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# DIFFERENTIABILITY OF DISTANCE FUNCTIONS IN $p$-NORMED SPACES

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ABSTRACT. The farthest point mapping in a $p$-normed space $X$ is studied in virtue of the Gateaux derivative and the Frechet derivative. Let $M$ be a closed bounded subset of $X$ having the uniformly $p$-Gateaux differentiable norm. Under certain conditions, it is shown that every maximizing sequence is convergent, moreover, if $M$ is a uniquely remotal set then the farthest point mapping is continuous and so $M$ is singleton. In addition, a Hahn–Banach type theorem in $p$-normed spaces is proved.

*Key words and phrases:* Frechet derivative, quasi-norm, $p$-normed space, farthest mapping, Hahn-Banach theorem, remotal set.

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## 1. INTRODUCTION

Let $X$ be a real linear space. A quasi-norm is a real-valued function on $X$ satisfying the following conditions:

(i) $\|x\| \geq 0$ for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \|\cdot\|)$ is called a quasi-normed space if $\|\cdot\|$ is a quasi-norm on $X$. The smallest possible $K$ is called the module of concavity of $\|\cdot\|$ . By a quasi-Banach space we mean a complete quasi-normed space, i.e., a quasi-normed space in which every Cauchy sequence converges in $X$.

This class includes Banach spaces. The most significant class of quasi-Banach spaces, which are not Banach spaces are $L_p$-spaces for $0 < p < 1$ equipped with the $L_p$-norms $\|\cdot\|_p$ .

A quasi-norm $\|\cdot\|$ is called a $p$-norm $(0 < p < 1)$ if $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$. In this case a quasi-normed (quasi Banach) space is called a $p$-normed ($p$-Banach) space. By the Aoki–Rolewicz theorem [7] each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms. See [1, 4, 8] and the references therein for more information.

Throughout the paper, $S(X)$ and $B(X)$ denote the unit sphere and the unit ball of $X$, respectivelty. If $x^*$ is in $X^*$, the dual of $X$, and $x \in X$ we write $x^*(x)$ as $\langle x^*, x\rangle$. We also consider quasi-norms with $K > 1$. The case where $K = 1$ turns out to be the classical normed spaces, so we will not discuss it and refer the interested reader to [2].

Let $k > 0$. A real valued function $f$ on $X$ is said to be $k$-Gateaux differentiable at a point $x$ of $X$ if there is an element $df(x)$ of $X^*$ such that for each $y$ in $X$,

$$\lim_{t\to 0} \operatorname{sgn}(t)|t|^{-k}(f(x + ty) - f(x)) = \langle df(x), y\rangle, \tag{1.1}$$

and we call $df(x)$ the $k$-Gateaux derivative of $f$ at $x$.

We say that $f$ is $k$-Frechet differentiable at a point $x$ of $X$ if there is an element $f'(x)$ of $X^*$ such that

$$\lim_{\|y\|\to 0} \|y\|^{-k}(f(x + y) - f(x) - \langle f'(x), y\rangle) = 0, \tag{1.2}$$

and we call $f'(x)$ the $k$ -Frechet derivative of $f$ at $x$. See [2].

The norm of $X$ is called uniformly $k$-Gateaux differentiable if for $f(x) = \|x\|^k$ equality (1.1) holds uniformly for $x, y \in X$, and the operator norm $\|df(x)\|$ is less than or equal 1. Then $df(x)$ is denoted by $D_k(x)$.

We say that a non-zero element $x^*$ of $X^*$ strongly exposes $B(X)$ at $x \in S(X)$, provided $\langle x^*, x\rangle = \|x^*\|$, and a sequence $\{y_n\}$ in $B(X)$ converges to $x$ whenever $\{\langle x^*, y_n\rangle\}$ converges to $\langle x^*, x\rangle$.

The reader is referred to [2, 5, 6] for analogue results concerning normed spaces and to the book of D.H. Hyers, G. Isac and Th.M. Rassias [3] for extensive theory and applications of nonlinear analysis methods.

In this paper we use some ideas of [2] to study the farthest point mapping in a $p$-normed space $X$ by virtue of the Gateaux derivative and the Frechet derivative. Let $M$ be a closed bounded subset of $X$ which has a uniformly $p$-Gateaux differentiable norm. Under certain conditions, we show that every maximizing sequence is convergent, moreover, if $M$ is a uniquely remotal set then the farthest point mapping is continuous and so $M$ is a singleton. In addition, we prove a Hahn–Banach type theorem for $p$-normed spaces and give an application.

## 2. MAIN RESULTS

We start this section with following definition.

**Definition 2.1.** Let $X$ be a $p$-normed space and $M$ be a bounded closed subset of $X$. Setting

$$\Psi(x) := \sup\{\|x-y\|^p : y \in M\} \qquad (x \in X),$$

$F(x) := \{y \in M : \|x-y\|^p = \Psi(x)\}$ is called the set of farthest points in $M$ form $x$ **(CHECK)**. We say that $M$ is a uniquely remotal set if $F$ is a singleton for each $x \in X$ and then we denote the single element of $F(x)$ by $F(x)$. The map $F$ is called the antiprojection or the farthest point mapping for $M$.
A center of a bounded set $M$ in a $p$-normed space $X$ is an element $c$ in $X$ such that

$$r(M) := \sup_{y \in M} \|c-y\|^p = \inf_{x \in X} \sup_{y \in M} \|x-y\|.$$

In fact $r(M)$, the so called Chebyshev radius of $M$, is the smallest ball in $X$ containing $M$.
We call a sequence $\{y_n\}$ in $M$ a maximizing sequence for $x$ provided $\|x-y_n\|^p \to \Psi(x)$ as $n \to \infty$.

**Lemma 2.1.** *If $M$ is a nonempty bounded subset of $X$ and $x, y \in X$ then*

$$|\Psi(x) - \Psi(y)| \leq \|x-y\|^p. \tag{2.1}$$

*Proof.*

$$\begin{aligned}
\Psi(x) - \Psi(y) &= \sup\{\|x-m\|^p : m \in M\} - \sup\{\|y-m\|^p : m \in M\} \\
&\leq \sup\{\|x-m-y+m\|^p : m \in M\} \\
&= \|x-y\|^p.
\end{aligned}$$

∎

**Lemma 2.2.** *If $x \in X$ is a point of $k$-Gateaux differentiability of $\Psi$ and $y \in F(x)$ then*

$$\langle d\Psi(x), \|x-y\|^{-1}(x-y)\rangle \leq \begin{cases} 0, & k < 1; \\ p\|x-y\|^{p-k}, & k = 1. \end{cases}$$

*Proof.* Since $y \in F(x)$ we have $\|x-y\|^p = \Psi(x)$. Hence for $0 < t < 1$, we get

$$\begin{aligned}
\Psi(x + t(x-y)) - \Psi(x) &\geq \|x + t(x-y) - y\|^p - \|x-y\|^p \\
&= (1+t)^p\|x-y\|^p - \|x-y\|^p \\
&= ((1+t)^p - 1)\|x-y\|^p.
\end{aligned} \tag{2.2}$$

Let $s \in [0, \infty)$. Then

$$\begin{aligned}
\langle d\Psi(x), sz\rangle &= \lim_{t \to 0^+} t^{-k}(\Psi(x+tsz) - \Psi(x)) \\
&= s^k s^{-k} \lim_{t \to 0^+} t^{-k}(\Psi(x+tsz) - \Psi(x)) \\
&= s^k \lim_{t \to 0^+} (s^{-k}t^{-k})(\Psi(x+tsz) - \Psi(x)) \\
&= s^k\langle d\Psi(x), z\rangle,
\end{aligned} \tag{2.3}$$

whence

$$\begin{aligned}\langle d\Psi(x), \|x-y\|^{-1}(x-y)\rangle &= \lim_{t\to 0}\operatorname{sgn}(t)|t|^{-k}(\|x-y\|^{-k})(\Psi(x+t(x-y))-\Psi(x))\\ &= \begin{cases} \geq \lim_{t\to 0^+}\frac{(|1+t|^p-1)}{t^k}\|x-y\|^{p-k}, & t>0;\\ \leq \lim_{t\to 0^-}\frac{(|1+t|^p-1)}{-(-t)^k}\|x-y\|^{p-k}, & t<0.\end{cases}\end{aligned}$$

Then

$$\langle d\Psi(x), \|x-y\|^{-1}(x-y)\rangle = \begin{cases} 0, & k<1;\\ p\|x-y\|^{p-k}, & k=1.\end{cases}$$

∎

**Theorem 2.3.** *Suppose that $M$ is a closed bounded subset of $X$ with a uniformly $p$-Gateaux differentiable norm, $\Psi$ is $p$-Gateaux differentiable at a point of $x \in X\backslash M$ with $y \in F(x)$ and $d\Psi(x)$ strongly exposes $B(X)$ at $\|x-y\|^{-1}(x-y)$. Then every maximizing sequence for $x$ converges to $y$. Moreover, if $M$ is a uniquely remotal set then $F$ is continuous at $x$.*

*Proof.* Suppose that $\{y_n\}$ is a maximizing sequence for $x$. For each $t$, we have

$$\begin{aligned}\Psi(x+tz)-\Psi(x) &\geq \sup_{y_n\in M}\|x+tz-y_n\|^p - \lim_{n\to\infty}\|x-y_n\|^P\\ &\geq \limsup_{n\to\infty}\big(\|x+tz-y_n\|^p-\|x-y_n\|^p\big).\end{aligned}\tag{2.4}$$

Fix $z\in X$. We have

$$\begin{aligned}\langle d\Psi(x), z\rangle &= \lim_{t\to 0^-}\operatorname{sgn}(t)|t|^{-p}(\Psi(x+tz)-\Psi(x))\\ &\leq \lim_{t\to 0^-}\big(-(-t)^{-p}\limsup_{n\to\infty}\big(\|x+tz-y_n\|^p-\|x-y_n\|^p\big)\big)\\ &= \lim_{t\to 0^-}\big(\liminf_{n\to\infty}\big(-(-t)^{-p}(\|x+tz-y_n\|^p-\|x-y_n\|^p)\big)\big)\\ &\leq \liminf_{n\to\infty}\langle D_p(x-y_n), z\rangle\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}\langle d\Psi(x), z\rangle &= \lim_{t\to 0^+}\operatorname{sgn}(t)|t|^{-p}(\Psi(x+tz)-\Psi(x))\\ &\geq \lim_{t\to 0^+}\big(t^{-p}\limsup_{n\to\infty}\big(\|x+tz-y_n\|^p-\|x-y_n\|^p\big)\big)\\ &= \lim_{t\to 0^+}\big(\limsup_{n\to\infty}\big(t^{-p}(\|x+tz-y_n\|^p-\|x-y_n\|^p)\big)\big)\\ &\geq \limsup_{n\to\infty}\langle D_p(x-y_n), z\rangle.\end{aligned}\tag{2.6}$$

Using (2.5) and (2.6) we get

$$\lim_{n\to\infty}\langle D_p(x-y_n), z\rangle = \langle d\Psi(x), z\rangle.\tag{2.7}$$

We also have

$$\begin{aligned}\langle D_p(x-y_n), x-y_n\rangle &= \lim_{t\to 0^+}\big(t^{-p}(\|x-y_n+t(x-y_n)\|^p-\|x-y_n\|^p)\big)\\ &= \lim_{t\to 0^+}\big(t^{-p}\|(x-y_n)\|^p((1+t)^p-1)\big).\end{aligned}\tag{2.8}$$

Using equalities (2.7) for $z = x - y_n$, (2.3) and (2.8) we obtain

$$\begin{aligned}\langle d\Psi(x), \|x-y_n\|^{-1}(x-y_n)\rangle &= \lim_{n\to\infty} \langle D_p(x-y_n), \|x-y_n\|^{-1}(x-y_n)\rangle \\ &= \lim_{n\to\infty}\lim_{t\to 0^+} \big(t^{-p}((1+t)^p - 1)\big) = 0 \\ &= \langle d\Psi(x), \|x-y\|^{-1}(x-y)\rangle.\end{aligned}$$

Since $d\Psi(x)$ strongly exposes $B(X)$ at $\|x-y\|^{-1}(x-y)$, we get

$$\lim_{n\to\infty} \|x-y_n\|^{-1}(x-y_n) = \|x-y\|^{-1}(x-y).$$

Since $\{y_n\}$ is a maximizing sequence for $x$, we therefore easily get

$$\lim_{n\to\infty} y_n = y. \tag{2.9}$$

Next, assume that $\{x_n\}$ converges to $x$. We shall show that $\{F(x_n)\}$ converges to $F(x)$.

Since $M$ is a uniquely remotal set we have $\|x_n - F(x_n)\|^p = \sup\{\|x_n - y\|^p; y \in M\}$. