal{R}$ is a nonvoid and total relator on X, then we already have $\tau_{\mathcal{R}^\vartriangle} = \mathcal{T}_{\mathcal{R}^\vartriangle} = \mathcal{E}_{\mathcal{R}^\vartriangle} \cup \{\emptyset\}$ and $\digamma_{\mathcal{R}^\vartriangle} = \mathcal{F}_{\mathcal{R}^\vartriangle} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.

In addition to Definition 9.1, we have the following.

Definition 9.2. If $\mathcal{R}$ is a relator on X then the relators

$$\mathcal{R}^\infty = \big\{R^\infty : \ R \in \mathcal{R}\big\} \qquad \text{and} \qquad \mathcal{R}^\partial = \big\{S \subset X^2 : \ S^\infty \in \mathcal{R}\big\}$$

are called the direct and inverse preorder modifications of $\mathcal{R}$, respectively.

Remark 9.7. Recall that if R is a relation on X, then R^∞ is the smallest preorder relation on X such that $R \subset R^\infty$ (see [82]).

Moreover, R^∞ is the largest relation on X such that $\mathcal{T}_R = \mathcal{T}_{R^\infty}$, resp. $\mathcal{F}_R = \mathcal{F}_{R^\infty}$ (see [115] and [206]).

By using the corresponding definitions, we can easily establish the following.

Theorem 9.6. *If $\Box = \infty$ or ∂, then $\Box$ is a modification operation on the family of all relators on X in the sense that:*

(1) $\mathcal{R}^\Box = \mathcal{R}^{\Box\Box}$ *for any relator $\mathcal{R}$ on X;*
(2) $\mathcal{R}^\Box \subset \mathcal{S}^\Box$ *if $\mathcal{R}$ and $\mathcal{S}$ are relators on X such that $\mathcal{R} \subset \mathcal{S}$.*

Remark 9.8. Now, we see that $\mathcal{R}^\infty \subset \mathcal{R}^{*\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^*$, and hence $\mathcal{R}^{*\infty} = \mathcal{R}^{\infty*\infty}$ and $\mathcal{R}^{\infty*} = \mathcal{R}^{*\infty*}$ for any relator $\mathcal{R}$ on X.

Moreover, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have $\mathcal{R}^\infty \subset \mathcal{S}$ if and only if $\mathcal{R} \subset \mathcal{S}^\partial$. Therefore, the mappings ∞ and ∂ establish an increasing Galois connection (see [47, p. 155]). Thus, $\infty\partial$ is already a closure operation.

Now, as a counterpart of Theorem 9.5, we can also state the following theorem which was first proved by Mala [115] (see also [118]).

Theorem 9.7. $\wedge\infty$ *is a modification operation on the family of all relators on X such that, for any relator $\mathcal{R}$ on X, $\mathcal{R}^{\wedge\infty}$ is the largest preorder relator on X such that*

$$\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\wedge\infty}}, \qquad \textit{resp.} \qquad \mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\wedge\infty}}.$$

Remark 9.9. Mala has also proved that if $\mathcal{R}$ is a relator on X, then in general there does not exist a largest relator $\mathcal{S}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{S}}$. This is another serious disadvantage of open sets to fat sets and interiors.

However, if $\mathcal{R}$ is a relator on X, then it can be shown that $\mathcal{R}^\sharp = \mathcal{R}^{\#\partial}$ is the largest relator on X such that $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^\sharp}$, resp. $\bar{\tau}_{\mathcal{R}} = \bar{\tau}_{\mathcal{R}^\sharp}$. However, the operation $\sharp$ is not stable. That is $\{X^2\} \neq \{X^2\}^\sharp$.

Analogous to the definition of $\mathcal{R}^\infty$, we introduce the following.

Definition 9.3. If $\mathcal{R}$ is a relator on X then the relator

$$\mathcal{R}^{-1} = \big\{ R^{-1} : \quad R \in \mathcal{R} \big\}$$

is called the inverse of $\mathcal{R}$.

Remark 9.10. It can be shown that $\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}$ and $\bar{\tau}_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$. Moreover, $\big(\mathcal{R}^\Box\big)^{-1} = \big(\mathcal{R}^{-1}\big)^\Box$ if $\Box \in \{*, \#, \infty, \partial\}$.

However, the operations $\wedge$ and $\vartriangle$ are not inversion compatible. Therefore, we also need the notations $\mathcal{R}^\vee = \big(\mathcal{R}^\wedge\big)^{-1}$ and $\mathcal{R}^\triangledown = \big(\mathcal{R}^\vartriangle\big)^{-1}$.

Definition 9.4. For any relator $\mathcal{R}$ on X, we write

$$\delta_{\mathcal{R}} = \bigcap \mathcal{R} \qquad \text{and} \qquad \sigma_{\mathcal{R}} = \bigcup \mathcal{R}.$$

Now, as an immediate consequence of Theorems 7.2 and 7.3, we can also state:

Theorem 9.8. *If $\mathcal{R}$ is a relator on X, then for any $x \in X$ we have*

$$\operatorname{cl}_{\mathcal{R}}\big(\{x\}\big) = \delta_{\mathcal{R}^{-1}}(x) = \delta_{\mathcal{R}}^{-1}(x).$$

Remark 9.11. Finally, we note that if $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then we may also define $\mathcal{R} \circ \mathcal{R} = \big\{ R \circ R : \ R \in \mathcal{R} \big\}$,

$$\mathcal{R} \circ \mathcal{S} = \big\{ R \circ S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\} \quad \text{and} \quad \mathcal{R} \wedge \mathcal{S} = \big\{ R \cap S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \big\}.$$

10. REFLEXIVE AND SYMMETRIC RELATORS

By using the corresponding definitions, we can easily prove the following.

Theorem 10.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is reflexive;*
(2) *$\delta_{\mathcal{R}}$ is reflexive;*
(3) $A \subset \mathrm{cl}_{\mathcal{R}}(A)$ $\big(\mathrm{int}_{\mathcal{R}}(A) \subset A \big)$ *for all* $A \subset X$.

Definition 10.1. A relator $\mathcal{R}$ on X is called

(1) quasi-topological if $x \in \mathrm{int}_{\mathcal{R}}\big(\mathrm{int}_{\mathcal{R}}\big(R(x)\big)\big)$ for all $x \in X$ and $R \in \mathcal{R}$;
(2) topological if for any $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subset R(x)$.

Remark 10.1. Note that the inclusion $x \in \mathrm{int}_{\mathcal{R}}\big(R(x)\big)$ trivially holds for all $x \in X$ and $R \in \mathcal{R}$.

Moreover, a singleton relator $\{R\}$ on X is topological (quasi-topological) if and only if R is a preorder (transitive) relation on X.

The appropriateness of Definition 10.1 is apparent from the following theorems which have been mostly proved in [191, 192].

Theorem 10.2. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topological;*
(2) $\mathrm{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$*;*
(3) $\mathrm{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}$ $\big(\mathrm{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}} \big)$ *for all* $A \subset X$.

Remark 10.2. By the above theorem, a relator $\mathcal{R}$ on X is called weakly (strongly) quasi-topological if $\mathrm{cl}_{\mathcal{R}}\big(\{x\}\big) \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$ $\big(R(x) \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R} \big)$.

Theorem 10.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological;*
(2) *$\mathcal{R}$ is reflexive and quasi-topological;*
(3) $\mathrm{int}_{\mathcal{R}}(A) = \bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A)$ $\big(\mathrm{cl}_{\mathcal{R}}(A) = \bigcap \mathcal{F}_{\mathcal{R}} \cap \mathcal{P}^{-1}(A) \big)$ *for all* $A \subset X$.

Remark 10.3. By the above theorem, a relator $\mathcal{R}$ on X is called weakly (strongly) topological if it is reflexive and weakly (strongly) quasi-topological.

The importance of strongly topological relators lies mainly in the following.

Theorem 10.4. *If $\mathcal{R}$ is a relator on X and R° is a relation on X, for each $R \in \mathcal{R}$, such that*

$$R^{\circ}(x) = \mathrm{int}_{\mathcal{R}}\big(R(x)\big)$$

for all $x \in X$, then $\mathcal{R}^{\circ} = \big\{ R^{\circ} : \ R \in \mathcal{R} \big\}$ is a strongly topological relator on X such that:

(1) *$\mathcal{R}$ is reflexive if and only if $\mathcal{R} \subset \big(\mathcal{R}^{\circ}\big)^{*}$;*
(2) *$\mathcal{R}$ is quasi-topological if and only if $\mathcal{R}^{\circ} \subset \mathcal{R}^{\wedge}$.*

Now, as an immediate consequence of Theorems 10.3 and 10.4, we can also state:

Corollary 10.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological;*
(2) *$\mathcal{R}$ is topologically equivalent to $\mathcal{R}^{\circ}$;*
(3) *$\mathcal{R}$ is topologically equivalent to a strongly topological relator.*

The following theorem also holds.

Theorem 10.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topological;*
(2) *$\mathcal{R}$ is topologically equivalent to $\mathcal{R}_{\mathcal{T}_{\mathcal{R}}}\left(\mathcal{R}^{\wedge\infty}\right)$;*
(3) *$\mathcal{R}$ is topologically equivalent to a preorder relator.*

The quasi-topologicalness properties are closely related to the various transitivity properties of relators. However, before listing some transitivity properties, we consider a few symmetry properties.

Definition 10.2. If $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called $\square$–symmetric if $\left(\mathcal{R}^{\square}\right)^{-1} \subset \mathcal{R}^{\square}$.

Remark 10.4. Now, the relator $\mathcal{R}$ may be called properly symmetric if it is symmetric with respect to the identity operation for relators. That is, $\mathcal{R}^{-1} \subset \mathcal{R}$, and thus $\mathcal{R} = \mathcal{R}^{-1}$.

On the other hand, the relator $\mathcal{R}$ is called weakly (strongly) symmetric if $\delta_{\mathcal{R}}$ (each member of $\mathcal{R}$) is symmetric.

The appropriateness of the above definitions is apparent from the following theorems which have been mostly proved in [192, 193].

Theorem 10.7. *If $\mathcal{R}$ is a relator on X and $\square$ is an inversion compatible closure operation for relators on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is $\square$–symmetric;*
(2) *$\mathcal{R}$ and $\mathcal{R}^{-1}$ are $\square$-equivalent;*
(3) *$\mathcal{R}$ is $\square$–equivalent to a properly symmetric relator.*

Remark 10.5. Note that, by the corresponding definitions, (1) means only that $\left(\mathcal{R}^{\square}\right)^{-1} \subset \mathcal{R}^{\square}$. And this is equivalent to $\mathcal{R}^{-1} \subset \mathcal{R}^{\square}$ by the assumed properties of $\square$.

Theorem 10.8. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically symmetric;*
(2) *$\mathcal{R}$ is topologically simple and weakly symmetric;*
(3) *$A \cap \mathrm{cl}_{\mathcal{R}}(B) \neq \emptyset$ implies $B \cap \mathrm{cl}_{\mathcal{R}}(A) \neq \emptyset$ for all $A, B \subset X$.*

Remark 10.6. The above theorem shows that topological symmetry is a rather restrictive symmetry property. Namely, even a strongly symmetric relator need not be topologically symmetric.

Therefore, a relator $\mathcal{R}$ may be called topologically semisymmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^{\wedge}$. Note that thus we have $\mathcal{R}^{\wedge} = \left(\mathcal{R}^{-1}\right)^{\wedge}$ if and only if both $\mathcal{R}$ and $\mathcal{R}^{-1}$ are topologically semisymmetric.

Theorem 10.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is weakly symmetric;*
(2) *$\mathcal{R}$ is topologically equivalent to $\mathcal{R}^{\wedge} \cap \mathcal{R}^{\vee}$;*
(3) *$\mathcal{R}$ is topologically equivalent to a properly symmetric relator.*

Remark 10.7. Note that if $\mathcal{R}$ is topologically semisymmetric, i.e., $\mathcal{R}^{-1} \subset \mathcal{R}^{\wedge}$, then by Theorem 9.2 $\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\mathcal{R}^{\wedge}} \subset \mathrm{cl}_{\mathcal{R}^{-1}}$. Hence, by Theorem 9.8, $\delta_{\mathcal{R}}^{-1} \subset \delta_{\mathcal{R}}$, and thus $\mathcal{R}$ is weakly symmetric.

Curiously enough, weak symmetry is a most important symmetry property of relators. It corresponds to a famous regularity axiom R_o introduced by Shanin [180] and Davis [48] (see also [193, 196]).

11. TRANSITIVE AND FILTERED RELATORS

Definition 11.1. If $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called $\square$–transitive if $\mathcal{R}^{\square\square} \subset \big(\mathcal{R}^{\square} \circ \mathcal{R}^{\square}\big)^{\square}$.

Remark 11.1. The relator $\mathcal{R}$ is called strictly $\square$–transitive if $\mathcal{R}^{\square\square} \subset \big(\mathcal{R}^{\square} \bar{\circ} \mathcal{R}^{\square}\big)^{\square}$.

The appropriateness of the above definition is apparent from the following theorems which have been mostly proved in [191, 192].

Theorem 11.1. *If $\mathcal{R}$ is a relator on X, then*

(1) *$\mathcal{R}$ is uniformly transitive if and only if $\mathcal{R} \subset \big(\mathcal{R} \circ \mathcal{R}\big)^{*}$;*
(2) *$\mathcal{R}$ is proximally transitive if and only if $\mathcal{R} \subset \big(\mathcal{R} \circ \mathcal{R}\big)^{\#}$;*
(3) *$\mathcal{R}$ is topologically transitive if and only if $\mathcal{R} \subset \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$.*

Remark 11.2. Due to the above theorem, a relator $\mathcal{R}$ is called strongly topologically transitive if $\mathcal{R} \subset (\mathcal{R} \circ \mathcal{R})^{\wedge}$.

From the above theorem, we can easily derive the following.

Corollary 11.2. *If $\mathcal{R}$ is a reflexive relator on X, then*

(1) *$\mathcal{R}$ is uniformly transitive if and only if $\mathcal{R}^{*} = \big(\mathcal{R} \circ \mathcal{R}\big)^{*}$;*
(2) *$\mathcal{R}$ is proximally transitive if and only if $\mathcal{R}^{\#} = \big(\mathcal{R} \circ \mathcal{R}\big)^{\#}$;*
(3) *$\mathcal{R}$ is topologically transitive if and only if $\mathcal{R}^{\wedge} = \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$.*

Hint. By the reflexivity of $\mathcal{R}$, we have $R = R \circ \Delta_X \subset R \circ S$ for all $R, S \in \mathcal{R}$. Therefore, $\mathcal{R} \circ \mathcal{R} \subset \mathcal{R}^{*}$, and thus $\mathcal{R}^{\wedge} \circ \mathcal{R} \subset \mathcal{R}^{\wedge} \circ \mathcal{R}^{\wedge} \subset \mathcal{R}^{\wedge *} = \mathcal{R}^{\wedge}$ also holds. ∎

We also have the following.

