The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 6, Issue 1, Article 14, pp. 1-9, 2009

FIXED POINTS AND STABILITY OF THE CAUCHY FUNCTIONAL EQUATION

CHOONKIL PARK AND THEMISTOCLES M. RASSIAS

Special Issue in Honor of the 100th Anniversary of S.M. Ulam

Received 26 November, 2008; accepted 21 December, 2008; published 4 September, 2009.

DEPARTMENT OF MATHEMATICS
HANYANG UNIVERSITY
SEOUL 133-791, REPUBLIC OF KOREA
baak@hanyang.ac.kr

DEPARTMENT OF MATHEMATICS
NATIONAL TECHNICAL UNIVERSITY OF ATHENS
ZOGRAFOU CAMPUS, 15780 ATHENS, GREECE
trassias@math.ntua.gr

ABSTRACT. Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the Cauchy functional equation.

Key words and phrases: Cauchy functional equation, Fixed point, Homomorphism in Banach algebra, Generalized Hyers-Ulam stability, Derivation on Banach algebra.

2000 *Mathematics Subject Classification.* Primary 39A10, 39B72; Secondary 47H10, 46B03.

ISSN (electronic): 1449-5910

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x * y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is: "How do the solutions of the inequality differ from those of the given functional equation?"

Hyers [7] gave the first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

Th.M. Rassias [27] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

Theorem 1.1 (Th.M. Rassias)**.** *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The inequality (1.1) has been influential in the development of what is now known as the *generalized Hyers-Ulam stability* or *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980, the topic of approximate homomorphisms, or the stability of the equation of homomorphisms, has been studied by a number of mathematicians. Găvruta [6], following Th.M. Rassias' approach for the stability of the linear mapping between Banach spaces obtained a generalization of Th.M. Rassias' Theorem. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [4], [9], [10], [11], [13] – [25], [28] – [30]).

We recall two fundamental results in fixed point theory.

Theorem 1.2 ([1, 2, 26]). *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,*

$$d(Jx, Jy) \leq Ld(x, y), \qquad \forall x, y \in X$$

for some Lipschitz constant $L < 1$. Then

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*
(2) *the fixed point x^* is globally attractive, i.e.,*

$$\lim_{n\to\infty} J^n x = x^*$$

for any starting point $x \in X$;
(3) *one has the following estimation inequalities:*

$$d(J^n x, x^*) \leq L^n d(x, x^*),$$
$$d(J^n x, x^*) \leq \frac{1}{1-L} d(J^n x, J^{n+1}x),$$
$$d(x, x^*) \leq \frac{1}{1-L} d(x, Jx)$$

for all nonnegative integers n and all $x \in X$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.3 ([5]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with a Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

This paper is organized as follows: In Section 2, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the Cauchy functional equation.

In Section 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the Cauchy functional equation.

Throughout this section, assume that A is a complex Banach algebra with norm $\|\cdot\|_A$ and that B is a complex Banach algebra with norm $\|\cdot\|_B$.

2. STABILITY OF HOMOMORPHISMS IN BANACH ALGEBRAS

For a given mapping $f : A \to B$, we define

$$D_\mu f(x, y) := \mu f(x + y) - f(\mu x) - f(\mu y)$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $x, y \in A$.

Note that a $\mathbb{C}$-linear mapping $H : A \to B$ is called a *homomorphism* in Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ such that*

$$\lim_{j\to\infty} 2^{-j}\varphi(2^j x, 2^j y) = 0, \tag{2.1}$$

$$\|D_\mu f(x,y)\|_B \le \varphi(x,y), \tag{2.2}$$

$$\|f(xy) - f(x)f(y)\|_B \le \varphi(x,y) \tag{2.3}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x,x) \le 2L\varphi(\frac{x}{2},\frac{x}{2})$ for all $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{1}{2-2L}\varphi(x,x) \tag{2.4}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to B\}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \le C\varphi(x,x), \quad \forall x \in A\}.$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in A$.

By Theorem 3.1 of [1],

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in X$.

Letting $\mu = 1$ and $y = x$ in (2.2), we get

$$\|f(2x) - 2f(x)\|_B \le \varphi(x,x) \tag{2.5}$$

for all $x \in A$. So

$$\|f(x) - \frac{1}{2}f(2x)\|_B \le \frac{1}{2}\varphi(x,x)$$

for all $x \in A$. Hence $d(f, Jf) \le \frac{1}{2}$.

