The Australian Journal of Mathematical Analysis and Applications

AJMAA

Volume 6, Issue 1, Article 12, pp. 1-8, 2009

STABILITY OF ALMOST MULTIPLICATIVE FUNCTIONALS

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Special Issue in Honor of the 100th Anniversary of S.M. Ulam

Received 1 December, 2008; accepted 21 December, 2008; published 4 September, 2009.

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ABSTRACT. Let δ and p be non-negative real numbers. Let $\mathbb{F}$ be the real or complex number field and $\mathcal{A}$ a normed algebra over $\mathbb{F}$. If a mapping $\phi\colon \mathcal{A} \to \mathbb{F}$ satisfies

$$|\phi(xy) - \phi(x)\phi(y)| \le \delta\|x\|^p\|y\|^p \qquad (x, y \in \mathcal{A}),$$

then we show that ϕ is multiplicative or $|\phi(x)| \le (1 + \sqrt{1 + 4\delta})\|x\|^p/2$ for all $x \in \mathcal{A}$. If, in addition, ϕ satisfies

$$|\phi(x + y) - \phi(x) - \phi(y)| \le \delta(\|x\|^p + \|y\|^p) \qquad (x, y \in \mathcal{A})$$

for some $p \neq 1$, then by using Hyers-Ulam-Rassias stability of additive Cauchy equation [22], we show that ϕ is a ring homomorphism or $|\phi(x)| \le 2\delta\|x\|^p/|2 - 2^p|$ for all $x \in \mathcal{A}$; In other words, ϕ is a ring homomorphism, or an approximately zero mapping. The results of this paper are inspired by Th.M. Rassias' stability theorem.

Key words and phrases: Almost multiplicative functional, Hyers-Ulam-Rassias stability.

2000 *Mathematics Subject Classification.* 39B72, 46J99.

ISSN (electronic): 1449-5910

1. INTRODUCTION

The stability problem of functional equations was first raised by S. M. Ulam (cf. [30, Chapter VI]). "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism? (An ε-automorphism of G means a transformation f of G into itself such that $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all $x, y \in G$.)" D. H. Hyers [10] gave an affirmative answer to the problem as follows.

Theorem A. *Let E_1 and E_2 be two real Banach spaces and $\varepsilon \geq 0$. If a mapping $f\colon E_1 \to E_2$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$, then there exists a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear, where $\mathbb{R}$ is the real number field.

This result is called the *Hyers-Ulam stability* of the *additive* Cauchy equation $g(x + y) = g(x) + g(y)$. Here we note that Hyers [10] calls any solution of this equation a "linear" function or transformation. Hyers considered only the *bounded* Cauchy difference $f(x+y)-f(x)-f(y)$.

T. Aoki [1] introduced an unbounded one and obtained the stability of an *additive* mapping, which generalizes a result [10, Theorem 1] of Hyers. Th.M. Rassias [22], who independently introduced the unbounded Cauchy difference, was the first to prove the stability of the linear mapping between Banach spaces. The concept of Hyers-Ulam-Rassias stability originated from Rassias' paper [22] for the stability of the linear mapping. Rassias [22] generalized Hyers' result as follows:

Theorem B. *Let E_1 and E_2 be two real Banach spaces, $\varepsilon \geq 0$ and $0 \leq p < 1$. If a mapping $f\colon E_1 \to E_2$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$, then there is a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|x\|^p$$

for all $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear.

This result is, what is called the *Hyers-Ulam-Rassias stability* of the additive Cauchy equation $g(x + y) = g(x) + g(y)$. The result of Hyers is simply the case where $p = 0$. Thus, the result of Rassias is a generalization to the case where $0 \leq p < 1$. It should be mentioned that it allows the Cauchy difference to be unbounded. During the 27th International Symposium on Functional Equations, Rassias raised the problem of whether a similar result holds for $1 \leq p$. Z. Gajda [6, Theorem 2] proved that Theorem B is valid for $1 < p$. In the same paper [6, Example], he also gave an example showing that a similar result to the above does not hold for $p = 1$. Later, Th.M. Rassias and P. Šemrl [23, Theorem 2] gave another counter example for $p = 1$. Note that if $p < 0$, then $\|0\|^p$ is obviously meaningless. However, if we assume that $\|0\|^p$ means ∞, then with minor changes in the proof given in [22], we can prove that the result is also valid for $p < 0$. Thus, the Hyers-Ulam-Rassias stability of the additive Cauchy equation holds for all $p \in \mathbb{R} \setminus \{1\}$.

J. A. Baker [4] considered the stability of the multiplicative Cauchy equation $f(xy) = f(x)f(y)$: If $\delta \geq 0$ and ϕ is a complex-valued function on a semigroup S such that:

$|\phi(xy)-\phi(x)\phi(y)| \leq \delta$ for all $x, y \in S$, then ϕ is multiplicative, or $|\phi(x)| \leq (1+\sqrt{1+4\delta}\,)/2$ for all $x \in S$.

Taking Rassias' result [22] into account, it seems natural to consider the unbounded multiplicative Cauchy difference $\phi(xy) - \phi(x)\phi(y)$. Let $\mathbb{F}$ be the real, or complex number field and $\mathcal{A}$ a normed algebra over $\mathbb{F}$.

We consider the functionals $\phi\colon \mathcal{A} \to \mathbb{F}$ satisfying

$$|\phi(xy) - \phi(x)\phi(y)| \leq \delta \|x\|^p \|y\|^p \qquad (x, y \in \mathcal{A}) \tag{1.1}$$

for some $\delta \geq 0$ and $p \geq 0$.

When ϕ satisfies (1.1) for $p = 1$, ϕ is said to be δ-multiplicative. The stability of δ-multiplicative *linear* functionals has been studied by many authors [14, 15, 16, 24, 29]. Moreover, stability results [28] for almost additive δ-multiplicative functionals are also known.

In this paper, we shall prove that a Baker type stability result holds for $\phi\colon \mathcal{A} \to \mathbb{F}$ satisfying (1.1) for some $\delta \geq 0$ and $p \geq 0$. If, in addition, ϕ is almost additive in the sense

$$|\phi(x+y) - \phi(x) - \phi(y)| \leq \delta(\|x\|^p + \|y\|^p) \qquad (x, y \in \mathcal{A}),$$

then we show that ϕ is a ring homomorphism, that is, ϕ is both additive and multiplicative, or $|\phi(x)| \leq (1+\sqrt{1+4\delta}\,)\|x\|^p/2$ for all $x \in \mathcal{A}$. As a corollary, by using Rassias' result [22], we will prove that if $p \neq 1$, then ϕ is a ring homomorphism, or $|\phi(x)| \leq 2\delta\|x\|^p/|2-2^p|$ for all $x \in \mathcal{A}$. The Hyers-Ulam-Rassias stability of ring homomorphisms was obtained by R. Badora [3], which is a generalization of a result [5] of D. G. Bourgin.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathbb{F}$ be the real or complex number field and $\mathcal{A}$ a normed algebra over $\mathbb{F}$. If a functional $\phi\colon \mathcal{A} \to \mathbb{F}$ satisfies*

$$|\phi(xy) - \phi(x)\phi(y)| \leq \delta \|x\|^p \, \|y\|^p \qquad (x, y \in \mathcal{A}) \tag{2.1}$$

for some $\delta \geq 0$ and $p \geq 0$, then ϕ is multiplicative or

$$|\phi(x)| \leq \frac{1+\sqrt{1+4\delta}}{2}\,\|x\|^p$$

for all $x \in \mathcal{A}$.

To prove Theorem 2.1, we need the following lemma, which we will also use to prove Theorem 2.3.

Lemma 2.2. *Let $\mathbb{F}$ be the real or complex number field and $\mathcal{A}$ a normed algebra over $\mathbb{F}$. Let $\phi\colon \mathcal{A} \to \mathbb{F}$ be a functional such that* (2.1) *holds for some $\delta \geq 0$ and $p \geq 0$. If there exists a constant k with $0 \leq k < \infty$ such that $|\phi(x)| \leq k\|x\|^p$ for every $x \in \mathcal{A}$, then*

$$|\phi(x)| \leq \frac{1+\sqrt{1+4\delta}}{2}\,\|x\|^p \tag{2.2}$$

for all $x \in \mathcal{A}$.

