ERGODIC SOLENOIDAL HOMOLOGY II: DENSITY OF ERGODIC SOLENOIDS

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ABSTRACT. A measured solenoid is a laminated space endowed with a tranversal measure invariant by holonomy. A measured solenoid immersed in a smooth manifold produces a closed current (known as a generalized Ruelle-Sullivan current). Uniquely ergodic solenoids are those for which there is a unique (up to scalars) transversal measure. It is known that for any smooth manifold, any real homology class is represented by a uniquely ergodic solenoid. In this paper, we prove that the currents associated to uniquely ergodic solenoids are dense in the space of closed currents, therefore proving the abundance of such objects.

1. INTRODUCTION

In [2], the authors introduced the concept of a measured solenoid as a laminated space S endowed with a transversal measure μ. Given a measured oriented solenoid S_μ, any smooth map $f : S \to M$ into a differentiable manifold M gives rise, by integration of forms, to a closed current

$$(S_\mu, f) \in \mathcal{C}_k(M),$$

whose associated homology class

$$[S_\mu, f] \in H_k(M, \mathbb{R})$$

is called the generalized Ruelle-Sullivan class.

A special role in the theory is played by the solenoids which have an ergodicity property. A uniquely ergodic solenoid is a solenoid S for which there is a unique (up to multiples) transversal measure, and this measure has to support the whole of the solenoid. In this situation, the "geometry" (that is, the solenoid itself) determines the "measure". This has a more precise significance if we note the following: for a uniquely ergodic immersed solenoid $f : S \to M$, take one leaf $l \subset S$, and an exhaustion l_R by compact sets such that $f(l_R)$ can be capped off by a "small" cap, thus defining a homology class $[\overline{f(l_R)}] \in H_k(M, \mathbb{Z})$. Then the normalizations

$$\frac{[\overline{f(l_R)}]}{\text{Vol}(f(l_R))} \longrightarrow [S_\mu, f].$$

In other words, any one leaf determines the whole solenoid.

The main result of [2] establishes that for any manifold M, any homology class $a \in H_k(M, \mathbb{R})$ can be represented by a uniquely ergodic oriented solenoid $f : S \to M$, which moreover is immersed (but we have to allow possible transversal self-intersections if $a \cup a \neq 0$).

In this note, we want to prove that for a given homology class $a \in H_k(M, \mathbb{R})$, the currents associated to immersed uniquely ergodic solenoids are dense in the space of closed currents in $\mathcal{C}_k(M)$ representing a.

2. DEFINITIONS

Let us review the main concepts introduced in [2].

Definition 2.1. A k-solenoid, where $k \geq 0$, of class $C^{r,s}$, is a compact Hausdorff space endowed with an atlas of flow-boxes $\mathcal{A} = \{(U_i, \varphi_i)\}$,

$$\varphi_i : U_i \to D^k \times K(U_i),$$

where D^k is the k-dimensional open ball, and $K(U_i) \subset \mathbb{R}^l$ is the transversal set of the flow-box. The changes of charts $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ are of the form

$$(2.1) \qquad \varphi_{ij}(x, y) = (X(x, y), Y(y)),$$

where $X(x, y)$ is of class $C^{r,s}$ and $Y(y)$ is of class C^s.

Let S be a k-solenoid, and $U \cong D^k \times K(U)$ be a flow-box for S. The sets $L_y = D^k \times \{y\}$ are called the (local) leaves of the flow-box. A leaf $l \subset S$ of the solenoid is a connected k-dimensional manifold whose intersection with any flow-box is a collection of local leaves. The solenoid is oriented if the leaves are oriented (in a transversally continuous way).

A transversal for S is a subset T which is a finite union of transversals of flow-boxes. Given two local transversals T_1 and T_2 and a path contained in a leaf from a point of T_1 to a point of T_2, there is a well-defined holonomy map $h : T_1 \to T_2$. The holonomy maps form a pseudo-group.

Definition 2.2. Let S be a k-solenoid. A transversal measure $\mu = (\mu_T)$ for S associates to any local transversal T a locally finite measure μ_T supported on T, which are invariant by the holonomy pseudogroup, i.e. if $h : T_1 \to T_2$ is a holonomy map, then $h_*\mu_{T_1} = \mu_{T_2}$.

We denote by S_μ a k-solenoid S endowed with a transversal measure $\mu = (\mu_T)$. We refer to S_μ as a measured solenoid. Observe that for any transversal measure $\mu = (\mu_T)$ the scalar multiple $c\,\mu = (c\,\mu_T)$, where $c > 0$, is also a transversal measure. Notice that there is no natural scalar normalization of transversal measures.

Definition 2.3. Let S be a k-solenoid. The solenoid S is uniquely ergodic if it has a unique (up to scalars) transversal measure μ and its support is the whole of S.

Now let M be a smooth manifold of dimension n. An immersion of a k-solenoid S into M, with $k < n$, is a smooth map $f : S \to M$ such that the differential restricted to the tangent spaces of leaves has rank k at every point of S. The solenoid $f : S \to M$ is transversally immersed if for any flow-box $U \subset S$ and chart $V \subset M$, the map $f : U = D^k \times K(U) \to V \subset \mathbb{R}^n$ is an embedding, and the images of the leaves intersect transversally in M. If moreover f is injective, then we say that the solenoid is embedded.

Note that under a transversal immersion, resp. an embedding, $f : S \to M$, the images of the leaves are immersed, resp. injectively immersed, submanifolds.

Denote by

$$\mathcal{C}_k(M)$$

the space of compactly supported currents of dimension k on M. We have the following.

Definition 2.4. Let S_μ be an oriented measured k-solenoid. An immersion $f : S \to M$ defines a generalized Ruelle-Sullivan current $(S_\mu, f) \in \mathcal{C}_k(M)$ as follows. Let $S = \bigcup_i S_i$ be a measurable partition such that each S_i is contained in a flow-box U_i. For $\omega \in \Omega^k(M)$, we define

$$\langle (S_\mu, f), \omega \rangle = \sum_i \int_{K(U_i)} \left(\int_{L_y \cap S_i} f^*\omega \right) \, d\mu_{K(U_i)}(y) \,,$$

where L_y denotes the horizontal disk of the flow-box.

In [2] it is proved that (S_μ, f) is a closed current. Therefore, it defines a real homology class

$$[S_\mu, f] \in H_k(M, \mathbb{R}) \,.$$

In their original article [3], Ruelle and Sullivan defined this notion for the restricted class of solenoids embedded in M.

In [2], it is proved that if (S_μ, f) is an embedded solenoid and the transversal measure has no atoms (for instance, if S has no compact leaves), then

$$[S_\mu, f] \cup [S_\mu, f] = 0 \,.$$

So if $a \in H_k(M, \mathbb{R})$ is a homology class with $a \cup a \neq 0$, then it cannot be represented by an embedded solenoid.

Now we introduce a Riemannian metric on M. Let (S_μ, f) be a uniquely ergodic immersed oriented solenoid. The leaf-wise volume of S (induced by the metric on M) together with the transversal measure, give a finite measure supported on S. We normalize it to have total mass 1. Note that this produces a unique transversal measure μ.

Let $l \subset S$ be a leaf of S. Suppose that there is an exhaustion l_R of l such that $f(l_R)$ has a small cap, that is, there is an oriented submanifold C_R of dimension k with boundary $\partial C_R = -\partial f(l_R)$, satisfying

$$(2.2) \qquad\qquad \frac{\mathrm{Vol}(C_R)}{\mathrm{Vol}\, f(l_R)} \longrightarrow 0\,.$$

Then consider the integration current $(l_R, f) \in \mathcal{C}_k(M)$, defined by $\langle (l_R, f), \omega \rangle = \int_{l_R} f^* \omega$. We have the following:

Proposition 2.1. *In the situation above, $(l_R, f)/\mathrm{Vol}\, f(l_R) \to (S_\mu, f)$.*

Proof. First note that the currents $(l_R, f)/\mathrm{Vol}\, f(l_R)$ are bounded. Therefore they accumulate. To see that the sequence converges, it is enough to check that there is only one accumulation point. So assume that $(l_R, f)/\mathrm{Vol}\, f(l_R)$ converges, say, to some current T.

First note that the current T is closed:

$$\begin{aligned}
\langle \partial T, \omega \rangle &= \lim_{R \to \infty} \frac{1}{\mathrm{Vol}\, f(l_R)} \langle \partial(l_R, f), \omega \rangle \\
&= \lim_{R \to \infty} \frac{1}{\mathrm{Vol}\, f(l_R)} \langle \partial C_R, \omega \rangle \\
&= \lim_{R \to \infty} \frac{1}{\mathrm{Vol}\, f(l_R)} \int_{C_R} d\omega = 0\,,
\end{aligned}$$

by (2.2).

Clearly, T is supported on S. By unique ergodicity, T should be a multiple of (S_μ, f). Let us see this: T is a current defined by a transversal measure if and only if it is a daval measure as defined in [2] (that is, locally it is the product of the Riemannian volume along leaves with a transversal measure). This is equivalent to T being invariant by the group G_S^0 consisting of diffeomorphisms of the solenoid isotopic to the identity. So we have to check that for a leaf-wise vector field X defined on the solenoid, $L_X T = 0$ holds. However, for any $(k+1)$-form β on S, $i_X \beta|_S = 0$, so $\int_{l_R} i_X \beta = 0$. This means that $\langle i_X(l_R, f), \beta \rangle = \langle (l_R, f), i_X \beta \rangle = 0$. Therefore $i_X T = 0$, and hence

$$L_X T = d\, i_X T + i_X d T = 0\,,$$

as claimed.

The argument above yields that $T = \lambda(S_\mu, f)$, for some $\lambda \in \mathbb{R}$. Clearly $\lambda \geq 0$, since both currents define the same orientation at a fixed point of the leaf l. Consider the norm dual to the C^0-norm on $\mathcal{C}_k(M)$. It is easy to see that $||(l_R, f)/\mathrm{Vol}\, f(l_R)|| = 1$. Therefore $||T|| = 1$. But our normalization implies that $||(S_\mu, f)|| = 1$. So $T = (S_\mu, f)$. ∎

3. Realization Theorem

Let M be a smooth compact oriented Riemannian C^∞ manifold and let $a \in H_k(M, \mathbb{R})$ be a non-zero real k-homology class. The main result of [2] is the construction of a uniquely ergodic oriented transversally immersed k-solenoid representing a positive multiple of a.

The construction is as follows. Take a collection $C_1, \ldots, C_{b_k} \in H_k(M, \mathbb{Z})$ which is a basis of $H_k(M, \mathbb{Q})$ and such that the C_i are represented by immersed submanifolds $S_i \subset M$ with a trivial normal bundle and such that all intersections are transversal. After switching the orientations of C_i if necessary, reordering the cycles and multiplying a by a suitable positive real number, we may suppose that

$$a = \lambda_1 C_1 + \cdots + \lambda_r C_r,$$

for some $r \geq 1$, $\lambda_i > 0$, $1 \leq i \leq r$, and $\sum \lambda_i = 1$.

Let $h : \mathbb{T} \to \mathbb{T}$ be a diffeomorphism of the circle which is a Denjoy counter-example with an irrational rotation number and of class $C^{2-\epsilon}$, for some $\epsilon > 0$. Hence h is uniquely ergodic. Let μ_K be the unique invariant probability measure, which has support on a Cantor set $K \subset \mathbb{T}$. Partition the Cantor set K into r disjoint compact subsets $K_1, \ldots, K_r$ in cyclic order, each of which with $\mu_K(K_i) = \lambda_i$.

Now construct the immersed solenoid $f : S \to M$ as follows. The central part (or core) of S lies inside a ball in M, and it is of the form $C = S^{k-1} \times [-1, 1] \times K$ embedded as

$$(x, t, y) \mapsto (x, t, h_t(y)),$$

where h_t is an isotopy of $\mathbb{T}$ from the identity to h.
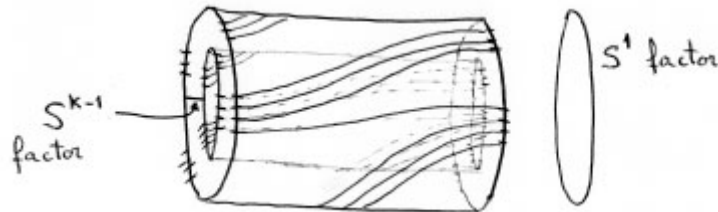
The rest of the solenoid consists of $(S_i - (D_i^+ \cup D_i^-)) \times K_i$ for $i = 1, \ldots, r$, where $S_i - (D_i^+ \cup D_i^-)$ is S_i with two balls removed, and we are taking parallel copies of S_i as leaves. We glue the boundaries $\partial D_i^\pm \times K_i$ to the boundaries $S^{k-1} \times \{\pm 1\} \times K_i \subset \partial C$.

This gives an oriented k-solenoid of class $C^{\infty, 2-\epsilon}$. That is, the changes of charts (2.1) are C^∞ in x and $C^{2-\epsilon}$ in y.

We have a global transversal $T = \{p\} \times K \subset S^{k-1} \times K \subset S$. Identifying $T \cong K$, the holonomy pseudo-group is generated by $h : K \to K$. Hence S is uniquely ergodic. Let μ denote the transversal measure corresponding to μ_K.
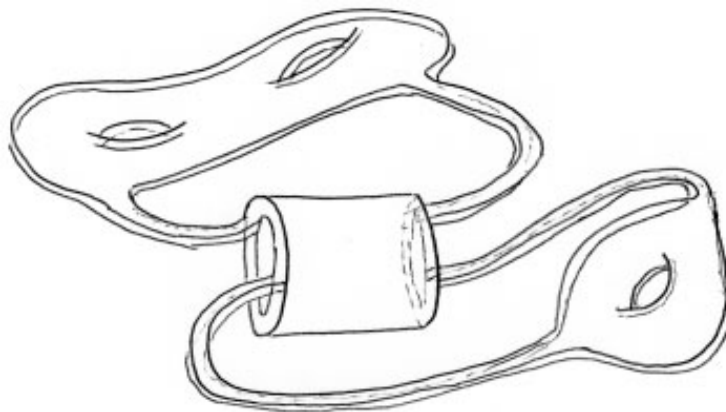

Figure 1: The solenoid S.

It remains to prove that $[S_\mu, f] = a$ (up to a scalar multiple). This is done by coupling against suitable test forms, as done in [2].

4. DENSITY OF UNIQUELY ERGODIC SOLENOIDS

Now we move on to the main result of this paper. We consider the space of currents $\mathcal{C}_k(M)$ with the weak topology (that is, the topology as a dual space to $\Omega^k(M)$ with the Fréchet topology given by the C^∞-convergence of forms).

Theorem 4.1. *Let $T \in \mathcal{C}_k(M)$ be a (compactly supported) closed current representing a homology class $a \in H_k(M, \mathbb{R})$. Then there is an oriented immersed uniquely ergodic solenoid $f : S \to M$ such that the current (S_μ, f) is as close to T as we want, and $[S_\mu, f] = a$.*

Proof. We start by considering the oriented, transversally immersed, measured solenoid $f_a : S_a \to M$, constructed in Section 3. It satisfies the condition $[S_a, f_a] = a$. Then

$$T - (S_a, f_a) \in \mathcal{C}_k(M)$$

is an exact current. Therefore there is some $T' \in \mathcal{C}_{k+1}(M)$ with

$$T - (S_a, f_a) = \partial T' \,.$$

By a regularization procedure (or using the density of the forms in the space of currents), we can find a smooth $(n - k - 1)$-form β (thought of as a current, by duality) such that $T' - \beta$ is very small. So

$$T \approx (S_a, f_a) + d\beta \,.$$

Now take an open cover $\{U_i\}$ of M by coordinate open sets, and a refinement $\{V_i\}$ such that $\overline{V}_i \subset U_i$. Consider a partition of unity ρ_i subordinated to V_i, and let $\beta_i = \rho_i \beta$. Writing in coordinates $(x_1, \ldots, x_n)$ for U_i, we have

$$\beta_i = \sum_{i_1 < \ldots < i_{n-k-1}} h_{i_1 \ldots i_{n-k-1}} \, dx_{i_1} \wedge \cdots \wedge dx_{i_{n-k-1}} \,.$$

Consider now a bump function $\tilde{\rho}_i$, which is one on V_i and zero off U_i. Then

$$\beta_i = \sum_{i_1 < \ldots < i_{n-k-1}} \tilde{\rho}_i \, h_{i_1 \ldots i_{n-k-1}} \, d(\tilde{\rho}_i \, x_{i_1}) \wedge \cdots \wedge d(\tilde{\rho}_i \, x_{i_{n-k-1}}) \,,$$

and

$$d\beta_i = \sum_{i_1 < \ldots < i_{n-k-1}} d(\tilde{\rho}_i \, h_{i_1 \ldots i_{n-k-1}}) \wedge d(\tilde{\rho}_i \, x_{i_1}) \wedge \cdots \wedge d(\tilde{\rho}_i \, x_{i_{n-k-1}}) \,.$$

As $d\beta = \sum d\beta_i$, we have that $d\beta$ can be written as a sum of terms of the form

$$(4.1) \qquad\qquad df_1^i \wedge \cdots \wedge df_{n-k}^i \,,$$

where $f_1^i, \ldots, f_{n-k}^i$ are (compactly supported) globally defined functions on M.

If $f_1, \ldots, f_{n-k}$ are compactly supported functions on M, write

$$(4.2) \qquad\qquad F = (f_1, \ldots, f_{n-k}) : M \to \mathbb{R}^{n-k} \,.$$

Then the form $\alpha = df_1 \wedge \cdots \wedge df_{n-k} = F^*\omega$, where $\omega = dx_1 \wedge \cdots \wedge dx_{n-k}$ is the volume form on $\mathbb{R}^{n-k}$.

Lemma 4.2. *If $\alpha = df_1 \wedge \cdots \wedge df_{n-k}$, with F as in (4.2), then there is an embedded solenoid with trivial holonomy and transversal Cantor sets whose associated current is as close to α as we want.*

Proof. Consider the set $C = \{x \in M \,;\, \alpha(x) = 0\}$. Then $F(C) \subset \mathbb{R}^{n-k}$ is the set of critical values of F, which has zero measure by Sard's theorem. Denote by $K = F(M) \subset \mathbb{R}^{n-k}$, and note that it is compact since F is compactly supported. Now let U be a small open neighbourhood of $F(C)$. We see that

$$(4.3) \qquad\qquad F^*\omega - F^*(\omega|_{K-U}) = F^*(\omega|_U)$$

is small (as a current in M). Fix $\epsilon > 0$, and consider $C_\epsilon = \{x \in M \,;\, |\alpha(x)| < \epsilon\}$. For $\beta \in \Omega^k(M)$, we have

$$(4.4) \qquad\qquad \left| \int_{F^{-1}(U) \cap C_\epsilon} F^*\omega \wedge \beta \right| \le \epsilon \, \mathrm{Vol}(\mathrm{Supp}(F)) |\beta| \,,$$

so it is small. In the complement $M - C_\epsilon$, the leaves $F^{-1}(x)$ have uniformly bounded volume. This follows from the fact that dF has norms $\geq \epsilon$ in normal directions to the leaves, and the total volume of $M - C_\epsilon$ is bounded (recall that M may be non-compact, but as F is compactly supported, $\overline{M - C}$ is compact, and hence $M - C_\epsilon$ is also compact). Therefore

$$(4.5) \qquad \left| \int_{F^{-1}(U) \cap (M - C_\epsilon)} F^* \omega \wedge \beta \right| = \left| \int_U \left(\int_{F^{-1}(x)} \beta \right) dx_1 \wedge \dots dx_{n-k} \right| \leq C \operatorname{Vol}(U) \,,$$

which is small by taking $U \supset C$ very small. Adding (4.4) and (4.5), we get that (4.3) is small.

The current $F^*(\omega|_{K-U})$ is actually a solenoid. The leaves $F^{-1}(x)$ are compact submanifolds, and the transversal measure is the pull-back of ω to the transversals. Note that $K - U$ is a global transversal, and that the holonomy is trivial. Now consider a measure μ, supported on a Cantor set which approximates $\omega|_{K-U}$. Then

$$\langle F^* \mu, \beta \rangle = \int_{K-U} \left(\int_{F^{-1}(y)} \beta \right) d\mu(y) = \langle \mu, F_* \beta \rangle \,.$$

Since the norm of dF is bigger than a fixed constant, the norm of $F_* \beta$ is bounded by a constant times the norm of β. Therefore

$$|\langle F^* \mu, \beta \rangle - \langle F^* \omega, \beta \rangle| = |\langle \mu - \omega, F_* \beta \rangle| \leq \varepsilon |F_* \beta| \leq C \varepsilon |\beta| \,,$$

for some small $\varepsilon > 0$. This completes the proof of the lemma. ∎

Using this lemma, we get a collection of solenoids, with transversals being Cantor sets, whose union defines the sought current. All but one have trivial holonomy, and the remaining one is uniquely ergodic. Our next task is to perform a surgery to obtain a connected (and uniquely ergodic) solenoid. First note that we may decompose a solenoid with trivial holonomy into smaller chunks so that its total (transversal) mass is smaller than that of the solenoid S_a.

Proposition 4.3. *Let $f_i : S_{i,\mu_i} \to M$, $i = 1, 2$, be two immersed oriented measured solenoids of the same dimension k with Cantor transversal structure. Suppose that there are two disjoint discs $D_i^{n-k} \subset M$ such that $T_i = D_i^{n-k} \cap S_i$ is a global transversal for S_i. Let h_i be the holonomy of S_i. Suppose also that there is a diffeomorphism $\varphi : T_1 \to T_2$ which preserves the measures. Then there is an immersed oriented measured solenoid $f : S_\mu \to M$ with global transversal $T = T_1$, and holonomy $h = \varphi^{-1} \circ h_2 \circ \varphi \circ h_1$, such that the associated currents satisfy*

$$(S_\mu, f) \approx (S_{1,\mu_1}, f_1) + (S_{2,\mu_2}, f_2) \,.$$

Proof. Take flow boxes $D^k \times T_i$ for S_i, with $\{0\} \times T_i$ corresponding to T_i, remove the interiors $D_{1/2}^k \times T_i$, and glue the boundaries, to get the required (abstract) solenoid

$$S = (S_1 - D_{1/2}^k \times T_1) \cup_{\partial D_{1/2}^k \times T_1 \cong_\varphi \partial D_{1/2}^k \times T_2} (S_2 - D_{1/2}^k \times T_2) \,.$$

The transversal measure is induced by μ_1 (or μ_2).

It remains to define the immersion $f : S \to M$. On $S_i - D_1^k \times T_i$, it is defined to be equal to f_i. Now fix $y_0 \in T_1$ and $y_0' = \varphi(y_0) \in T_2$, and consider a path $\gamma : [0, 1] \to M$ from y_0 to y_0' transversal to the leaves through the end-points. Fatten up γ to a map $\hat{\gamma} : D^k \times [0, 1] \to M$, which matches the leaves at the end-points. We can take a small open set $U \subset D_1^{n-k}$ and the corresponding $\varphi(U) \subset D_2^{n-k}$, so that we can extend $\hat{\gamma}$ to a map

$$\tilde{\gamma} : D^k \times [0, 1] \times (U \cap T_1) \to M \,.$$

Repeating this for a finite cover of T_1, we obtain an open subset $V \supset T_1$, and a map

$$\tilde{\gamma} : D^k \times [0, 1] \times T_1 \to M \,.$$

Now define $f : S \to M$ as follows. Firstly, send $(D_1^k - D_{3/4}^k) \times T_i$ to $f_i((D^k - D_\epsilon^k) \times T_i)$, and secondly, define f on $((D_{3/4}^K - D_{1/2}^k) \times T_1) \cup_\varphi ((D_{3/4}^k - D_{1/2}^k) \times T_2)$ as $\tilde{\gamma}|_{S_\epsilon^k \times [0,1] \times T_1}$. Finally, one has to smooth out the corners, but this is achieved with a very small perturbation.

To end the proof of the proposition, we need to consider the difference current

$$T'' = (S_\mu, f) - (S_{1,\mu_1}, f_1) - (S_{2,\mu_2}, f_2).$$

The leaves L_y'' of this solenoidal current are diffeomorphic to the middle portion, plus two small caps

$$L_y'' = \tilde{\gamma}(S_\epsilon^k \times [0,1] \times \{y\}) \cup f_1(D_\epsilon^k \times \{y\}) \cup f_2(D_\epsilon^k \times \{\varphi(y)\}), \qquad y \in T_1.$$

So

$$\langle T'', \beta \rangle = \int_{T_1} \left(\int_{L_y''} \beta \right) d\mu(y).$$

The volume of L_y'' is bounded by a constant times ϵ, so it is very small for ϵ small, as required. ∎

We conclude the proof of Theorem 4.1 as follows. We have the solenoid (S_a, f_a) and a collection of solenoids (S_i, f_i) approaching (4.1). Decompose S_i into smaller solenoids so that we can assume that the total transversal measure of S_i does not exceed that of S_a. We can use Proposition 4.3 to glue each S_i (once at a time) to S_a (note that Proposition 4.3 works with the transversal of S_i and a sub-transversal of S_a). Note here that we have to construct a diffeomorphism between the transversal Cantor sets. For this it may be necessary to arrange the transversal Cantor set for S_i in such a way that it is diffeomorphic to the transversal Cantor set of S_a, but this is not problematic, since we were only requiring that the transversal measure of S_i approximates the smooth transversal measure of the foliation that S_i is approximating.

The holonomy of the resulting solenoid is that of S_a, hence it is uniquely ergodic. This completes the proof of Theorem 4.1. ∎

Remark 4.1. An immersed solenoid $f : S \to M$ is said to have a trapping region if there is a ball B in M such that all holonomy phenomenon of S occur inside B, and the holonomy is generated by a single map (see [2] for a precise definition). All the solenoids constructed in Sections 3 and 4 have a trapping region. This is relevant since the more restricted the class of solenoids that we use is, the better it is for possible future applications.

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