Theorem 11.3. *If $\mathcal{R}$ is a nonvoid relator on X then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-topological;*
(2) *$\mathcal{R}$ is topologically transitive.*

Hint. If (2) holds, then by Theorem 11.1 we have $\mathcal{R} \subset \big(\mathcal{R}^{\wedge} \circ \mathcal{R}\big)^{\wedge}$. Therefore, for any $R \in \mathcal{R}$ and $x \in X$ there exist $S \in \mathcal{R}$ and $T \in \mathcal{R}^{\wedge}$ such that $T\big[S(x)\big] = (T \circ S)(x) \subset R(x)$. This implies that $T(u) \subset R(x)$ for all $u \in S(x)$. Therefore, $S(x) \subset \operatorname{int}_{\mathcal{R}^{\wedge}}\big(R(x)\big) = \operatorname{int}_{\mathcal{R}}\big(R(x)\big)$. Hence, we can see that $x \in \operatorname{int}_{\mathcal{R}}\big(S(x)\big) \subset \operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big)$, and thus (1) also holds. ∎

Now, as a vague analogue of Theorem 10.4, we state the following.

Theorem 11.4. *If $\mathcal{R}$ is a reflexive, topologically semisymmetic and strongly topologically transitive relator on X and R^{-} is a relation on X, for each $R \in \mathcal{R}$, such that*

$$R^{-}(x) = \operatorname{cl}_{\mathcal{R}}\big(R(x)\big)$$

for all $x \in X$, then $\mathcal{R}^{-} = \big\{R^{-} : \quad R \in \mathcal{R}\big\}$ is a reflexive, strongly topologically transitive relator on X such that $\mathcal{R}$ and $\mathcal{R}^{-}$ are topologically equivalent.

Remark 11.3. If $\mathcal{R}$ is a reflexive, topologically semisymmetric and uniformly transitive relator on X, then we can more easily prove that $\mathcal{R}$ and $\mathcal{R}^{-}$ are uniformly equivalent. Thus $\mathcal{R}^{-}$ is also uniformly transitive.

Now, by using Theorems 10.1 and 11.4 and Remark 11.3, we can easily prove the following theorems.

Theorem 11.5. *If $\mathcal{R}$ is a reflexive, topologically semisymmetic and strongly topologically transitive relator on X, then for any sequence $\big(A_n\big)_{n=1}^{\infty}$ of subsets of X we have*

(1) $\lim\limits_{n\to\infty}{}_{\mathcal{R}} A_n = \lim\limits_{n\to\infty}{}_{\mathcal{R}} \operatorname{cl}_{\mathcal{R}}(A_n)$;

(2) $\operatorname{adh}\limits_{n\to\infty}{}_{\mathcal{R}} A_n = \operatorname{adh}\limits_{n\to\infty}{}_{\mathcal{R}} \operatorname{cl}_{\mathcal{R}}(A_n)$.

Proof. Assume that $x \in \operatorname{adh}\limits_{n\to\infty}{}_{\mathcal{R}} A_n$, $R \in \mathcal{R}$ and $n \in \mathbb{N}$. Then, by Theorem 11.4, there exists $S \in \mathcal{R}$ such that $\operatorname{cl}_{\mathcal{R}}\big(S(x)\big) \subset R(x)$. Moreover, by Definition 8.2, there exists $k \in \mathbb{N}$ such that $A_k \subset S(x)$. Hence, it is already clear that $\operatorname{cl}_{\mathcal{R}}(A_k) \subset \operatorname{cl}_{\mathcal{R}}\big(S(x)\big) \subset R(x)$. Therefore, $x \in \operatorname{adh}\limits_{n\to\infty}{}_{\mathcal{R}} \operatorname{cl}_{\mathcal{R}}(A_n)$ is also true. Thus, we have proved that $\operatorname{adh}\limits_{n\to\infty}{}_{\mathcal{R}} A_n \subset \operatorname{adh}\limits_{n\to\infty}{}_{\mathcal{R}} \operatorname{cl}_{\mathcal{R}}(A_n)$. The converse inclusion is quite obvious from Definition 8.2 by Theorem 10.1. ∎

Theorem 11.6. *If $\mathcal{R}$ is a reflexive, topologically semisimmetric and uniformly transitive relator on X and $\big(A_n\big)_{n=1}^{\infty}$ is a sequence of subsets of $X(\mathcal{R})$, then the following assertions are equivalent:*

(1) $\big(A_n\big)_{n=1}^{\infty}$ *is infinitesimal;*

(2) $\big(\operatorname{cl}_{\mathcal{R}}(A_n)\big)_{n=1}^{\infty}$ *is infinitesimal.*

Proof. By Remark 11.3, for any $R \in \mathcal{R}$, there exists $S \in \mathcal{R}$ such that $S^{-} \subset R$. This implies that $\operatorname{cl}_{\mathcal{R}}\big(S(x)\big) = S^{-}(x) \subset R(x)$ for all $x \in X$. Moreover, if (1) holds, then by Definition 8.1 there exist $x \in X$ and $n \in \mathbb{N}$ such that $A_n \subset S(x)$. Hence, it is already clear that $\operatorname{cl}_{\mathcal{R}}(A_n) \subset \operatorname{cl}_{\mathcal{R}}\big(S(x)\big) \subset R(x)$. Therefore, (2) is also true. The converse implication is quite obvious from Definition 8.1 by Theorem 10.1. ∎

From this theorem, by Remark 8.1, it is clear that we also have:

Corollary 11.7. *If $\mathcal{R}$ is as in the above theorem and A is a subset of $X(\mathcal{R})$, then the following assertions are equivalent:*

(1) *A is infinitesimal;*

(2) *$\operatorname{cl}_{\mathcal{R}}(A)$ is infinitesimal.*

Analogous to Definition 11.1, we may also introduce the following:

Definition 11.2. If $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called $\square$–filtered if $\big(\mathcal{R}^{\square} \wedge \mathcal{R}^{\square}\big)^{\square} \subset \mathcal{R}^{\square\square}$.

Remark 11.4. Now, the relator $\mathcal{R}$ is said to be properly filtered if it is filtered with respect to the identity operation for relators. That is, $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}$, and thus $\mathcal{R} = \mathcal{R} \wedge \mathcal{R}$.

On the other hand, the relator $\mathcal{R}$ is said to be totally filtered if for any $R, S \in \mathcal{R}$ we have either $R \subset S$ or $S \subset R$. Thus, a totally filtered relator is properly filtered.

The appropriateness of the above definitions is apparent from the following theorems which have been mostly proved in [192].

Theorem 11.8. *If $\mathcal{R}$ is a relator on X and $\square = *$ or $\wedge$, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is $\square$–filtered;*

(2) *$\mathcal{R}^{\square}$ is properly filtered;*

(3) *$\mathcal{R}$ and $\mathcal{R} \wedge \mathcal{R}$ are $\square$–equivalent.*

Remark 11.5. If $A \subset X$ and R and S are relations on X, then in general we only have $(R \cap S)[A] \subset R[A] \cap S[A]$.

Therefore, a relator $\mathcal{R}$ on X is called weakly proximally filtered if for any $A \subset X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T[A] \subset R[A] \cap S[A]$. This is actually more important than the proximal filteredness.

Theorem 11.9. *If $\mathcal{R}$ is a relator on X then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is topologically filtered;*
(2) $\operatorname{cl}_{\mathcal{R}}(A \cup B) = \operatorname{cl}_{\mathcal{R}}(A) \cup \operatorname{cl}_{\mathcal{R}}(B)$ *for all* $A, B \subset X$*;*
(3) $\operatorname{int}_{\mathcal{R}}(A \cap B) = \operatorname{int}_{\mathcal{R}}(A) \cap \operatorname{int}_{\mathcal{R}}(B)$ *for all* $A, B \subset X$.

Hence, by Definition 7.3 and Theorem 10.3, we also have:

Corollary 11.10. *If $\mathcal{R}$ is a topological relator on X then the following assertions are equivalent.*

(1) *$\mathcal{R}$ is topologically filtered;*
(2) $A \cap B \in \mathcal{T}_{\mathcal{R}}$ *for all* $A, B \in \mathcal{T}_{\mathcal{R}}$*;*
(3) $A \cup B \in \mathcal{F}_{\mathcal{R}}$ *for all* $A, B \in \mathcal{F}_{\mathcal{R}}$.

Remark 11.6. For an arbitrary relator $\mathcal{R}$ on X, assertions (2) and (3) are equivalent to the $\wedge\infty$-filteredness of $\mathcal{R}$.

To briefly state a useful theorem on convergence Cauchy sequences, in addition to Definitions 10.2 and 11.1, we shall need:

Definition 11.3. A relator $\mathcal{R}$ on X is called locally uniform if for each $R \in \mathcal{R}$ and $x \in X$ there exist $S, T \in \mathcal{R}$ such that

$$\big(S \circ S^{-1} \circ T\big)(x) \subset R(x).$$

Remark 11.7. Note that if $\mathcal{R}$ is a reflexive, strongly symmetric and strictly uniformly transitive relator on X, then $\mathcal{R}$ is already locally uniform.

Moreover, note that if $\mathcal{R}$ is a uniformly filtered, strongly topologically transitive and proximally symmetric relator on X, then $\mathcal{R}$ is also locally uniform.

Theorem 11.11. *If $\mathcal{R}$ is a locally uniform relator on X, then for any convergence Cauchy sequence A of non-void subsets of $X(\mathcal{R})$ we have*

$$\lim_{\mathcal{R}}\big(A\big) = \operatorname{adh}_{\mathcal{R}}\big(A\big).$$

Proof. By Remark 8.3, we need only show that $\operatorname{adh}_{\mathcal{R}}(A) \subset \lim_{\mathcal{R}}(A)$. For this, assume that $x \in \operatorname{adh}_{\mathcal{R}}(A)$ and $R \in \mathcal{R}$. Then, by the local uniformity of $\mathcal{R}$, there exist $S, T \in \mathcal{R}$ such that

$$\big(S \circ S^{-1} \circ T\big)(x) \subset R(x).$$

Moreover, since A is convergence Cauchy, there exist $u \in X$ and $n \in \mathbb{N}$ such that $A_k \subset S(u)$ for all $k \in \mathbb{N}$ with $k \geq n$. Furthermore, since $x \in \operatorname{adh}_{\mathcal{R}}(A)$, there exists $l \in \mathbb{N}$, with $l \geq n$, such that $A_l \subset T(x)$.

Now, if $k \in \mathbb{N}$ such that $k \geq n$, then by choosing $a \in A_l$, we can see that $a \in T(x)$ and $a \in S(u)$, i.e., $u \in S^{-1}(a)$. Therefore, $u \in S^{-1}[T(x)]$, and thus

$$A_k \subset S(u) \subset S\big[S^{-1}[T(x)]\big] = \big(S \circ S^{-1} \circ T\big)(x) \subset R(x).$$

Hence, it is clear that $x \in \lim_{\mathcal{R}}(A)$ also holds. ∎

Now, as an immediate consequence of the corresponding definitions and the above theorem, we can also state:

Corollary 11.12. *If $\mathcal{R}$ is a sequentially convergence-adherence point-complete (set-complete) locally uniform relator on X, then $\mathcal{R}$ is already sequentially convergence point-complete (set-complete).*

12. SEPARATED AND DIRECTED RELATORS

Definition 12.1. A relator $\mathcal{R}$ on X is called

(1) T_0–separating if for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that either $y \notin R(x)$ or $x \notin R(y)$;
(2) T_1–separating if for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $y \notin R(x)$;
(3) T_2–separating if for any $x, y \in X$, with $x \neq y$, there exist $R, S \in \mathcal{R}$ such that $R(x) \cap S(y) = \emptyset$.

Remark 12.1. The relator $\mathcal{R}$ is called strictly T_2–separating if for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $R(x) \cap R(y) = \emptyset$.

Moreover, the relator $\mathcal{R}$ is called quasi–T_2–separating if for any $x, y \in X$, with $\operatorname{cl}_{\mathcal{R}}(\{x\}) \neq \operatorname{cl}_{\mathcal{R}}(\{x\})$, there exist $R, S \in \mathcal{R}$ such that $R(x) \cap R(y) = \emptyset$.

The following theorems have mostly been proved in [192] (see also [124, Ch. 3]).

Theorem 12.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is T_0–separating;*
(2) $\delta_{\mathcal{R}^{-1} \wedge \mathcal{R}} \subset \Delta_X$;
(3) *$\mathcal{R}$ is weakly antisymmetric.*

Theorem 12.2. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is T_1–separating:*
(2) $\delta_{\mathcal{R}} \subset \Delta_X$;
(3) $\{x\} \in \mathcal{F}_{\mathcal{R}}$ *for all* $x \in X$.

Remark 12.2. It is clear that a T_1–separating relator is weakly transitive and weakly quasi-topological.

As an immediate consequence of the above theorems, we can also state:

Corollary 12.3. *A relator $\mathcal{R}$ on X is T_1–separating if and only if it is T_0–separating and weakly symmetric.*

Theorem 12.4. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is T_2–separating;*
(2) $\delta_{\mathcal{R}^{-1} \circ \mathcal{R}} \subset \Delta_X$;
(3) $\Delta_X \in \mathcal{F}_{\mathcal{R} \boxtimes \mathcal{R}}$.

Theorem 12.5. *If $\mathcal{R}$ is a T_0–separating, topologically semisymmetric and strongly topologically transitive relator on X, then $\mathcal{R}$ is already T_2–separating.*

Proof. If $x, y \in X$ such that $x \neq y$, then by the T_0–separatingness of $\mathcal{R}$ there exists $R \in \mathcal{R}$ such that either $y \notin R(x)$ or $x \notin R(y)$. Assume, for instance, that $y \notin R(x)$. Now, by the strong topological transitivity of $\mathcal{R}$, there exist $S, T \in \mathcal{R}$ such that $T[S(x)] \subset R(x)$. Therefore, $y \notin T[S(x)]$, and thus $S(x) \cap T^{-1}(y) = \emptyset$. Moreover, since $\mathcal{R}$ is topologically semisymmetric, there exists $U \in \mathcal{R}$ such that $U(y) \subset T^{-1}(y)$. Therefore, $S(x) \cap U(y) = \emptyset$, and thus $\mathcal{R}$ is T_2–separating. ∎

We can also easily prove the following theorems concerning T_2–separating relators.

Theorem 12.6. *If $\mathcal{R}$ is a T_2–separating relator on X, then for any sequence A of nonvoid subsets of X the inclusions $x \in \lim_{\mathcal{R}}(A)$ and $y \in \operatorname{adh}_{\mathcal{R}}(A)$ imply $x = y$.*

Proof. Assume on the contrary that $x \neq y$. Then, by Definition 12.1, there exist $R, S \in \mathcal{R}$ such that $R(x) \cap S(y) = \emptyset$. Moreover, by Definition 8.2, there exists $n \in \mathbb{N}$ such that $A_k \subset R(x)$ for all $k \in \mathbb{N}$ with $k \geq n$. Furthermore, by Definition 8.2, there exists $k \in \mathbb{N}$, with $k \geq n$, such that $A_k \subset S(y)$. Therefore, $A_k \subset R(x) \cap S(y) = \emptyset$, and thus $A_k = \emptyset$. This contradiction proves the theorem. ∎

Corollary 12.7. *If $\mathcal{R}$ is a T_2–separating relator on X, then $\lim_{\mathcal{R}}(A)$ is at most a singleton for any sequence A of nonvoid subsets of X.*

Theorem 12.8. *If A is an infinitesimal subset of a relator space $X(\mathcal{R})$ such that the relator $\mathcal{R}^{-1}$ is strictly T_2–separating, then A is at most a singleton.*

Proof. Assume, on the contrary, that there exist $a, b \in A$ such that $a \neq b$. Then, since $\mathcal{R}^{-1}$ is strictly T_2–separating, there exists $R \in \mathcal{R}$ such that $R^{-1}(a) \cap R^{-1}(b) = \emptyset$. Moreover, since A is infinitesimal, there exists $x \in X$ such that $A \subset R(x)$. Hence, it follows that $a, b \in R(x)$, and thus $x \in R^{-1}(a) \cap R^{-1}(b) = \emptyset$. This contradiction proves the theorem. ∎

The following also holds.

Theorem 12.9. *If $\mathcal{R}$ is a uniformly symmetric relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is strictly T_2–separating;*
(2) *$\mathcal{R}^{-1}$ is strictly T_2–separating.*

Proof. If (1) holds, then for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $R(x) \cap R(y) = \emptyset$. Moreover, since $\mathcal{R}$ is uniformly symmetric, there exists $S \in R$ such that $S \subset R^{-1}$. Hence, it follows that $S^{-1} \subset R$, and thus $S^{-1}(x) \subset R(x)$ and $S^{-1}(y) \subset R(y)$. Therefore, we also have $S^{-1}(a) \cap S^{-1}(b) = \emptyset$, and thus (2) also holds. ∎

Now, in contrast to T_2-separatedness, we introduce some directedness properties.

Definition 12.2. A relator $\mathcal{R}$ on X is called

(1) semi-directed if $R(x) \cap S(y) \neq \emptyset$ for all $x, y \in X$ and $R, S \in \mathcal{R}$;
(2) quasi-directed if $R(x) \cap S(y) \in \mathcal{E}_{\mathcal{R}}$ for all $x, y \in X$ and $R, S \in \mathcal{R}$.

Remark 12.3. The relator $\mathcal{R}$ is said to be directed if it is both total and quasi-directed.

Note that thus a directed relator is semi-directed, and a semi-directed relator is necessarily total.

As a preliminary illustration of the above definitions, we immediately state the following:

Example 12.1. If $\prec$ is a relation on X, then the relator $\{\prec\}$ is semi-directed if and only if for each $x, y \in X$ there exists $z \in X$ such that $x \prec z$ and $y \prec z$.

Moreover, if $\prec$ is a transitive relation on X, then the relator $\{\prec\}$ is directed if and only if it is semi-directed.

The appropriateness of the above definitions is also apparent from the following theorems which have mostly been proved in [192].

Theorem 12.10. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is semi-directed;*
(2) *$\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$;*
(3) *$A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}_{\mathcal{R}}$.*

Remark 12.4. In the light of Theorem 12.4, the following assertions are also equivalent to the semi-directedness of $\mathcal{R}$:

(1) $\mathcal{R}^{-1} \circ \mathcal{R} = \{X^2\}$;
(2) $\delta_{\mathcal{R}^{-1} \circ \mathcal{R}} = X^2$;
(3) $\Delta_X \in \mathcal{D}_{\mathcal{R} \boxtimes \mathcal{R}}$.

Theorem 12.11. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is quasi-directed;*
(2) *$A \cap B \in \mathcal{E}_{\mathcal{R}}$ for all $A, B \in \mathcal{E}_{\mathcal{R}}$;*
(3) *$A \cap B \in \mathcal{D}_{\mathcal{R}}$ for all $A \in \mathcal{E}_{\mathcal{R}}$ and $B \in \mathcal{D}_{\mathcal{R}}$.*

13. A FEW BASIC FACTS ON VECTOR RELATORS

According to [205], we have the following:

Definition 13.1. A nonvoid family $\mathcal{R}$ of relations on a vector space X is called a vector relator on X if

(1) $R(x) = x + R(0)$ for all $x \in X$ and $R \in \mathcal{R}$;
(2) $R(0)$ is an absorbing balanced subset of X for all $R \in \mathcal{R}$;
(3) for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(0) + S(0) \subset R(0)$.

Remark 13.1. In this case, we say that $\mathcal{R}$ is a proper vector relator on X. Namely, if $\square$ is a unary operation for relators on X, then a relator $\mathcal{R}$ on X is called a $\square$–vector relator if it is $\square$-equivalent to a proper vector relator.

The appropriateness of the above definition is apparent from the following.

Example 13.1. If $\mathcal{P}$ is a nonvoid family of preseminorms on X, then by Theorem 6.9 it is clear that the family $\mathcal{R}_{\mathcal{P}}$ of all surrounding B_r^p, where $p \in \mathcal{P}$ and $r > 0$, is a vector relator on X.

Remark 13.2. Clearly, vector relators are somewhat more general objects than vector topologies. Namely, each vector topology $\mathcal{T}$ on X can be derived from a nonvoid, directed family $\mathcal{P}$ of preseminorms on X (see [237] and [121]).

From Definition 13.1, by using our former results on translation relations, we can easily derive several basic properties of vector relators.

Theorem 13.1. *If $\mathcal{R}$ is a vector relator on X, then*

(1) *each member of $\mathcal{R}$ is an absorbing, balanced translation relation on X;*
(2) *$\mathcal{R}$ is a strictly uniformly transitive, well-chained tolerance relator on X.*

Proof. From Definition 13.1, by Theorems 4.1 and 6.6, it is clear that (1) is true. Hence, by Corollaries 5.18 and 5.15, we can see that $\mathcal{R}$ is a well-chained tolerance relator on X.

Moreover, if $R \in \mathcal{R}$, then by Definition 13.1 there exists $S \in \mathcal{R}$ such that $S(0) + S(0) \subset R(0)$. Hence, by Theorem 5.14, it follows that $S^2 = S \circ S \subset R$. Therefore, $\mathcal{R}$ is strictly uniformly transitive. ∎

From the above theorem, by using Theorem 11.3 and Corollary 5.19, we can immediately derive:

Corollary 13.2. *If $\mathcal{R}$ is a vector relator on X, then $\mathcal{R}$ is a topological relator on X such that $\tau_{\mathcal{R}} = \bar{\tau}_{\mathcal{R}} = \{\emptyset, X\}$.*

Remark 13.3. The vector relator $\mathcal{R}$ need not be strongly topological. Moreover, if $\mathcal{R} \neq \{X^2\}$, then $\mathcal{R}$ is not proximal. Namely, if $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \neq X$, then by Corollary 13.2 there is no $V \in \tau_{\mathcal{R}}$ such that $x \in V \subset R(x)$.

However, note that if $\mathcal{P}$ is a nonvoid family of preseminorms on X, then by Remark 6.5 we have $B_r^p(x) \in \mathcal{T}_{\mathcal{R}_{\mathcal{P}}}$ for all $x \in X$, $r > 0$ and $p \in \mathcal{P}$. Therefore, the induced vector relator $\mathcal{R}_{\mathcal{P}}$ is already strongly topological.

Theorem 13.3. *If $\mathcal{R}$ is a vector relator on a vector space X over $\mathbb{K}$, then*

(1) *for any $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) + S(y) \subset R(x+y)$ for all $x, y \in X$;*
(2) *for any $R \in \mathcal{R}$ and $n \in \mathbb{N}$ there exists $S \in \mathcal{R}$ such that $\lambda S(x) \subset R(\lambda x)$ for all $x \in X$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq n$.*

Proof. To prove (2), note that if $R \in \mathcal{R}$ and $n \in \mathbb{N}$, then from Definition 13.1, by induction, we can see that there exists $S \in \mathcal{R}$ such that

$$\sum_{k=1}^{2^n} S(0) \subset R(0).$$

Hence, since $0 \in S(0)$, it follows that $\sum_{k=1}^{n} S(0) \subset R(0)$. Now, if $\lambda \in \mathbb{K}$ such that $|\lambda| \leq n$, then by using the balanced property of $S(0)$ we can see that

$$\lambda S(0) \subset n S(0) \subset \sum_{k=1}^{n} S(0) \subset R(0).$$

Thus, by the corresponding definitions, it is clear that

$$\lambda S(x) = \lambda\big(x + S(0)\big) = \lambda x + \lambda S(0) \subset \lambda x + R(0) = R(\lambda x)$$

for all $x \in X$. ∎

Remark 13.4. Note that in the above theorem, we can also write any subsets A and B of X in place of the points x and y.

Now, concerning vector relators, we can also easily prove:

Theorem 13.4. *If $\mathcal{R}$ is a vector relator on a vector space X over $\mathbb{K}$ and $\mathfrak{F} = \operatorname{cl}_{\mathcal{R}}$ or $\operatorname{int}_{\mathcal{R}}$, then*

(1) *$\mathfrak{F}(A) + \mathfrak{F}(B) \subset \mathfrak{F}(A+B)$ for all $A, B \subset X$;*
(2) *$\mathfrak{F}(x + A) = x + \mathfrak{F}(A)$ for all $x \in X$ and $A \subset X$;*
(3) *$\mathfrak{F}(\lambda A) = \lambda \mathfrak{F}(A)$ for all $A \subset X$ and $\lambda \in \mathbb{K}$ with $\lambda \neq 0$.*

Remark 13.5. By using Theorems 10.1, 7.4, 13.4 and 10.2, we can see that $\operatorname{cl}_{\mathcal{R}}(A+B) = \operatorname{cl}_{\mathcal{R}}\big(\operatorname{cl}_{\mathcal{R}}(A) + \operatorname{cl}_{\mathcal{R}}(B)\big)$ for all $A, B \subset X$.

Moreover, by Theorems 9.8 and 13.1, it is clear that $\delta_{\mathcal{R}}(x) = \operatorname{cl}_{\mathcal{R}^{-1}}\big(\{x\}\big) = \operatorname{cl}_{\mathcal{R}}\big(\{x\}\big)$ for all $x \in X$. Hence, by using Theorems 13.4 and 10.2, we can easily see that $\delta_{\mathcal{R}}$ is a closed-valued, linear equivalence relation on X.

By using Theorem 13.4, we can also establish the following:

Theorem 13.5. *If $\mathcal{R}$ is a vector relator on a vector space X over $\mathbb{K}$ and $\mathcal{A} \in \{\mathcal{T}_{\mathcal{R}}, \mathcal{F}_{\mathcal{R}}, \mathcal{E}_{\mathcal{R}}, \mathcal{D}_{\mathcal{R}}\}$, then*

(1) *$x + A \in \mathcal{A}$ for all $x \in X$ and $A \in \mathcal{A}$;*
(2) *$\lambda A \in \mathcal{A}$ for all $A \in \mathcal{A}$ and $\lambda \in \mathbb{K}$ with $\lambda \neq 0$.*

In addition to Definition 13.1, we also introduce:

Definition 13.2. A vector relator $\mathcal{R}$ on X is called

(1) convex if $R(0)$ is a convex subset of X for all $R \in \mathcal{R}$;
(2) separating if for any $x \in X$, with $x \neq 0$, there exists R such that $x \notin R(0)$;
(3) filtered if for any $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(0) \subset R(0) \cap S(0)$.

The appropriateness of this definition is apparent from:

Example 13.2. If $\mathcal{P}$ is a nonvoid family of seminorms on X, then the vector relator $\mathcal{R}_{\mathcal{P}}$ is convex. If $\mathcal{P}$ is a nonvoid, directed (separating) family of preseminorms on X, then the vector relator $\mathcal{R}_{\mathcal{P}}$ is filtered (separating).

Moreover, as useful consequences of the corresponding definitions and theorems, we have the following theorems.

Theorem 13.6. *If $\mathcal{R}$ is a vector relator on X, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is convex;*
(2) *each member of $\mathcal{R}$ is a convex relation on X.*

Theorem 13.7. *If $\mathcal{R}$ is a vector relator on X, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is separating;*
(2) *$\mathcal{R}$ is T_0–separating;*
(3) *$\mathcal{R}$ is strictly T_2-separating.*

Proof. If (2) holds, then from Theorems 13.1 and 12.5 we can see that $\mathcal{R}$ is T_2–separating. Hence, it is clear that (1) also holds.

On the other hand, if (1) holds, then for any $x, y \in X$, with $x \neq y$, there exists $R \in \mathcal{R}$ such that $x - y \notin R(0)$. Hence, it follows that $x \notin R(0) + y = y + R(0) = R(y)$. Moreover, by Theorem 13.1, there exists $S \in \mathcal{R}$ such that $S^2 \subset R$. This implies that $S[S(y)] = S^2(y) \subset R(y)$. Therefore, we also have $x \notin S[S(y)]$, which implies that $S(x) \cap S(y) = S^{-1}(x) \cap S(y) = \emptyset$. Therefore, (3) also holds. ∎

By Theorems 12.2 and 13.5, we also have:

Corollary 13.8. *If $\mathcal{R}$ is a vector relator on X, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is separating;*
(2) *$\{0\} \in \mathcal{F}_{\mathcal{R}}$;*
(3) *$\{x\} \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$;*
(4) *$\{x\} \in \mathcal{F}_{\mathcal{R}}$ for some $x \in X$.*

By using Theorem 5.14, we can establish:

Theorem 13.9. *If $\mathcal{R}$ is a vector relator on X, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is filtered;*
(2) *$\mathcal{R}$ is uniformly filtered;*
(3) *$\mathcal{R}$ is topologically filtered.*

Remark 13.6. By Corollaries 13.2 and 11.10, it is clear that $\mathcal{T}_{\mathcal{R}}$ is a vector topology on X if and only if $\mathcal{R}$ is filtered.

Finally, we note that the following theorem is also true.

Theorem 13.10. *If $\mathcal{R}$ is a vector relator on X, then the following assertions are equivalent:*
(1) *$\mathcal{R}$ is semi-directed;*
(2) *$R(x) \cap R(0) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$;*
(3) *$\mathcal{R} = \{X^2\}$;*
(4) *$\mathcal{R}$ is directed.*

Hint. If $R \in \mathcal{R}$, then by Theorem 13.1 there exists $S \in \mathcal{R}$ such that $S \circ S \subset R$. Moreover, if (2) holds, then for any $x \in X$ we have $S(x) \cap S(0) \neq \emptyset$. Hence, it follows that $x \in S^{-1}\big[S(0)\big] = S\big[S(0)\big] = \big(S \circ S\big)(0) \subset R(0)$. Therefore, $R(0) = X$, and thus $R(x) = x + R(0) = x + X = X$ also holds. Hence, it is clear that $R = X^2$, and thus (3) also holds. ∎

14. FURTHER RESULTS ON SEQUENCES OF SETS

Theorem 14.1. *If A is a sequence of subsets of a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ and $\lambda \in \mathbb{K} \setminus \{0\}$, then*

(1) $\lim_{\mathcal{R}}(\lambda A) = \lambda \lim_{\mathcal{R}}(A)$;
(2) $\operatorname{adh}_{\mathcal{R}}(\lambda A) = \lambda \operatorname{adh}_{\mathcal{R}}(A)$.

Proof. Suppose that $x \in \operatorname{adh}_{\mathcal{R}}(A)$ and $R \in \mathcal{R}$. Then, by Theorem 13.3, there exists $S \in \mathcal{R}$ such that

$$\lambda S(x) \subset R(\lambda x).$$

Moreover, by Definition 8.2, for each $n \in \mathbb{N}$ there exists $k \geq n$ such that $A_k \subset S(x)$. Hence,

$$\big(\lambda A\big)_k = \lambda A_k \subset \lambda S(x) \subset R(\lambda x).$$

Therefore, $\lambda x \in \operatorname{adh}_{\mathcal{R}}(\lambda A)$ also holds.

Thus, we have proved that $\lambda \operatorname{adh}_{\mathcal{R}}(A) \subset \operatorname{adh}_{\mathcal{R}}(\lambda A)$. Now, by writing λ^{-1} in place of λ and λA in place of A, we can see that

$$\lambda^{-1} \operatorname{adh}_{\mathcal{R}}(\lambda A) \subset \operatorname{adh}_{\mathcal{R}}\big(\lambda^{-1}\lambda A\big) = \operatorname{adh}_{\mathcal{R}}(A),$$

and thus $\operatorname{adh}_{\mathcal{R}}(\lambda A) \subset \lambda \operatorname{adh}_{\mathcal{R}}(A)$ also holds. ∎

Corollary 14.2. *If A is a convergence (adherence) null sequence of subsets of a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then λA is also a convergence (adherence) null sequence of subsets of $X(\mathcal{R})$.*

Theorem 14.3. *If a is a point and B is a sequence of subsets of a vector relator space $X(\mathcal{R})$, then*

(1) $\lim_{\mathcal{R}}(a + B) = a + \lim_{\mathcal{R}}(B)$;
(2) $\operatorname{adh}_{\mathcal{R}}(a + B) = a + \operatorname{adh}_{\mathcal{R}}(B)$.

Proof. If $x \in a + \operatorname{adh}_{\mathcal{R}}(B)$, then there exists $b \in \operatorname{adh}_{\mathcal{R}}(B)$ such that $x = a + b$. Moreover, for any $R \in \mathcal{R}$ and $n \in \mathbb{N}$, there exists $k \geq n$ such that $B_k \subset R(b)$. Hence, it is clear that

$$a + B_k \subset a + R(b) = R(a + b) = R(x).$$

Therefore, $x \in \operatorname{adh}_{\mathcal{R}}(a + B)$ also holds.

Thus, we have proved that $a + \operatorname{adh}_{\mathcal{R}}(B) \subset \operatorname{adh}_{\mathcal{R}}(a + B)$. Now, by writing $-a$ in place of a and $a + B$ in place of B, we can see that

$$-a + \operatorname{adh}_{\mathcal{R}}(a + B) \subset \operatorname{adh}_{\mathcal{R}}(-a + a + B) = \operatorname{adh}_{\mathcal{R}}(B),$$

and thus $\operatorname{adh}_{\mathcal{R}}(a + B) \subset a + \operatorname{adh}_{\mathcal{R}}(B)$ also holds. ∎

Theorem 14.4. *If A and B are sequences of subsets of a vector relator space $X(\mathcal{R})$, then*

(1) $\lim_{\mathcal{R}}(A) + \lim_{\mathcal{R}}(B) \subset \lim_{\mathcal{R}}(A + B)$;
(2) $\lim_{\mathcal{R}}(A) + \operatorname{adh}_{\mathcal{R}}(B) \subset \operatorname{adh}_{\mathcal{R}}(A + B)$.

Proof. If $x \in \lim_{\mathcal{R}}(A) + \operatorname{adh}_{\mathcal{R}}(B)$, then there exist $a \in \lim_{\mathcal{R}}(A)$ and $b \in \operatorname{adh}_{\mathcal{R}}(B)$ such that $x = a + b$. Moreover, if $R \in \mathcal{R}$, then by Theorem 13.3, there exists $S \in \mathcal{R}$ such that

$$S(a) + S(b) \subset R(a + b) = R(x).$$

Now, by the corresponding definitions, we can see that there exists $n_1 \in \mathbb{N}$ such that $A_k \subset S(a)$ for all $k \in \mathbb{N}$ with $k \geq n_1$. Moreover, if $n \in \mathbb{N}$ and $m = \max\{n_1, n\}$, then there exists $k \geq m$ such that $B_k \subset S(b)$. Hence, it is clear that $k \geq n$ such that

$$A_k + B_k \subset S(a) + S(b) \subset R(x).$$

Therefore, $x \in \operatorname{adh}_{\mathcal{R}}(A + B)$ also holds. ∎

Corollary 14.5. *If A is a convergent and B is a convergence (adherence) null sequence of subsets of a vector relator space $X(\mathcal{R})$, then $A+B$ is a convergence (adherence) null sequence of subsets of $X(\mathcal{R})$.*

Analogous to Theorem 14.1, we can also prove the following theorems.

Theorem 14.6. *If A is an infinitesimal sequence of subsets of a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then λA is also an infinitesimal sequence of subsets of $X(\mathcal{R})$.*

Corollary 14.7. *If A is an infinitesimal subset of a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then λA is also an infinitesimal subset of $X(\mathcal{R})$.*

Theorem 14.8. *If A is a convergence (adherence) Cauchy sequence of subsets of a vector relator space $X(\mathcal{R})$ over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then λA is also a convergence (adherence) Cauchy sequence of subsets of $X(\mathcal{R})$.*

Proof. If $R \in \mathcal{R}$, then by Theorem 13.3 there exists $S \in \mathcal{R}$ such that

$$\lambda S(x) \subset R(\lambda x)$$

for all $x \in X$. Moreover, if A is adherence Cauchy, then by Definition 8.3, there exists $x \in X$ such that for each $n \in \mathbb{N}$ there exists $k \geq n$ such that $A_k \subset S(x)$. Hence, it is clear that

$$\big(\lambda A\big)_k = \lambda A_k \subset \lambda S(x) \subset R(\lambda x).$$

Therefore, the sequence λA is also adherence Cauchy. ∎

Theorem 14.9. *If A is an infinitesimal subset and B is an infinitesimal sequence of subsets of a vector relator space $X(\mathcal{R})$, then $A+B$ is also an infinitesimal sequence of subsets of $X(\mathcal{R})$.*

Corollary 14.10. *If A and B are infinitesimal subsets of a vector relator space $X(\mathcal{R})$, then $A+B$ is also an infinitesimal subset of $X(\mathcal{R})$.*

Theorem 14.11. *If A is a convergence Cauchy and B is a convergence (adherence) Cauchy sequence of subsets of a vector relator space $X(\mathcal{R})$, then $A+B$ is also a convergence (adherence) Cauchy sequence of subsets of $X(\mathcal{R})$.*

Proof. If $R \in \mathcal{R}$, then by Theorem 13.3 there exists $S \in \mathcal{R}$ such that

$$S(x) + S(y) \subset R(x+y)$$

for all $x, y \in X$. Moreover, by Definition 8.3, there exist $x \in X$ and $n_o \in \mathbb{N}$ such that $A_k \subset S(x)$ for all $k \in \mathbb{N}$ with $k \geq n_o$. Furthermore, if B is adherence Cauchy, then by Definition 8.3 there exists $y \in X$ such that for each $n \in \mathbb{N}$ there exists $k \geq n$ such that $B_k \subset S(y)$. Therefore, if $n \in \mathbb{N}$ and $m = \max\{n_o, n\}$, then there exists $k \geq m$ such that $B_k \subset S(y)$. Hence,

$$\big(A+B\big)_k = A_k + B_k \subset S(x) + S(y) \subset R(x+y)$$

and therefore, the sequence $A+B$ is also adherence Cauchy. ∎

In the sequel, we will need the following curious theorem on the pointwise union of sequences of sets.

Theorem 14.12. *If A is convergence (adherence) Cauchy and B and C are convergence null sequences of subsets of a vector relator space $X(\mathcal{R})$, then $\big(A+B\big) \cup \big(A+C\big)$ is also a convergence (adherence) Cauchy sequence of subsets of $X(\mathcal{R})$.*

Proof. If $R \in \mathcal{R}$, then by Theorem 13.3 there exists $S \in \mathcal{R}$ such that

$$S(x) + S(y) \subset R(x+y)$$

for all $x, y \in X$. Moreover, since B and C are convergence null, there exist $n_1, n_2 \in \mathbb{N}$ such that $B_k \subset S(0)$ and $C_l \subset S(0)$ for all $k, l \in \mathbb{N}$ with $k \geq n_1$ and $l \geq n_2$. Furthermore, if A is adherence Cauchy, then there exists $x \in X$ such that for any $n \in \mathbb{N}$ there exists $k \geq n$ such that $A_k \subset S(x)$. Therefore, if $n \in \mathbb{N}$ and $m = \max\{n_1, n_2, n\}$, then there exists $k \geq m$ such that $A_k \subset S(x)$. Hence, it is clear that

$$(A+B)_k = A_k + B_k \subset S(x) + S(0) \subset R(x)$$

and

$$(A+C)_k = A_k + C_k \subset S(x) + S(0) \subset R(x).$$

Therefore, we also have

$$\Big(\big(A+B\big) \cup \big(A+C\big)\Big)_n = (A+B)_n \cup (A+C)_n \subset R(x).$$

Thus, the sequence $\big(A+B\big) \cup \big(A+C\big)$ is also adherence Cauchy. ∎

15. THE ASSOCIATED HYERS SEQUENCES

According to [61], we introduce the following:

Definition 15.1. If F is a relation on a groupoid U to a vector space X, then for each $n \in \mathbb{N}$ we define a relation F_n on U to X such that

$$F_n(u) = \frac{1}{2^n} F\big(2^n u\big)$$

for all $u \in U$. Moreover, in accordance with this notation, we write $F_0 = F$.

Remark 15.1. In recognition of the pioneering work of Hyers [90], the sequence $(F_n)_{n=1}^{\infty}$ will be called the Hyers sequence associated with F.

Functional generalizations of Hyers' sequences have formerly been given by Rassias [158], Lee and Jun [111], Gilányi, Kaiser and Páles [81].

We can easily prove the following theorems concerning Hyers' sequences.

Theorem 15.1. *If F is a relation of a groupoid U to a vector space X, then for any $A \subset U$ and $n \in \mathbb{N}$ we have*

$$F_n[A] = \frac{1}{2^n} F\big[2^n A\big].$$

Proof. By using the corresponding definitions and the fact that unions are preserved under relations, we see that

$$\begin{aligned} F_n[A] &= \bigcup_{u \in A} F_n(u) = \bigcup_{u \in A} \frac{1}{2^n} F\big(2^n u\big) = \frac{1}{2^n} \bigcup_{u \in A} F\big(2^n u\big) \\ &= \frac{1}{2^n} \bigcup_{u \in A} F\big(\{2^n u\}\big) = \frac{1}{2^n} F\Bigg(\bigcup_{u \in A} \{2^n u\}\Bigg) = \frac{1}{2^n} F[2^n A]. \end{aligned}$$

∎

Theorem 15.2. *If F is a relation of a groupoid U to a vector space X and G is a relation of X to another vector space Y, then for any $n \in \mathbb{N}$ we have*

$$\big(G \circ F\big)_n = G_n \circ F_n.$$

Proof. By the corresponding definitions and Theorem 15.1,

$$\begin{aligned}\big(G\circ F\big)_n(u) &= \frac{1}{2^n}(G\circ F)\big(2^n u\big)\\ &= \frac{1}{2^n}G\left[F\big(2^n u\big)\right]\\ &= \frac{1}{2^n}G\Big[2^n\frac{1}{2^n}F\big(2^n u\big)\Big]\\ &= \frac{1}{2^n}G\big[2^n F_n(u)\big]\\ &= G_n\big[F_n(u)\big] = \big(G_n\circ F_n\big)(u)\end{aligned}$$

for all $u\in U$. Therefore, the required equality is also true. ∎

Theorem 15.3. *If F is a relation of one vector space X onto another Y, then for any $n\in\mathbb{N}$ we have*

$$\big(F_n\big)^{-1} = \big(F^{-1}\big)_n .$$

Proof. By the corresponding definitions, for any $x\in X$ and $y\in Y$, we have

$$\begin{aligned}x\in\big(F_n\big)^{-1}(y) &\iff y\in F_n(x)\\ &\iff y\in\frac{1}{2^n}F\big(2^n x\big)\\ &\iff 2^n y\in F\big(2^n x\big)\\ &\iff 2^n x\in F^{-1}\big(2^n y\big)\\ &\iff x\in\frac{1}{2^n}F^{-1}\big(2^n y\big)\\ &\iff x\in\big(F^{-1}\big)_n(y).\end{aligned}$$

∎

We note the following:

Theorem 15.4. *If F is a relation of a semigroup U to a vector space X, then for any $n,m\in\mathbb{N}$ we have*

$$\big(F_n\big)_m = F_{n+m} .$$

Proof. By Definition 15.1 and Theorem 3.1, it is clear that

$$\begin{aligned}\big(F_n\big)_m(u) &= \frac{1}{2^m}F_n\big(2^m u\big)\\ &= \frac{1}{2^m}\left(\frac{1}{2^n}F\Big(2^n\big(2^m u\big)\Big)\right)\\ &= \left(\frac{1}{2^n}\frac{1}{2^m}\right)F\Big(\big(2^n 2^m\big)u\Big)\\ &= \frac{1}{2^{n+m}}F\big(2^{n+m}u\big) = F_{n+m}(u)\end{aligned}$$

for all $u\in U$. ∎

Corollary 15.5. *If F is a relation of a semigroup U to a vector space X, then for any $u\in U$ and $n\in\mathbb{N}$ we have*

$$F_n(2u) = 2F_{n+1}(u).$$

Proof. By Definition 15.1 and Theorem 15.4, it is clear that

$$F_n(2u) = 2\frac{1}{2}F_n(2u) = 2\big(F_n\big)_1(u) = 2F_{n+1}(u).$$

∎

Remark 15.2. If F is a relation of a groupoid U to a vector space X, then by the corresponding definitions we also have $F_0(2u) = 2F_1(u)$ for all $u \in U$.

Theorem 15.6. *If F is a relation of a semigroup U to a vector space X, then for each $n \in \mathbb{N}$ there exists a selection $f_{_{(n)}}$ of F_n such that*

$$f_{_{(n)}}(2u) = 2f_{_{(n+1)}}(u)$$

for all $u \in U$ and $n \in \mathbb{N}$.

Proof. If $f_{_{(n)}}$ is a selection of F_n for some $n \in \mathbb{N}$ and $f_{_{(n+1)}} = \big(f_{_{(n)}}\big)_1$, then by Definition 15.1 and Corollary 15.5 we have

$$f_{_{(n+1)}}(u) = \big(f_{_{(n)}}\big)_1(u) = \frac{1}{2}f_{_{(n)}}(2u) \in \frac{1}{2}F_n(2u) = \frac{1}{2}2F_{n+1}(u) = F_{n+1}(u)$$

for all $u \in U$. Therefore, $f_{_{(n+1)}}$ is a selection of F_{n+1}. Moreover, by the above computation, we also have $f_{_{(n)}}(2u) = 2f_{_{(n+1)}}(u)$ for all $u \in U$. Hence, by induction, the required assertion is also true. ∎

16. FURTHER RESULTS ON HYERS' SEQUENCES

We now prove the following theorems concerning Hyers' sequences.

Theorem 16.1. *If F and G are relations of a groupoid U to a vector space X such that $F \subset G$, then $F_n \subset G_n$ for all $n \in \mathbb{N}$.*

Corollary 16.2. *If F is a relation of a groupoid U to a vector space X and f is a selection of F, then f_n is a selection of F_n for all $n \in \mathbb{N}$.*

Theorem 16.3. *If F and G are relations of a groupoid U to a vector space X over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then for any $n \in \mathbb{N}$ we have*

(1) $(F+G)_n = F_n + G_n$;
(2) $(\lambda F)_n = \lambda F_n$.

Proof. Clearly,

$$\begin{aligned}(F+G)_n(u) &= \frac{1}{2^n}(F+G)\big(2^n u\big)\\ &= \frac{1}{2^n}\,\Big(F\big(2^n u\big) + G\big(2^n u\big)\Big)\\ &= \frac{1}{2^n}F\big(2^n u\big) + \frac{1}{2^n}G\big(2^n u\big)\\ &= F_n(u) + G_n(u) = \big(F_n + G_n\big)(u)\end{aligned}$$

for all $u \in U$. Therefore, (1) is true. ∎

Assertion (2) can be immediately derived from the following.

Theorem 16.4. *Suppose that F is a relation of a groupoid U to a vector space X and $*$ is a unary operation on $\mathcal{P}(X)$ such that*

$$\left(\frac{1}{2}A\right)^* = \frac{1}{2}A^*$$

for all $A \subset X$. Define a relation F^* on U to X such that $F^*(u) = F(u)^*$ for all $u \in U$. Then, for any $n \in \mathbb{N}$, we have

$$\big(F^*\big)_n = \big(F_n\big)^*.$$

Proof. Now, by induction,

$$\left(\frac{1}{2^n}A\right)^* = \frac{1}{2^n}A^*$$

holds for all $n \in \mathbb{N}$ and $A \subset X$. Therefore, we have

$$\begin{aligned}\big(F^*\big)_n(u) &= \frac{1}{2^n}F^*\big(2^n u\big)\\ &= \frac{1}{2^n}F\big(2^n u\big)^*\\ &= \left(\frac{1}{2^n}F\big(2^n u\big)\right)^*\\ &= \big(F_n(u)\big)^* = \big(F_n\big)^*(u)\end{aligned}$$

for all $u \in U$. Thus, the required equality is also true. ∎

By Theorem 13.4, we also have:

Theorem 16.5. *If F is a relation of a groupoid U to a vector relator space $X(\mathcal{R})$ and*

$$F^\circ(u) = \operatorname{int}_{\mathcal{R}}\big(F(u)\big) \qquad \textit{and} \qquad F^-(u) = \operatorname{cl}_{\mathcal{R}}\big(F(u)\big)$$

for all $u \in U$, then for any $n \in \mathbb{N}$ we have

(1) $\big(F^\circ\big)_n = \big(F_n\big)^\circ$;
(2) $\big(F^-\big)_n = \big(F_n\big)^-$.

Finally, we note that the following theorem is also true.

Theorem 16.6. *If $(F_{(i)})_{i\in I}$ is a family of relations of a groupoid U to a vector space X, then for any $n \in \mathbb{N}$*

(1) $F = \bigcup_{i\in I} F_{(i)}$ *implies* $F_n = \bigcup_{i\in I} \big(F_{(i)}\big)_n$;
(2) $F = \bigcap_{i\in I} F_{(i)}$ *implies* $F_n = \bigcap_{i\in I} \big(F_{(i)}\big)_n$.

Proof. If the condition of (2) holds, then

$$\begin{aligned} F_n(u) &= \frac{1}{2^n} F\big(2^n u\big) \\ &= \frac{1}{2^n} \Big(\bigcap_{i \in I} F_{(i)}\Big)\big(2^n u\big) \\ &= \frac{1}{2^n} \bigcap_{i \in I} F_{(i)}\big(2^n u\big) \\ &= \bigcap_{i \in I} \frac{1}{2^n} F_{(i)}\big(2^n u\big) \\ &= \bigcap_{i \in I} \big(F_{(i)}\big)_n (u) = \Big(\bigcap_{i \in I} \big(F_{(i)}\big)_n\Big)(u) \end{aligned}$$

for all $u \in U$. Therefore, the conclusion of (2) also holds. ∎

Corollary 16.7. *If $(F_{(i)})_{i=1}^{\infty}$ is a sequence of relations of a groupoid U to a vector space X, then for any $n \in \mathbb{N}$*

(1) $F = \underline{\lim}_{i \to \infty} F_{(i)}$ *implies* $F_n = \underline{\lim}_{i \to \infty} \big(F_{(i)}\big)_n$;
(2) $F = \overline{\lim}_{i \to \infty} F_{(i)}$ *implies* $F_n = \overline{\lim}_{i \to \infty} \big(F_{(i)}\big)_n$.

Proof. If the condition of (2) holds, then by Theorem 16.6 we also have

$$\begin{aligned} F_n &= \Big(\overline{\lim_{i \to \infty}} F_{(i)}\Big)_n \\ &= \Big(\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} F_{(j)}\Big)_n \\ &= \bigcap_{i=1}^{\infty} \Big(\bigcup_{j=i}^{\infty} F_{(j)}\Big)_n \\ &= \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \big(F_{(j)}\big)_n = \overline{\lim_{i \to \infty}} \big(F_{(i)}\big)_n . \end{aligned}$$

∎

Remark 16.1. The latter corollary can be naturally extended to nets of relations. However, in the remainder of this paper we shall only be interested in sequences of relations.

17. APPROXIMATELY SUBHOMOGENEOUS RELATIONS

Definition 17.1. A relation F on one groupoid U to another V is called n–homogeneous, for some $n \in \mathbb{N}$, if

$$F(nu) = nF(u)$$

for all $u \in U$. Moreover, F is called $\mathbb{N}$–homogeneous if it is n–homogeneous for all $n \in \mathbb{N}$.

Example 17.1. If f is an additive function of one groupoid U to another V, then by induction it can be easily seen that f is $\mathbb{N}$–homogeneous.

Remark 17.1. If p is only a 2–homogeneous and subadditive (or Jensen convex) function of a vector space X, then p is already $\mathbb{N}$–homogeneous (for the proofs, see [35]).

Example 17.2. If A is a non-void subset of a semigroup U and $F(n) = nA$ for all $n \in \mathbb{N}$, then by Theorem 3.1, F is an $\mathbb{N}$–homogeneous relation of $\mathbb{N}$ to U.

Theorem 17.1. *If F is a relation of a groupoid U to a vector space X, then the following assertions are equivalent:*

(1) $F_1 = F$;
(2) F *is* 2*–homogeneous.*

Proof. For any $u \in U$, we have

$$F_1(u) = F(u) \iff \frac{1}{2}F(2u) = F(u) \iff F(2u) = 2F(u).$$

∎

Theorem 17.2. *If F is a* 2*-homogeneous relation of a semigroup U to a vector space X, then $F_n = F$ for all $n \in \mathbb{N}$.*

Proof. By Theorems 15.4 and 17.1, for any $n \in \mathbb{N}$, we have

$$F_{n+1} = F_{1+n} = \big(F_1\big)_n = F_n.$$

Hence, by induction, $F_n = F_1 = F$ also holds. ∎

Corollary 17.3. *If F is a relation of a semigroup U to a vector space X and f is a* 2*–homogeneous selection of F, then f is also a selection of the relation $G = \bigcap_{n=1}^{\infty} F_n$.*

Proof. By Theorem 17.2, $f = f_n$. Moreover, by Corollary 16.2, f_n is a selection of F_n. Therefore, f is also a selection of F_n. ∎

Definition 17.2. A relation F on one groupoid U to another V is called n–subhomogeneous, for some $n \in \mathbb{N}$, if

$$F(nu) \subset nF(u)$$

for all $u \in U$. Moreover, F is called $\mathbb{N}$–subhomogeneous if it is n–subhomogeneous for all $n \in \mathbb{N}$.

Example 17.3. If R is a translation relation on a vector space X such that $n^{-1}R(0) \subset R(0)$ for all $n \in \mathbb{N}$, then R is $\mathbb{N}$–subhomogeneous.

Namely, by Theorem 4.1, we have

$$R(nx) = nx + R(0) = n\left(x + \frac{1}{n}R(0)\right) \subset n\big(x + R(0)\big) = nR(x)$$

for all $x \in X$ and $n \in \mathbb{N}$.

Definition 17.3. A relation F on one groupoid U to another V is called Φ–approximately n–subhomogeneous, for some relation Φ on U to V and some $n \in \mathbb{N}$, if

$$F(nu) \subset nF(u) + \Phi(u)$$

for all $u \in U$. Moreover, F is called Φ–approximately $\mathbb{N}$–subhomogeneous if it is Φ–approximately n–subhomogeneous for all $n \in \mathbb{N}$.

Remark 17.2. If F is a relation of a groupoid U to a groupoid V with zero and $\Theta(u) = \{0\}$ for all $u \in U$, then for any $n \in \mathbb{N}$ the following assertions are equivalent:

(1) F is n–subhomogeneous;
(2) F is Θ–approximately n–subhomogeneous.

It is important to note that Definition 17.3 is a natural generalization of Definition 4.1 of [88].

Example 17.4. If f is a function of a groupoid U to a normed space X, φ is a non-negative function of U and Φ_φ is a relation of U to X such that

$$\Phi_\varphi(u) = \bar{B}_{\varphi(u)}(0)$$

for all $u \in U$, then for any $n \in \mathbb{N}$ the following assertions are equivalent:

(1) f is φ–approximately n–homogeneous;
(2) f is Φ_φ–approximately n–subhomogeneous.

Namely, by the corresponding definition, (1) means only that

$$\big\| f(nu) - nf(u) \big\| \le \varphi(u)$$

for all $u \in U$. And this is equivalent to the requirement that

$$f(nu) - nf(u) \in \bar{B}_{\varphi(u)}(0),$$

or equivalently

$$f(nu) \in nf(u) + \bar{B}_{\varphi(u)}(0) = nf(u) + \Phi_\varphi(u)$$

for all $u \in U$.

Remark 17.3. In addition to the above example, note that

$$\begin{aligned}\big(\Phi_\varphi\big)_n(u) &= \frac{1}{2^n}\Phi_\varphi(2^n u)\\ &= \frac{1}{2^n}\bar{B}_{\varphi(2^n u)}(0)\\ &= \bar{B}_{\frac{1}{2^n}\varphi(2^n u)}(0) = \bar{B}_{\varphi_n(u)}(0) = \Phi_{\varphi_n}(u)\end{aligned}$$

for all $u \in U$ and $n \in \mathbb{N}$. Thus, $\big(\Phi_\varphi\big)_n = \Phi_{\varphi_n}$ also holds for all $n \in \mathbb{N}$.

Theorem 17.4. *If F and Φ are relations of a groupoid U to a vector space X, then the following assertions are equivalent:*

(1) $F_1 \subset F + \frac{1}{2}\Phi$;
(2) *F is Φ-approximately 2–subhomogeneous.*

Proof. For any $u \in U$, we have

$$\begin{aligned} F_1(u) \subset \left(F + \frac{1}{2}\Phi\right)(u) \quad &\Longleftrightarrow \quad \frac{1}{2}F\big(2u\big) \subset F(u) + \frac{1}{2}\Phi(u)\\ &\Longleftrightarrow \quad F(2u) \subset 2F(u) + \Phi(u). \end{aligned}$$

∎

The $\Phi = U \times \{0\}$ particular case of the above theorem immediately yields

Corollary 17.5. *If F is a relation of a groupoid U to a vector space X, then the following assertions are equivalent:*

(1) $F_1 \subset F$;
(2) *F is 2–subhomogeneous.*

18. Further Results on Subhomogeneous Relations

Theorem 18.1. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector space X, then for any $n \in \{0\} \cup \mathbb{N}$ we have*

$$F_{n+1} \subset F_n + \frac{1}{2} \Phi_n .$$

Proof. By Theorem 17.4 and Remark 15.1, we have

$$F_1 \subset F + \frac{1}{2} \Phi = F_0 + \frac{1}{2} \Phi_0 .$$

Therefore, the required inclusion is true for $n = 0$. Moreover, if the required inclusion is true for some $n \in \{0\} \cup \mathbb{N}$, then by Theorems 15.4, 16.1 and 16.3,

$$F_{n+2} = \big(F_{n+1}\big)_1 \subset \left(F_n + \frac{1}{2} \Phi_n\right)_1 = (F_n)_1 + \frac{1}{2} \big(\Phi_n\big)_1 = F_{n+1} + \frac{1}{2} \Phi_{n+1} .$$

Therefore, the required inclusion is also true for $n+1$. ∎

The $\Phi = U \times \{0\}$ particular case of the above theorem immediately yields:

Corollary 18.2. *If F is a 2–subhomogeneous relation of a semigroup U to a vector space X, then $\big(F_n\big)_{n=1}^{\infty}$ is a decreasing sequence of subsets of F.*

In this respect, it is also worth proving the following.

Theorem 18.3. *If F is a 2–subhomogeneous relation on a semigroup U to a vector space X, then the relation $G = \bigcap_{n=1}^{\infty} F_n$ is already 2–homogeneous.*

Proof. If $u \in U$, then by Corollaries 15.5 and 18.2 we have

$$\begin{aligned} G\big(2u\big) &= \left(\bigcap_{n=1}^{\infty} F_n \right) \big(2u\big) = \bigcap_{n=1}^{\infty} F_n\big(2u\big) = \bigcap_{n=1}^{\infty} 2F_{n+1}(u) \\ &= 2 \bigcap_{n=1}^{\infty} F_{n+1}(u) = 2 \bigcap_{n=2}^{\infty} F_n(u) = 2 \bigcap_{n=1}^{\infty} F_n(u) = 2 \left(\bigcap_{n=1}^{\infty} F_n \right) (u) = 2G(u). \end{aligned}$$

Namely, we have $F_2(u) \subset F_1(u)$, and thus $F_1(u) \cap F_2(u) = F_2(u)$. ∎

Theorem 18.4. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector space X, then F_n is Φ_n–approximately 2–subhomogeneous for all $n \in \mathbb{N}$.*

Proof. By Theorems 15.4 and 18.1, we have

$$\big(F_n\big)_1 = F_{n+1} \subset F_n + \frac{1}{2} \Phi_n$$

for all $n \in \mathbb{N}$. Therefore, Theorem 17.4 can be applied. ∎

The $\Phi = U \times \{0\}$ particular case of the above theorem immediately yields:

Corollary 18.5. *If F is a 2–subhomogeneous relation of a semigroup U to a vector space X, then F_n is also 2–subhomogeneous for all $n \in \mathbb{N}$.*

By using Theorem 18.1, we can also prove:

Theorem 18.6. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector space X, then for any $n \in \mathbb{N}$ we have*

$$F_n \subset F + \frac{1}{2} \sum_{i=0}^{n-1} \Phi_i .$$

Proof. By Theorem 17.4 and Remark 15.1, we have

$$F_1 \subset F_0 + \frac{1}{2}\Phi_0 = F + \frac{1}{2}\sum_{i=0}^{1-1}\Phi_i .$$

Therefore, the required inclusion is true for $n = 1$. Moreover, if the required inclusion is true for some $n \in \mathbb{N}$, then by Theorem 18.1 we also obtain

$$F_{n+1} \subset F_n + \frac{1}{2}\Phi_n \subset F + \frac{1}{2}\sum_{i=0}^{n-1}\Phi_i + \frac{1}{2}\Phi_n = F + \frac{1}{2}\left(\sum_{i=0}^{n-1}\Phi_i + \Phi_n\right) = F + \frac{1}{2}\sum_{i=0}^{n+1-1}\Phi_i .$$

Therefore, the required inclusion is also true for $n + 1$. ∎

Now, as a common generalization of Theorems 18.1 and 18.6, we have:

Theorem 18.7. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector space X, then for any $n \in \{0\} \cup \mathbb{N}$ and $k \in \mathbb{N}$,*

$$F_{n+k} \subset F_n + \frac{1}{2}\sum_{i=n}^{n+k-1}\Phi_i .$$

Proof. By Theorems 15.4, 18.6, 16.1 and 16.3, we obtain

$$\begin{aligned} F_{n+k} &= \big(F_k\big)_n \subset \left(F + \frac{1}{2}\sum_{i=0}^{k-1}\Phi_i\right)_n \\ &= F_n + \frac{1}{2}\sum_{i=0}^{k-1}\big(\Phi_i\big)_n = F_n + \frac{1}{2}\sum_{i=0}^{k-1}\Phi_{n+i} = F_n + \frac{1}{2}\sum_{j=n}^{n+k-1}\Phi_j . \end{aligned}$$

∎

Definition 18.1. If Φ is a relation on one groupoid U to another V, then we define

$$\Gamma_\Phi = \bigcup_{k=0}^{\infty}\sum_{i=0}^{k}\Phi_i .$$

Theorem 18.8. *If Φ is a relation of a semigroup U to a vector space X, then for any $n \in \mathbb{N}$ we have*

$$\Gamma_{\Phi_n} = \big(\Gamma_\Phi\big)_n = \bigcup_{k=0}^{\infty}\sum_{i=n}^{n+k}\Phi_i .$$

Proof. By Theorem 15.4, we obtain

$$\Gamma_{\Phi_n} = \bigcup_{k=0}^{\infty}\sum_{i=0}^{k}\big(\Phi_n\big)_i = \bigcup_{k=0}^{\infty}\sum_{i=0}^{k}\Phi_{n+i} = \bigcup_{k=0}^{\infty}\sum_{j=n}^{n+k}\Phi_j .$$

Moreover, by Theorems 15.4, 16.3 and 16.6, we also have

$$\Gamma_{\Phi_n} = \bigcup_{k=0}^{\infty}\sum_{i=0}^{k}\Phi_{n+i} = \bigcup_{k=0}^{\infty}\sum_{i=0}^{k}\big(\Phi_i\big)_n = \left(\bigcup_{k=0}^{\infty}\sum_{i=0}^{k}\Phi_i\right)_n = \big(\Gamma_\Phi\big)_n .$$

∎

Now, as an immediate consequence of Theorems 18.7 and 18.8, we can also state:

Theorem 18.9. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector space X, then for any $n \in \{0\} \cup \mathbb{N}$ and $k \in \mathbb{N}$,*

$$F_{n+k} \subset F_n + \frac{1}{2} \Gamma_{\Phi_n}.$$

Proof. Namely, by Theorems 18.7 and 18.8, we have

$$F_{n+k} \subset F_n + \frac{1}{2} \sum_{i=n}^{n+k-1} \Phi_i \subset F_n + \frac{1}{2} \bigcup_{l=1}^{\infty} \sum_{i=n}^{n+l-1} \Phi_i = F_n + \frac{1}{2} \bigcup_{j=0}^{\infty} \sum_{i=n}^{n+j} \Phi_i = F_n + \frac{1}{2} \Gamma_{\Phi_n}.$$

∎

The $n = 0$ particular case of the above theorem immediately yields:

Corollary 18.10. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector space X, then for any $k \in \mathbb{N}$,*

$$F_k \subset F + \frac{1}{2} \Gamma_{\Phi}.$$

19. Regular and Normal Relations

Analogous to Definition 3.1 of [88], we introduce the following.

Definition 19.1. A relation F on a groupoid U to a vector relator space $X(\mathcal{R})$ will be called:

(1) convergence (adherence) null-regular if the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is convergence (adherence) null for all $u \in U$;
(2) convergence (adherence) regular if the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is convergent (adherent) for all $u \in U$;
(3) convergence (adherence) quasi-regular if the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is convergence (adherence) Cauchy for all $u \in U$;
(4) semi-regular if the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is infinitesimal for all $u \in U$.

Remark 19.1. By the corresponding definitions and Theorem 8.6, it is clear that any one of the above properties implies the subsequent one.

Now, in addition to Definition 19.1, we introduce the following.

Definition 19.2. A relation Φ on a groupoid U to a vector relator space $X(\mathcal{R})$ will be called:

(1) convergence (adherence) null-normal if the relation Γ_Φ is convergence (adherence) null-regular;
(2) convergence (adherence) normal if the relation Γ_Φ is convergence (adherence) regular;
(3) convergence (adherence) quasi-normal if the relation Γ_Φ is convergence (adherence) quasi-regular;
(4) semi-normal if the relation Γ_Φ is semi-regular.

Remark 19.2. By Remark 19.1, any one of the above properties implies the subsequent one.

Moreover, we can easily prove:

Theorem 19.1. *If Φ is a relation of a semigroup U to a vector relator space $X(\mathcal{R})$, then the following assertions are equivalent:*

(1) *Φ is convergence null-normal;*

(2) *for each* $u \in U$ *and* $R \in \mathcal{R}$ *there exists* $n \in \mathbb{N}$ *such that for any* $k, l \in \mathbb{N}$, *with* $n \leq k \leq l$, *we have*

$$\sum_{i=k}^{l} \Phi_i(u) \subset R(0).$$

Proof. If (1) holds and $u \in U$, then by the corresponding definitions and Theorem 18.8 we have

$$0 \in \lim_{n\to\infty}{}_{\mathcal{R}} \big(\Gamma_\Phi\big)_n(u) = \lim_{n\to\infty}{}_{\mathcal{R}} \left(\bigcup_{j=0}^{\infty} \sum_{i=n}^{n+j} \Phi_i \right)(u)$$
$$= \lim_{n\to\infty}{}_{\mathcal{R}} \bigcup_{j=0}^{\infty} \sum_{i=n}^{n+j} \Phi_i(u).$$

Therefore, for each $R \in \mathcal{R}$, there exists $n \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, with $k \geq n$, we have

$$\bigcup_{j=0}^{\infty} \sum_{i=k}^{k+j} \Phi_i(u) \subset R(0).$$

Hence, for any $l \in \mathbb{N}$ with $l \geq k$, we have $\sum_{i=k}^{l} \Phi_i(u) \subset R(0)$. Thus, (2) also holds. The converse implication (2) $\Longrightarrow$ (1) can be similarly proved. ∎

Now, as an immediate consequence of the above theorem, we can also state:

Corollary 19.2. *If* Φ *is a convergence null-normal relation of a semigroup* U *to a vector relator space* $X(\mathcal{R})$, *then* Φ *is convergence null-regular.*

Proof. If $u \in U$ and $R \in \mathcal{R}$, then by Theorem 19.1 there exists $n \in \mathbb{N}$ such that

$$\Phi_k(u) = \sum_{i=k}^{k} \Phi_i(u) \subset R(0)$$

for all $k \in \mathbb{N}$ with $k \geq n$. Therefore, $0 \in \lim_{n\to\infty}{}_{\mathcal{R}} \Phi_n(u)$. Thus, the assertion is also true. ∎

Using a particular case of Definition 3.9 of [88], we can also state the following:

Example 19.1. If φ is a null-normal, non-negative function of a semigroup U in the sense that

$$S_\varphi(u) = \sum_{i=0}^{\infty} \varphi_i(u) < +\infty$$

for all $u \in U$, then the relation Φ_φ considered in Example 17.4 is convergence null-normal. Moreover, we have

$$\Gamma_{\Phi_\varphi} \subset \Phi_{S_\varphi}.$$

Namely, if $u \in U$ and $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that

$$\sum_{i=k}^{l} \varphi_i(u) < \frac{\varepsilon}{2}$$

for all $k, l \in \mathbb{N}$ with $n \leq k \leq l$. Therefore, by Remark 17.3, we also have

$$\sum_{i=k}^{l} \big(\Phi_\varphi\big)_i(u) = \sum_{i=k}^{l} \bar{B}_{\varphi_i(u)}(0) \subset \bar{B}_{\sum\limits_{i=k}^{l} \varphi_i(u)}(0) \subset \bar{B}_{\varepsilon/2}(0) \subset B_\varepsilon(0)$$

for all $k, l \in \mathbb{N}$ with $n \le k \le l$. Hence, by Theorem 19.1, the relation Φ_φ is convergence null-normal. Similarly, we can see that

$$\Gamma_{\Phi_\varphi}(u) = \bigcup_{k=0}^{\infty} \sum_{i=0}^{k} (\Phi_\varphi)_i(u) \subset \bigcup_{k=0}^{\infty} \bar{B}_{\sum_{i=0}^{k} \varphi_i(u)}(0) \subset \bigcup_{k=0}^{\infty} \bar{B}_{S_\varphi(u)}(0) = \bar{B}_{S_\varphi(u)}(0) = \Phi_{S_\varphi}(u)$$

for all $u \in U$. Therefore, the inclusion $\Gamma_{\Phi_\varphi} \subset \Phi_{S_\varphi}$ is also true.

20. An Approximate Selection Theorem

By using Theorems 14.8 and 14.9, we can easily establish the following.

Theorem 20.1. *If F and Φ are relations on a groupoid U to a vector relator space $X(\mathcal{R})$ such that at least one of the following conditions holds:*

(1) *F is convergence (adherence) quasi-regular and Φ is convergence quasi-normal;*
(2) *F is convergence quasi-regular and Φ is convergence (adherence) quasi-normal;*

then the relation $F + \frac{1}{2}\Gamma_\Phi$ is convergence (adherence) quasi-regular.

Proof. If $u \in U$ and (1) holds, then by Definition 19.1 the sequence $\big(F_n(u)\big)_{n=1}^{\infty}$ is convergence (adherence) Cauchy. Moreover, by Definitions 19.2 and 19.1, the sequence $\big((\Gamma_\Phi)_n(u)\big)_{n=1}^{\infty}$ is convergence Cauchy. Furthermore, by Theorem 16.3, we have

$$\left(F + \frac{1}{2}\Gamma_\Phi\right)_n(u) = F_n(u) + \frac{1}{2}(\Gamma_\Phi)_n(u)$$

for all $n \in \mathbb{N}$. Hence, by Theorems 14.8 and 14.9, the sequence $\big((F + \frac{1}{2}\Gamma_\Phi)_n(u)\big)_{n=1}^{\infty}$ is also convergence (adherence) Cauchy. Therefore, by Definition 19.1, the relation $F + \frac{1}{2}\Gamma_\Phi$ is convergence (adherence) quasi-regular. ∎

Since "quasi-regular convergence (adherence)" implies "semi-infinitesimal", by using Theorem 18.9, we can also prove the following.

Theorem 20.2. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a vector relator space $X(\mathcal{R})$ such that the relation $F + \frac{1}{2}\Gamma_\Phi$ is semi-regular, then F is convergence quasi-regular.*

Proof. If $u \in U$, then by Definition 19.1 the sequence $\big((F + \frac{1}{2}\Gamma_\Phi)_n(u)\big)_{n=1}^{\infty}$ is infinitesimal. Thus, by Definition 8.1, for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that

$$\left(F + \frac{1}{2}\Gamma_\Phi\right)_n(u) \subset R(x).$$

Now, if $m \in \mathbb{N}$ such that $m > n$, then by Theorems 18.9, 18.8 and 16.3,

$$\begin{aligned} F_m(u) &\subset \left(F_n + \frac{1}{2}\Gamma_{\Phi_n}\right)(u) = \left(F_n + \frac{1}{2}(\Gamma_\Phi)_n\right)(u) \\ &= \left(F + \frac{1}{2}\Gamma_\Phi\right)_n(u) \subset R(x). \end{aligned}$$

Therefore, by Definitions 8.3 and 8.2, the sequence $\big(f_n(u)\big)_{n=1}^{\infty}$ is convergence Cauchy. Thus, by Definition 19.1, the function f is convergence quasi-regular. ∎

By the above theorem, we have:

Corollary 20.3. *If F is as in the above theorem and $f_{(n)}$ is a selection of F_n for all $n \in \mathbb{N}$, then $\big(f_{(n)}(u)\big)_{n=1}^{\infty}$ is a convergence Cauchy sequence in $X(\mathcal{R})$ for all $u \in U$.*

Now, analogous to Theorem 5.5 of [88], we can prove the following.

Theorem 20.4. *If F is a Φ–approximately 2–subhomogeneous relation of a semigroup U to a separated, sequentially convergence point-complete vector relator space $X(\mathcal{R})$ such that the relation $F+\frac{1}{2}\Gamma_\Phi$ is semi-regular, then there exists a unique 2–homogeneous function f of U to X such that*

$$f(u)\in \operatorname{cl}_{\mathcal{R}}\left(F(u)+\frac{1}{2}\Gamma_\Phi(u)\right)$$

for all $u\in U$.

Proof. By Theorem 15.6, for each $n\in\mathbb{N}$, there exists a selection $f_{(n)}$ of F_n such that

$$f_{(n)}(2u)=2f_{(n+1)}(u)$$

for all $u\in U$ and $n\in\mathbb{N}$. Moreover, by Corollary 20.3, $\big(f_{(n)}(u)\big)_{n=1}^{\infty}$ is a convergence Cauchy sequence in $X(\mathcal{R})$ for all $u\in U$. Therefore, by Definitions 8.4 and 8.3, $\lim\limits_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(u)\neq\emptyset$ for all $u\in U$. Thus, by the axiom of choice, there exists a function f of U to X such that $f(u)\in\lim\limits_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(u)$ for all $u\in U$. Moreover, by Theorem 13.7 and Corollary 12.7, we necessarily have $\{f(u)\}=\lim\limits_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(u)$ for all $u\in U$. Hence, by identifying singletons with their elements, we may also write

$$f(u)=\lim_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(u)$$

for all $u\in U$. Now, by Theorem 14.1 and Definition 8.2, it is clear that

$$f(2u)=\lim_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(2u)=\lim_{n\to\infty}{}_{\mathcal{R}} 2f_{(n+1)}(u)=2\lim_{n\to\infty}{}_{\mathcal{R}} f_{(n+1)}(u)=2f(u)$$

for all $u\in U$. Therefore, f is 2–homogeneous. Moreover, by Corollary 18.10, we have

$$f_{(n)}(u)\in F_n(u)\subset\left(F+\frac{1}{2}\Gamma_\Phi\right)(u)=F(u)+\frac{1}{2}\Gamma_\Phi(u)$$

for all $u\in U$ and $n\in\mathbb{N}$. Hence, by Remark 8.3 and Corollary 8.5,

$$f(u)=\lim_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(u)\in\operatorname*{adh}_{n\to\infty}{}_{\mathcal{R}} f_{(n)}(u)\subset\operatorname{cl}_{\mathcal{R}}\left(F(u)+\frac{1}{2}\Gamma_\Phi(u)\right)$$

also holds for all $u\in U$. ∎

The unicity of the above function f is immediate by:

Theorem 20.5. *If F and Φ are relations of a semigroup U to a separated vector relator space $X(\mathcal{R})$ such that the relation $F+\frac{1}{2}\Gamma_\Phi$ is semi-regular, and f is a 2–homogeneous function of U to X such that*

$$f(u)\in \operatorname{cl}_{\mathcal{R}}\left(F(u)+\frac{1}{2}\Gamma_\Phi(u)\right)$$

for all $u\in U$, then we necessarily have

$$\{f(u)\}=\bigcap_{n=1}^{\infty}\operatorname{cl}_{\mathcal{R}}\left(F_n(u)+\frac{1}{2}\Gamma_{\Phi_n}(u)\right)$$

for all $u\in U$.

Proof. Now, we have

$$f(u) \in \operatorname{cl}_{\mathcal{R}}\Big(F(u) + \frac{1}{2}\Gamma_\Phi(u)\Big) = \operatorname{cl}_{\mathcal{R}}\Big(\Big(F + \frac{1}{2}\Gamma_\Phi\Big)(u)\Big) = \Big(F + \frac{1}{2}\Gamma_\Phi\Big)^{-}(u)$$

for all $u \in U$. Hence, by using Corollary 17.3 and Theorem 16.5, we can infer that

$$f(u) \in \bigg(\Big(F + \frac{1}{2}\Gamma_\Phi\Big)^{-}\bigg)_n(u) = \bigg(\Big(F + \frac{1}{2}\Gamma_\Phi\Big)_n\bigg)^{-}(u) = \operatorname{cl}_{\mathcal{R}}\bigg(\Big(F + \frac{1}{2}\Gamma_\Phi\Big)_n(u)\bigg)$$

for all $u \in U$ and $n \in \mathbb{N}$, Therefore, f is a selection of the relation G defined by

$$G(u) = \bigcap_{n=1}^{\infty} \operatorname{cl}_{\mathcal{R}}\bigg(\Big(F + \frac{1}{2}\Gamma_\Phi\Big)_n(u)\bigg)$$

for all $u \in U$. Moreover, we note that if $u \in U$, then by Definition 19.1 the sequence $\big(\big(F + \frac{1}{2}\Gamma_\Phi\big)_n(u)\big)_{n=1}^{\infty}$ is infinitesimal. Hence, by Theorems 13.1 and 11.6, the sequence $\big(\operatorname{cl}_{\mathcal{R}}\big(\big(F + \frac{1}{2}\Gamma_\Phi\big)_n(u)\big)\big)_{n=1}^{\infty}$ is also infinitesimal. Thus, by Remark 8.1, $G(u)$ is an infinitesimal subset of $X(\mathcal{R})$. Hence, by Theorems 13.1, 13.7 and 12.8, $G(u)$ is at most a singleton. Therefore, we necessarily have $G(u) = \{f(u)\}$. Moreover, by using Theorems 16.3 and 18.8, we can see that

$$G(u) = \bigcap_{n=1}^{\infty} \operatorname{cl}_{\mathcal{R}}\bigg(\Big(F + \frac{1}{2}\Gamma_\Phi\Big)_n(u)\bigg) = \bigcap_{n=1}^{\infty} \operatorname{cl}_{\mathcal{R}}\Big(F_n(u) + \frac{1}{2}\Gamma_{\Phi_n}(u)\Big)$$

also holds for all $u \in U$. ∎

Remark 20.1. Note that if f is a function and Φ is a relation of a groupoid U to a vector relator space $X(\mathcal{R})$, then by Theorem 13.4 we have

$$\operatorname{cl}_{\mathcal{R}}\Big(f_n(u) + \frac{1}{2}\Gamma_{\Phi_n}(u)\Big) = f_n(u) + \frac{1}{2}\operatorname{cl}_{\mathcal{R}}\Big(\Gamma_{\Phi_n}(u)\Big)$$

for all $u \in U$ and $n \in \{0\} \cup \mathbb{N}$.

21. Further Simplifications in the Functional Case

By using our former results, in addition to Remark 20.1, we have the following.

Theorem 21.1. *If f is a Φ–approximately 2–subhomogeneous function of a semigroup U to a relator space $X(\mathcal{R})$ and the relation Φ is semi-normal, then the function f is convergence quasi-regular.*

Proof. If $u \in U$, then by Definitions 19.2 and 19.1 the sequence $\big(\big(\Gamma_\Phi\big)_n(u)\big)_{n=1}^{\infty}$ is infinitesimal. Hence, by Theorem 14.6, the sequence $\big(\frac{1}{2}\big(\Gamma_\Phi\big)_n(u)\big)_{n=1}^{\infty}$ is also infinitesimal. Thus, by Definition 8.1, for each $R \in \mathcal{R}$ there exist $x \in X$ and $n \in \mathbb{N}$ such that

$$\frac{1}{2}\big(\Gamma_\Phi\big)_n(u) \subset R(x).$$

Now, if $m \in \mathbb{N}$ such that $m > n$, then by Theorem 18.9 and 18.8,

$$\begin{aligned} f_m(u) \in \Big(f_n + \frac{1}{2}\Gamma_{\Phi_n}\Big)(u) &= f_n(u) + \frac{1}{2}\Gamma_{\Phi_n}(u) \\ &= f_n(u) + \frac{1}{2}\big(\Gamma_\Phi\big)_n(u) \subset f_n(u) + R(x) \\ &= R\big(f_n(u) + x\big). \end{aligned}$$

Therefore, by Definitions 8.2 and 8.3, the sequence $\big(f_n(u)\big)_{n=1}^{\infty}$ is convergence Cauchy. Thus, by Definition 19.1, the function f is convergence quasi-regular. ∎

Hence, by Theorem 20.1, we also have:

Corollary 21.2. *If f is as in Theorem 21.1 and the relation Φ is adherence quasi-normal, then the relation $f+\frac{1}{2}\Gamma_\Phi$ is adherence quasi-regular.*

Now, by using the results of Section 20, we can establish the following.

Theorem 21.3. *If f is a Φ–approximately 2–subhomogeneous function of a semigroup U to a separated, sequentially convergence point-complete vector relator space $X(\mathcal{R})$ and the relation Φ is adherence quasi-normal, then there exists a 2–homogeneous function g of U to X such that*

$$g(u) \in f(u) + \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\Gamma_\Phi(u)\big)$$

for all $u \in U$. Moreover, we have

$$\{g(u)\} = \bigcap_{n=1}^{\infty}\Big(f_n(u) + \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\Gamma_{\Phi_n}(u)\big)\Big)$$

for all $u \in U$.

Proof. Now, by Corollary 21.2 and Remark 19.1, the relation $f+\frac{1}{2}\Gamma_\Phi$ is semi-regular. Thus, by Theorem 20.4, there exists a unique 2–homogeneous function g of U to X such that

$$g(u) \in \mathrm{cl}_{\,\mathcal{R}}\Big(f(u) + \frac{1}{2}\Gamma_\Phi(u)\Big)$$

for all $u \in U$. Moreover, by Theorem 20.5, we have

$$\{g(u)\} = \bigcap_{n=1}^{\infty}\mathrm{cl}_{\,\mathcal{R}}\Big(f_n(u) + \frac{1}{2}\Gamma_{\Phi_n}(u)\Big)$$

for all $u \in U$. Hence, by Remark 20.1, the required assertions are also true. ∎

In addition to the latter theorem, we prove the following.

Theorem 21.4. *If f is a function and Φ is a relation of a semigroup U to a separated vector relator space $X(\mathcal{R})$ such that Φ is convergence null-normal, and g is a 2–homogeneous function of U to X such that*

$$g(u) \in f(u) + \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\Gamma_\Phi(u)\big)$$

for all $u \in U$, then we necessarily have

$$g(u) = \lim_{n\to\infty}{}_{\mathcal{R}}\, f_n(u)$$

for all $u \in U$.

Proof. For any $u \in U$, we have

$$g(u) \in f(u) + \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\Gamma_\Phi(u)\big) = f(u) + \frac{1}{2}\big(\Gamma_\Phi\big)^{-}(u) = \Big(f + \frac{1}{2}\big(\Gamma_\Phi\big)^{-}\Big)(u).$$

By using Corollary 17.3 and Theorems 16.3 and 16.5, we can infer that

$$\begin{aligned} g(u) \in \Big(f + \frac{1}{2}\big(\Gamma_\Phi\big)^-\Big)_n(u) &= \Big(f_n + \frac{1}{2}\big(\big(\Gamma_\Phi\big)^-\big)_n\Big)(u) \\ &= \Big(f_n + \frac{1}{2}\big(\big(\Gamma_\Phi\big)_n\big)^-\Big)(u) \\ &= f_n(u) + \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\big(\Gamma_\Phi\big)_n(u)\big) \end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, by Definitions 19.2 and 19.1, we have

$$0 \in \lim_{n\to\infty}{}_{\mathcal{R}}\, \big(\Gamma_\Phi\big)_n(u).$$

Hence, by using Theorems 13.1, 11.5 and 14.1, we can infer that

$$0 \in \lim_{n\to\infty}{}_{\mathcal{R}}\, \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\big(\Gamma_\Phi\big)_n(u)\big).$$

Thus, by Definition 8.2, for any $R \in \mathcal{R}$ there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\big(\Gamma_\Phi\big)_k(u)\big) \subset R(0)$$

for all $k \in \mathbb{N}$ with $k \geq n$. Hence,

$$g(u) \in f_k(u) + \frac{1}{2}\,\mathrm{cl}_{\,\mathcal{R}}\big(\big(\Gamma_\Phi\big)_k(u)\big) \subset f_k(u) + R(0) = R\big(f_k(u)\big),$$

and thus

$$f_k(u) \in R^{-1}\big(g(u)\big) = R\big(g(u)\big)$$

for all $k \in \mathbb{N}$ with $k \geq n$. Therefore,

$$g(u) \in \lim_{n\to\infty}{}_{\mathcal{R}} f_n(u)$$

also holds. By Theorem 13.7 and Corollary 12.7, it is clear that the corresponding equality is also true. ∎

Now, to illustrate the above results, we prove the following counterpart of a straightforward extension of Găvruţă's theorem.

Example 21.1. If f is a φ-approximately 2–homogeneous function of a semigroup U to a Banach space X and φ is null-normal, then there exists a 2–homogeneous function g of U to X such that

$$\|f(u) - g(u)\| \leq \frac{1}{2} S_\varphi(u)$$

for all $u \in U$. Moreover, we have

$$g(u) = \lim_{n\to\infty} f_n(u)$$

for all $u \in U$.

To derive this, note that by Example 17.4, f is Φ_φ–approximately 2–homogeneous. Moreover, by Example 19.1, Φ_φ is convergence null-normal. Therefore, by Theorem 21.3, there exists a 2–homogeneous function g of U to X such that

$$g(u) \in f(u) + \frac{1}{2}\,\mathrm{cl}_{\mathcal{R}}\big(\Gamma_{\Phi_\varphi}(u)\big)$$

for all $u \in U$. By Theorem 21.4, we also have $g(u) = \lim\limits_{n\to\infty} f_n(u)$ for all $u \in U$ and Example 19.1 shows that

$$\Gamma_{\Phi_\varphi}(u) \subset \Phi_{S_\varphi}(u) = \bar{B}_{S_\varphi(u)}(0)$$

for all $u \in U$. Hence,

$$g(u) \in f(u) + \frac{1}{2}\operatorname{cl}_{\mathcal{R}}\big(\bar{B}_{S_\varphi(u)}(0)\big) = f(u) + \frac{1}{2}\bar{B}_{S_\varphi(u)}(0) = f(u) + \bar{B}_{\frac{1}{2}S_\varphi(u)}(0),$$

for all $u \in U$ and thus

$$\|f(u) - g(u)\| \leq \frac{1}{2}S_\varphi(u)$$

holds for all $u \in U$.

22. APPROXIMATELY SUBADDITIVE RELATIONS

Definition 22.1. A relation F on one groupoid U to another V is called Ψ–approximately subadditive, for some relation Ψ of U^2 to V, if

$$F(u+v) \subset \big(F(u) + F(v)\big) + \Psi(u,v)$$

for all $u, v \in U$.

The following example shows that this definition is a generalization of Definition 6.1 of [88].

Example 22.1. If f is a function of a groupoid U to a normed space X, ψ is a non-negative function of U^2 and Ψ_ψ is a relation of U^2 to X such that

$$\Psi_\psi(u,v) = \bar{B}_{\psi(u,v)}(0)$$

for all $u, v \in U$, then the following assertions are equivalent:

(1) f is ψ–approximately additive;
(2) f is Ψ_ψ– approximately subadditive.

By the corresponding definition, (1) simply means that

$$\big\|f(u+v) - \big(f(u) + f(v)\big)\big\| \leq \psi(u,v)$$

for all $u, v \in U$. This is equivalent to the requirement that

$$f(u+v) - \big(f(u) + f(v)\big) \in \bar{B}_{\psi(u,v)}(0),$$

or

$$f(u+v) \in f(u) + f(v) + \bar{B}_{\psi(u,v)}(0) = f(u) + f(v) + \Psi_\psi(u,v)$$

for all $u, v \in U$. That is, (2) holds.

We now establish the following result concerning approximately subadditive functions.

Theorem 22.1. *If F is a Ψ–approximately subadditive relation of a commutative semigroup U to a vector space X, then F_n is Ψ_n– approximately subadditive for all $n \in \mathbb{N}$.*

Proof. By the corresponding definitions and Theorem 3.2, we have

$$\begin{aligned}
F_n(u+v) &= \frac{1}{2^n}F\big(2^n(u+v)\big)\\
&= \frac{1}{2^n}F\big(2^nu + 2^nv\big)\\
&\subset \frac{1}{2^n}\Big(F\big(2^nu\big) + F\big(2^nv\big) + \Psi\big(2^nu,\ 2^nv\big)\Big)\\
&= \frac{1}{2^n}F\big(2^nu\big) + \frac{1}{2^n}F\big(2^nv\big) + \frac{1}{2^n}\Psi\big(2^n(u,v)\big)\\
&= F_n(u) + F_n(v) + \Psi_n(u,v)
\end{aligned}$$

for all $u, v \in U$ and $n \in \mathbb{N}$. Therefore, the required assertion is true. ∎

Definition 22.2. If U and V are sets and Ψ is a relation on U^2 to V, then we define

$$\Phi_\Psi(u) = \Psi(u, u)$$

for all $u \in U$.

Simple applications of the corresponding definitions immediately yield the following theorems.

Theorem 22.2. *If U is a groupoid and Ψ is a relation of U^2 to a vector space X, then for any $n \in \mathbb{N}$,*

$$\Phi_{\Psi_n} = \big(\Phi_\Psi\big)_n.$$

Proof. We evidently have

$$\Phi_{\Psi_n}(u) = \Psi_n(u, u) = \frac{1}{2^n}\Psi\big(2^n u, 2^n u\big) = \frac{1}{2^n}\Phi_\Psi\big(2^n u\big) = \big(\Phi_\Psi\big)_n(u)$$

for all $u \in U$. Therefore, the required equality is true. ∎

Theorem 22.3. *If F is a midconvex-valued, Ψ–approximately subadditive relation of a groupoid U to a vector space X, then F is Φ_Ψ–approximately 2–subhomogeneous.*

Proof. Namely, we have

$$\begin{aligned} F(2u) &= F(u+u) \subset F(u) + F(u) + \Psi(u, u) \\ &= 2\left(\frac{1}{2}F(u) + \frac{1}{2}F(u)\right) + \Phi_\Psi(u) \subset 2F(u) + \Phi_\Psi(u) \end{aligned}$$

for all $u \in U$. Therefore, by Definition 17.3, the required assertion is true. ∎

Now, analogous to Theorem 6.8 of [88], we can also prove the following:

Theorem 22.4. *If F is a midconvex-valued, Ψ–approximately subadditive relation of a commutative semigroup U to a separated, sequentially convergence point-complete vector relator space $X(\mathcal{R})$ such that F is adherence quasi-regular, Ψ is convergence null-regular and Φ_Ψ is convergence null-normal, then there exists an additive function f of U to X such that*

$$f(u) \in \mathrm{cl}_{\,\mathcal{R}}\left(F(u) + \frac{1}{2}\Gamma_{\Phi_\Psi}(u)\right)$$

for all $u \in U$. Moreover, we have

$$\{f(u)\} = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\,\mathcal{R}}\Big(F_n(u) + \frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)\Big)$$

for all $u \in U$.

Proof. Now, by Theorem 22.3, F is Φ_Ψ–approximately 2–subhomogeneous, and by Theorem 20.1, the relation $F + \frac{1}{2}\Gamma_{\Phi_\Psi}$ is adherence quasi-regular and hence it is semi-regular. Therefore, by Theorem 20.2, the relation F is actually convergence quasi-regular. Moreover, by Theorem 20.4, there exists a unique 2–homogeneous function f of U to X such that

$$f(u) \in \mathrm{cl}_{\,\mathcal{R}}\Big(F(u) + \frac{1}{2}\Gamma_{\Phi_\Psi}(u)\Big)$$

for all $u \in U$. By Theorems 20.5 and 22.2, we have

$$\{f(u)\} = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\,\mathcal{R}}\Big(F_n(u) + \frac{1}{2}\Gamma_{(\Phi_\Psi)_n}(u)\Big) = \bigcap_{n=1}^{\infty} \mathrm{cl}_{\,\mathcal{R}}\Big(F_n(u) + \frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)\Big)$$

for all $u \in U$. Therefore, we need only show that f is additive. For this, note that if $u, v \in U$, then by the above equality and Theorem 22.1 we have

$$f(u+v) \in \operatorname{cl}_{\mathcal{R}}\Big(F_n(u+v)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u+v)\Big)$$
$$\subset \operatorname{cl}_{\mathcal{R}}\Big(F_n(u)+F_n(v)+\Psi_n(u,v)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u+v)\Big)$$

for all $n \in \mathbb{N}$. By using Theorem 13.4, we can also see that

$$f(u)+f(v) \in \operatorname{cl}_{\mathcal{R}}\Big(F_n(u)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)\Big)+\operatorname{cl}_{\mathcal{R}}\Big(F_n(v)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(v)\Big)$$
$$\subset \operatorname{cl}_{\mathcal{R}}\Big(F_n(u)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)+F_n(v)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(v)\Big)$$
$$=\operatorname{cl}_{\mathcal{R}}\Big(F_n(u)+F_n(v)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(v)\Big)$$

for all $n \in \mathbb{N}$. Hence, by defining

$$A_n = F_n(u)+F_n(v),$$

$$B_n = \Psi_n(u,v)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u+v) \qquad \text{and} \qquad C_n = \frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(v)$$

for all $n \in \mathbb{N}$, we note that

$$\big\{f(u+v),\ f(u)+f(v)\big\} \subset \bigcap_{n=1}^{\infty}\operatorname{cl}_{\mathcal{R}}\Big(\big(A_n+B_n\big)\cup\big(A_n+C_n\big)\Big).$$

Now, to complete the proof, we need only note the following facts. Since the relation F is convergence quasi-regular, $\big(F_n(u)\big)_{n=1}^{\infty}$ and $\big(F_n(v)\big)_{n=1}^{\infty}$ are convergence Cauchy sequences in $X(\mathcal{R})$. Thus, by Theorem 14.11, $\big(A_n\big)_{n=1}^{\infty}$ is also a convergence Cauchy sequence in $X(\mathcal{R})$. Since Ψ is convergence null-regular, $\big(\Psi_n(u,v)\big)_{n=1}^{\infty}$ is a convergence null-sequence in $X(\mathcal{R})$. As Φ_Ψ is convergence null-normal and

$$\Gamma_{\Phi_{\Psi_n}} = \Gamma_{(\Phi_\Psi)_n} = \big(\Gamma_{\Phi_\Psi}\big)_n$$

for all $n \in \mathbb{N}$, we also note that $\big(\Gamma_{\Phi_{\Psi_n}}(u)\big)_{n=1}^{\infty}$, $\big(\Gamma_{\Phi_{\Psi_n}}(v)\big)_{n=1}^{\infty}$ and $\big(\Gamma_{\Phi_{\Psi_n}}(u+v)\big)_{n=1}^{\infty}$ are convergence null sequences in $X(\mathcal{R})$. Thus, by Corollaries 14.2 and 14.5, $\big(B_n\big)_{n=1}^{\infty}$ and $\big(C_n\big)_{n=1}^{\infty}$ are also convergence null-sequences in $X(\mathcal{R})$.

Hence, by Theorem 14.12, we can see that $\big(\big(A_n+B_n\big)\cup\big(A_n+C_n\big)\big)_{n=1}^{\infty}$ is a convergence Cauchy sequence in $X(\mathcal{R})$. Thus, it is infinitesimal and by Theorems 13.1 and 11.6,

$$\Big(\operatorname{cl}_{\mathcal{R}}\big(\big(A_n+B_n\big)\cup\big(A_n+C_n\big)\big)\Big)_{n=1}^{\infty}$$

is also an infinitesimal sequence in $X(\mathcal{R})$. Now, by Remark 8.1, it is clear that $\{f(u+v),\ f(u)+f(v)\}$ is an infinitesimal subset of $X(\mathcal{R})$. Therefore, by Theorems 13.1, 13.7 and 12.8, it is at most a singleton. Hence, $f(u+v)=f(u)+f(v)$ also holds. ∎

Remark 22.1. Note that if f is a function of a groupoid U to a vector relator space $X(\mathcal{R})$ and Ψ is a relation of U^2 to $X(\mathcal{R})$, then by Theorem 13.4 we have

$$\operatorname{cl}_{\mathcal{R}}\Big(f_n(u)+\frac{1}{2}\Gamma_{\Phi_{\Psi_n}}(u)\Big) = f_n(u)+\frac{1}{2}\operatorname{cl}_{\mathcal{R}}\Big(\Gamma_{\Phi_{\Psi_n}}(u)\Big)$$

for all $u \in U$ and $n \in \{0\}\cup\mathbb{N}$.

23. Further Simplifications in the Functional Case

By using our former results, in addition to Remark 22.1, we can prove the following.

Theorem 23.1. *If f is a Ψ–approximately subadditive function of a commutative semigroup U to a separated, sequentially convergence point-complete vector relator space $X(\mathcal{R})$, Ψ is convergence null-regular and Φ_Ψ is convergence null-normal, then there exists an additive function g of U to X such that*

$$g(u) \in f(u) + \frac{1}{2}\operatorname{cl}_{\mathcal{R}}\big(\Gamma_{\Phi_\Psi}(u)\big)$$

for all $u \in U$. Moreover, we have

$$g(u) = \lim_{n\to\infty}{}_{\mathcal{R}} f_n(u)$$

for all $u \in U$.

Proof. Now, f is a convex valued relation. Thus, by Theorem 22.3, f is Φ_Ψ–approximately 2–subhomogeneous, and Φ_Ψ is semi-normal. By Theorem 21.1, the function f is convergence quasi-regular and hence it is adherence quasi-regular. Thus, by Theorem 22.4 and Remark 22.1, there exists a unique additive function g of U to X such that

$$g(u) \in f(u) + \frac{1}{2}\operatorname{cl}_{\mathcal{R}}\big(\Gamma_{\Phi_\Psi}(u)\big)$$

for all $u \in U$. Hence, by Example 17.1 and Theorem 21.4, it is clear that

$$g(u) = \lim_{n\to\infty}{}_{\mathcal{R}} f_n(u)$$

also holds for all $u \in U$. ∎

Now, to illustrate the above results, we prove the following straightforward extension of Găvruţă's theorem.

Example 23.1. If f is a ψ-approximately additive function of a commutative semigroup U to a Banach space X, the function ψ is null-regular and the function φ_ψ, defined by

$$\varphi_\psi(u) = \psi(u,u)$$

for all $u \in U$, is null-normal, then there exists an additive function g of U to X such that

$$\|f(u) - g(u)\| \leq \frac{1}{2} S_\varphi(u)$$

for all $u \in U$. Moreover, we have

$$g(u) = \lim_{n\to\infty} f_n(u)$$

for all $u \in U$.

To derive this, note that by Example 22.1, the function f is Ψ_ψ–approximately subadditive. Moreover, by Remark 17.3,

$$\big(\Psi_\psi\big)_n(u,v) = \bar{B}_{\psi_n(u,v)}(0)$$

for all $u, v \in U$ and $n \in \mathbb{N}$. Hence, since ψ is now null-regular in the sense that

$$\lim_{n\to\infty} \psi_n(u,v) = 0$$

for all $u, v \in U$, it is clear that

$$0 \in \lim_{n\to\infty} \big(\Psi_\psi\big)_n(u,v)$$

also holds for all $u, v \in U$. Therefore, the relation Ψ_ψ is convergence null-regular. Moreover, we can see that

$$\Phi_{\Psi_\psi}(u) = \Psi_\psi(u,u) = \bar{B}_{\psi(u,u)}(0) = \bar{B}_{\varphi_\psi(u)}(0) = \Phi_{\varphi_\psi}(u)$$

for all $u \in U$, and thus $\Phi_{\Psi_\psi} = \Phi_{\varphi_\psi}$. Hence, by Example 19.1, it is clear that the relation Φ_{Ψ_ψ} is convergence null-normal. By Theorem 23.1, there exists an additive function g of U to X such that

$$g(u) \in f(u) + \frac{1}{2}\,\mathrm{cl}\big(\Gamma_{\Phi_{\Psi_\psi}}(u)\big)$$

for all $u \in U$. Moreover, we have $g(u) = \lim_{n\to\infty} f_n(u)$ for all $u \in U$. Example 19.1 also shows that

$$\Gamma_{\Phi_{\varphi_\psi}}(u) \subset \Phi_{S_{\varphi_\psi}}(u) = \bar{B}_{S_{\varphi_\psi}(u)}(0)$$

for all $u \in U$. Hence,

$$g(u) \in f(u) + \frac{1}{2}\,\mathrm{cl}\big(\bar{B}_{S_{\varphi_\psi}(u)}(0)\big) = f(u) + \frac{1}{2}\bar{B}_{S_{\varphi_\psi}(u)}(0) = f(u) + \bar{B}_{\frac{1}{2}S_{\varphi_\psi}(u)}(0),$$

for all $u \in U$. Therefore,

$$\|f(u) - g(u)\| \le \frac{1}{2}S_{\varphi_\psi}(u)$$

holds for all $u \in U$.

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