Proof. Set

$$m = \begin{cases} \sup\limits_{x \in \mathcal{A}} |\phi(x)| & \text{if } p = 0, \\ \sup\limits_{x \in \mathcal{A}\setminus\{0\}} \dfrac{|\phi(x)|}{\|x\|^p} & \text{if } p \neq 0, \end{cases}$$

then $m \leq k < \infty$ by hypothesis. Note that if $p \neq 0$, then $\phi(0) = 0$ since $|\phi(x)| \leq k\|x\|^p$. Thus, we have $|\phi(x)| \leq m\|x\|^p$ for all $x \in \mathcal{A}$. It follows from (2.1) that $|\phi(x^2) - \phi(x)^2| \leq \delta\|x\|^{2p}$, and so

$$
\begin{aligned}
|\phi(x)|^2 &\leq \delta\|x\|^{2p} + |\phi(x^2)| \\
&\leq \delta\|x\|^{2p} + m\|x^2\|^p \\
&\leq (\delta + m)\|x\|^{2p}
\end{aligned}
$$

for all $x \in \mathcal{A}$. This implies that $m^2 \leq \delta + m$. Now it is obvious that $m \leq (1 + \sqrt{1 + 4\delta}\,)/2$, and so we have (2.2) for all $x \in \mathcal{A}$. ∎

It should be mentioned that the following proof is based on those of [27, Proposition 2.2] and [28, Theorem 4].

Proof of Theorem 2.1. Suppose that ϕ is not multiplicative, that is, there are $a, b \in \mathcal{A}$ such that $\phi(ab) \neq \phi(a)\phi(b)$. Take $x \in \mathcal{A}$ arbitrarily. It follows from (2.1) that

$$
\begin{aligned}
&|\phi(x)|\,|\phi(ab) - \phi(a)\phi(b)| \\
&\qquad \leq |\phi(x)\phi(ab) - \phi(x(ab))| + |\phi(x(ab)) - \phi(xa)\phi(b)| \\
&\qquad\qquad + |\phi(xa)\phi(b) - \phi(x)\phi(a)\phi(b)| \\
&\qquad \leq \delta(\|x\|^p\,\|ab\|^p + \|xa\|^p\,\|b\|^p + \|x\|^p\,\|a\|^p|\phi(b)|) \\
&\qquad \leq \delta\|a\|^p\,(2\|b\|^p + |\phi(b)|)\,\|x\|^p,
\end{aligned}
$$

and hence

$$
|\phi(x)| \leq \frac{\delta\|a\|^p\,(2\|b\|^p + |\phi(b)|)}{|\phi(ab) - \phi(a)\phi(b)|}\,\|x\|^p.
$$

Since $x \in \mathcal{A}$ was arbitrary, Lemma 2.2 yields that

$$
|\phi(x)| \leq \frac{1 + \sqrt{1 + 4\delta}}{2}\,\|x\|^p
$$

for all $x \in \mathcal{A}$, and the proof is complete. ∎

Remark 2.1. One can also consider a mapping ϕ between two normed algebras $\mathcal{A}$ and $\mathcal{B}$ such that

$$
\|\phi(xy) - \phi(x)\phi(y)\| \leq \delta\|x\|^p\,\|y\|^p \qquad (x, y \in \mathcal{A})
$$

for some $\delta \geq 0$ and $p \geq 0$. If, in addition, the norm $\|\cdot\|$ of $\mathcal{B}$ satisfies

$$
\|fg\| = \|f\|\,\|g\| \qquad (f, g \in \mathcal{B}), \tag{2.3}
$$

then we see that the above proofs work well. Thus we have that a result similar to Theorem 2.1 holds for a mapping ϕ between normed algebras $\mathcal{A}$ and $\mathcal{B}$ with the property (2.3). On the other hand, the norm condition (2.3) is quite restrictive. In fact, if $\mathcal{B}$ is a unital real normed algebra, then (2.3) implies $\mathcal{B} = \mathbb{R}$, or $\mathcal{B} = \mathbb{C}$, or $\mathcal{B}$ is the quaternion field. It seems that the result was proved first by S. Mazur [19]. Moreover, some generalizations are obtained (cf. [2]). Although the above result is well-known, if $\mathcal{B}$ is a unital commutative complex Banach algebra, then we can give a simple proof, which is essentially due to I. Gelfand, D. Raikov and G. Shilov [7, Theorem 1 of §10]. Indeed, let $f \in \mathcal{B} \setminus \{0\}$. Take a boundary point λ of the spectrum $\hat{f}(M_{\mathcal{B}})$ of f, where $M_{\mathcal{B}}$ denotes the maximal ideal space of $\mathcal{B}$ and $\hat{f}$ denotes the Gelfand transform of

f. Let e be a unit element of $\mathcal{B}$ and $\mathbb{N}$ the set of all natural numbers. If $\{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C} \setminus \hat{f}(M_{\mathcal{B}})$ converges to λ, then $f - \lambda_n e$ is invertible, and so

$$\begin{aligned}\|(f-\lambda_n e)^{-1}\| &\geq \sup_{\varphi\in M_{\mathcal{B}}} \frac{1}{|\hat{f}(\varphi)-\lambda_n|}\\ &\geq \frac{1}{|\lambda-\lambda_n|} \to \infty \qquad (n\to\infty).\end{aligned}$$

Thus, it follows from (2.3) that $\|f-\lambda_n e\| \to 0$ as $n \to \infty$. Set

$$g_n = \frac{(f-\lambda_n e)^{-1}}{\|(f-\lambda_n e)^{-1}\|},$$

then $\|g_n\| = 1$ for all $n \in \mathbb{N}$, and so we get

$$\begin{aligned}\|(f-\lambda e)g_n\| &\leq \|(f-\lambda_n e)g_n\| + \|(\lambda_n-\lambda)g_n\|\\ &= \|f-\lambda_n e\| + |\lambda_n-\lambda| \to 0 \qquad (n\to\infty).\end{aligned}$$

Since $\|g_n\| = 1$, (2.3) shows that $f - \lambda e = 0$, proving $\mathcal{B} = \mathbb{C}$.

Theorem 2.3. *Let $\mathbb{F}$ be the real, or complex number field and $\mathcal{A}$ a normed algebra over $\mathbb{F}$. If $\phi\colon \mathcal{A} \to \mathbb{F}$ is a functional such that*

$$|\phi(xy)-\phi(x)\phi(y)| \leq \delta\|x\|^p\,\|y\|^p \qquad (x,y\in\mathcal{A}) \tag{2.4}$$

$$|\phi(x+y)-\phi(x)-\phi(y)| \leq \delta(\|x\|^p+\|y\|^p) \qquad (x,y\in\mathcal{A}), \tag{2.5}$$

for some $\delta \geq 0$ and $p \geq 0$, then ϕ is a ring homomorphism, or

$$|\phi(x)| \leq \frac{1+\sqrt{1+4\delta}}{2}\,\|x\|^p \tag{2.6}$$

for all $x \in \mathcal{A}$.

Proof. Suppose that ϕ is not a ring homomorphism. We will show that (2.6) holds for all $x \in \mathcal{A}$. There are two possibilities for ϕ. If ϕ is not multiplicative, then by Theorem 2.1 we have (2.6). If ϕ is not additive, then we will show that $|\phi(x)| \leq k\|x\|^p$ $(x \in \mathcal{A})$ for some constant k with $0 \leq k < \infty$. Indeed, if ϕ is not additive, then there exist $a, b \in \mathcal{A}$ such that

$$\phi(a+b) = \phi(a) + \phi(b).$$

For each $x \in \mathcal{A}$, it follows from (2.4) and (2.5) that

$$\begin{aligned}&|\phi(x)||\phi(a+b)-\phi(a)-\phi(b)|\\ &\quad\leq |\phi(x)\phi(a+b)-\phi(xa+xb)| + |\phi(xa+xb)-\phi(xa)-\phi(xb)|\\ &\qquad\quad + |\phi(xa)-\phi(x)\phi(a)| + |\phi(xb)-\phi(x)\phi(b)|\\ &\quad\leq \delta(\|x\|^p\,\|a+b\|^p + \|xa\|^p + \|xb\|^p + \|x\|^p\,\|a\|^p + \|x\|^p\,\|b\|^p)\\ &\quad\leq \delta(\|a+b\|^p + 2\|a\|^p + 2\|b\|^p)\|x\|^p,\end{aligned}$$

which implies that

$$|\phi(x)| \leq \frac{\delta(\|a+b\|^p + 2\|a\|^p + 2\|b\|^p)}{|\phi(a+b)-\phi(a)-\phi(b)|}\,\|x\|^p$$

for all $x \in \mathcal{A}$, as claimed. It follows from Lemma 2.2 that (2.6) holds for all $x \in \mathcal{A}$, and so the proof is complete. ∎

Corollary 2.4. *Let $\mathbb{F}$ be the real, or complex number field and $\mathcal{A}$ a normed algebra over $\mathbb{F}$. If a functional $\phi\colon \mathcal{A} \to \mathbb{F}$ satisfies* (2.4) *and* (2.5) *for some $\delta \geq 0$ and $p \geq 0$ with $p \neq 1$, then ϕ is a ring homomorphism, or*

$$|\phi(x)| \leq \frac{2\delta}{|2-2^p|}\,\|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Since ϕ is approximately additive in the sense of (2.5), it follows from [22] that there exists a unique additive mapping $T\colon \mathcal{A} \to \mathbb{F}$ such that

$$|\phi(x) - T(x)| \leq \frac{2\delta}{|2-2^p|}\,\|x\|^p \tag{2.7}$$

for all $x \in \mathcal{A}$. Suppose that ϕ is not a ring homomorphism. Then, by Theorem 2.3, we have (2.6) for every $x \in \mathcal{A}$. It follows from (2.7) that

$$|T(x)| \leq |\phi(x)| + \frac{2\delta}{|2-2^p|}\,\|x\|^p \leq k\|x\|^p \tag{2.8}$$

for all $x \in \mathcal{A}$, where

$$k = \frac{1+\sqrt{1+4\delta}}{2} + \frac{2\delta}{|2-2^p|}.$$

We show that $T(x) = 0$ for every $x \in \mathcal{A}$. To do this, take $x \in \mathcal{A}$ arbitrarily. Set $s = |1-p|/(1-p)$, then $s = \pm 1$. By (2.8), we have, for each natural number n, that

$$|T(n^s x)| \leq k\|n^s x\|^p = n^{sp}k\|x\|^p.$$

On the other hand, since T is additive, it is easy to see that $T(n^s x) = n^s T(x)$ for every n. It follows that

$$|T(x)| \leq n^{s(p-1)}k\|x\|^p \to 0 \qquad (\text{as } n \to \infty)$$

since $s(p-1) = -|1-p| < 0$. Since $x \in \mathcal{A}$ was arbitrary, we have $T(x) = 0$ for every $x \in \mathcal{A}$. By (2.7), we have $|\phi(x)| \leq 2\delta\|x\|^p/|2-2^p|$ for all $x \in \mathcal{A}$. ∎

Remark 2.2.

(i) Set $\phi(x) = (1+\sqrt{1+4\delta})x/2$ for every $x \in \mathbb{R}$. It is obvious that $\phi(x+y) = \phi(x)+\phi(y)$ and

$$\phi(xy) - \phi(x)\phi(y) = -\delta xy$$

hold for every $x, y \in \mathbb{R}$, and so ϕ satisfies the conditions (2.4) and (2.5) for $p = 1$. Although ϕ is additive, we see that ϕ is not multiplicative unless $\delta = 0$.

(ii) Let $p \geq 0$. Set $\phi(x) = |\sin x|^p$ for $x \in \mathbb{R}$. Since $|\sin x| \leq |x|$ for every $x \in \mathbb{R}$, we have

$$|\phi(xy) - \phi(x)\phi(y)| \leq |\sin(xy)|^p + |\sin x|^p\,|\sin y|^p \leq 2|x|^p\,|y|^p$$

for all $x, y \in \mathbb{R}$. First, let us consider the case when $p \leq 1$. Since

$$|x+y|^p \leq |x|^p + |y|^p \qquad (x, y \in \mathbb{R}),$$

we also have that

$$\begin{aligned}|\phi(x+y) - \phi(x) - \phi(y)| &\leq |\sin(x+y)|^p + |\sin x|^p + |\sin y|^p \\ &\leq |x+y|^p + |x|^p + |y|^p \leq 2(|x|^p + |y|^p)\end{aligned}$$

for all $x, y \in \mathbb{R}$. Although ϕ is neither additive nor multiplicative, ϕ satisfies (2.4) and (2.5) for $\delta = 2$.

We next consider the case when $p > 1$. In this case, we see that

$$|x + y|^p \leq 2^{p-1}(|x|^p + |y|^p)$$

holds for all $x, y \in \mathbb{R}$, and so we have

$$|\phi(x + y) - \phi(x) - \phi(y)| \leq (2^{p-1} + 1)(|x|^p + |y|^p)$$

for all $x, y \in \mathbb{R}$. This implies that ϕ satisfies (2.4) and (2.5) for $\delta = 2^{p-1} + 1$.

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