By Theorem 1.3, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.6}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.6) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x,x)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.7}$$

for all $x \in A$.
(3) $d(f, H) \le \frac{1}{1-L} d(f, Jf)$, which implies,

$$d(f, H) \le \frac{1}{2 - 2L}.$$

This implies that the inequality (2.4) holds.

It follows from (2.1), (2.2) and (2.7) that

$$\begin{aligned} &\|H(x+y) - H(x) - H(y)\|_B \\ &\quad = \lim_{n\to\infty} \frac{1}{2^n} \|f(2^n(x+y)) - f(2^n x) - f(2^n y)\|_B \\ &\quad \le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$H(x+y) = H(x) + H(y) \tag{2.8}$$

for all $x, y \in A$.

Letting $y = x$ in (2.2), we get

$$\mu f(2x) = f(\mu 2x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. By a similar method to that above, we obtain

$$\mu H(2x) = H(2\mu x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Thus one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.3) that

$$\begin{aligned} \|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n xy) - f(2^n x) f(2^n y)\|_B \\ &\le \lim_{n\to\infty} \frac{1}{4^n} \varphi(2^n x, 2^n y) \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y) = 0 \end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Thus $H : A \to B$ is a homomorphism satisfying (2.4), as desired. ∎

Corollary 2.2. *Let $r < \frac{1}{2}$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\|D_\mu f(x, y)\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{2.9}$$

$$\|f(xy) - f(x)f(y)\|_B \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{2.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{\theta}{2 - 4^r} \|x\|_A^{2r}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. ∎

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying (2.2) and (2.3) such that*

$$\lim_{j\to\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0 \tag{2.11}$$

for all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x, x) \le \frac{1}{2} L\varphi(2x, 2x)$ for all $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{L}{2 - 2L}\varphi(x, x) \tag{2.12}$$

for all $x \in A$.

Proof. We consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in A$.

It follows from (2.5) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\|_B \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{2}\varphi(x, x)$$

for all $x \in A$. Hence $d(f, Jf) \le \frac{L}{2}$.

By Theorem 1.3, there exists a mapping $H : A \to B$ such that:
(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.13}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.13) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x, x)$$

for all $x \in A$.
(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x)$$

for all $x \in A$.
(3) $d(f, H) \le \frac{1}{1-L} d(f, Jf)$, which implies,

$$d(f, H) \le \frac{L}{2 - 2L},$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of Theorem 2.1. ∎

Corollary 2.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying (2.9) and (2.10). Then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \le \frac{\theta}{4^r - 2}\|x\|_A^{2r}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

for all $x, y \in A$. Then $L = 2^{1-2r}$ and we get the desired result. ∎

3. STABILITY OF DERIVATIONS ON BANACH ALGEBRAS

Note that a $\mathbb{C}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation $D_\mu f(x,y) = 0$.

Theorem 3.1. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying (2.1) such that*

$$\|D_\mu f(x,y)\|_A \le \varphi(x,y), \tag{3.1}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \varphi(x,y) \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. If there exists an $L < 1$ such that $\varphi(x,x) \le 2L\varphi\left(\frac{x}{2}, \frac{x}{2}\right)$ for all $x \in A$, then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{1}{2-2L}\varphi(x,x) \tag{3.3}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned}
&\|\delta(xy) - \delta(x)y - x\delta(y)\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)\cdot 2^n y - 2^n x f(2^n y)\|_A \\
&\quad \le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). ∎

Corollary 3.2. *Let $r < \frac{1}{2}$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping such that*

$$\|D_\mu f(x,y)\|_A \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r, \tag{3.4}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \theta \cdot \|x\|_A^r \cdot \|y\|_A^r \tag{3.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then there exists a unique derivation $\delta : A \to A$ such that

$$\|f(x) - \delta(x)\|_A \le \frac{\theta}{2-4^r}\|x\|_A^{2r}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

for all $x, y \in A$. Then $L = 2^{2r-1}$ and we get the desired result. ∎

Theorem 3.3. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying (2.11), (3.1) and (3.2). If there exists an $L < 1$ such that $\varphi(x, x) \le \frac{1}{2} L\varphi(2x, 2x)$ for all $x \in A$, then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{L}{2 - 2L}\varphi(x, x)$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.3 and 3.1. ∎

Corollary 3.4. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying (3.4) and (3.5). Then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{\theta}{4^r - 2}\|x\|_A^{2r}$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta \cdot \|x\|_A^r \cdot \|y\|_A^r$$

for all $x, y \in A$. Then $L = 2^{1-2r}$ and we get the desired result. ∎

REFERENCES

[1] L. CĂDARIU and V. RADU, Fixed points and the stability of Jensen's functional equation, *J. Inequal. Pure Appl. Math.*, **4**, no. 1, Art. 4 (2003).

[2] L. CĂDARIU and V. RADU, On the stability of the Cauchy functional equation: a fixed point approach, *Grazer Math. Ber.*, **346** (2004), 43–52.

[3] S. CZERWIK, *Functional Equations and Inequalities in Several Variables*, World Scientific Publ. Co., New Jersey, London, Singapore and Hong Kong, 2002.

[4] S. CZERWIK, *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press Inc., Palm Harbor, Florida, 2003.

[5] J. DIAZ and B. MARGOLIS, A fixed point theorem of the alternative for contractions on a generalized complete metric space, *Bull. Amer. Math. Soc.* **74** (1968), 305–309.

[6] P. GĂVRUTA, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431–436.

[7] D.H. HYERS, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.

[8] D. H. HYERS, G. ISAC and Th. M. RASSIAS, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[9] D.H. HYERS and Th.M. RASSIAS, Approximate homomorphisms, *Aequationes Math.*, **44** (1992), 125–153.

[10] S. JUNG, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press Inc., Palm Harbor, Florida, 2001.

[11] M.S. MOSLEHIAN, On the orthogonal stability of the Pexiderized quadratic equation, *J. Difference Equ. Appl.*, **11** (2005), 999–1004.

[12] A. NAJATI and C. PARK, The Pexiderized Apollonius-Jensen type additive mapping and isomorphisms between C^*-algebras, *J. Difference Equ. Appl.* (to appear).

[13] C. PARK, On the stability of the linear mapping in Banach modules, *J. Math. Anal. Appl.*, **275** (2002), 711–720.

[14] C. PARK, Modified Trif's functional equations in Banach modules over a C^*-algebra and approximate algebra homomorphisms, *J. Math. Anal. Appl.*, **278** (2003), 93–108.

[15] C. PARK, On an approximate automorphism on a C^*-algebra, *Proc. Amer. Math. Soc.*, **132** (2004), 1739–1745.

[16] C. PARK, Lie $*$-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras, *J. Math. Anal. Appl.*, **293** (2004), 419–434.

[17] C. PARK, Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations, *J. Lie Theory*, **15** (2005), 393–414.

[18] C. PARK, Homomorphisms between Poisson JC^*-algebras, *Bull. Braz. Math. Soc.*, **36** (2005), 79–97.

[19] C. PARK, Hyers-Ulam-Rassias stability of a generalized Euler-Lagrange type additive mapping and isomorphisms between C^*-algebras, *Bull. Belgian Math. Soc.-Simon Stevin,* **13** (2006), 619–631.

[20] C. PARK, Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras, *Fixed Point Theory and Applications,* **2007**, Art. ID 50175 (2007).

[21] C. PARK and J. CUI, Generalized stability of C^*-ternary quadratic mappings, *Abstract Appl. Anal.*, **2007**, Art. ID 23282 (2007).

[22] C. PARK and J. HOU, Homomorphisms between C^*-algebras associated with the Trif functional equation and linear derivations on C^*-algebras, *J. Korean Math. Soc.*, **41** (2004), 461–477.

[23] C. PARK and A. NAJATI, Homomorphisms and derivations in C^*-algebras, *Abstract Appl. Anal.*, **2007**, Art. ID 80630 (2007).

[24] C. PARK and J. PARK, Generalized Hyers-Ulam stability of an Euler-Lagrange type additive mapping, *J. Difference Equ. Appl.*, **12** (2006), 1277–1288.

[25] C. PARK and Th.M. RASSIAS, Isometric additive mappings in generalized quasi-Banach spaces, *Banach J. Math. Anal.*, **2** (2008), 59–69.

[26] V. RADU, The fixed point alternative and the stability of functional equations, *Fixed Point Theory,* **4** (2003), 91–96.

[27] Th.M. RASSIAS, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.,* **72** (1978), 297–300.

[28] Th.M. RASSIAS, Problem 16; 2, *Report of the 27^{th} International Symp. on Functional Equations, Aequationes Math.*, **39** (1990), 292–293; 309.

[29] Th.M. RASSIAS, The problem of S.M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.*, **246** (2000), 352–378.

[30] Th.M. RASSIAS, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.*, **251** (2000), 264–284.

[31] Th.M. RASSIAS, On the stability of functional equations and a problem of Ulam, *Acta Appl. Math.*, **62** (2000), 23–130.

[32] S.M. ULAM, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.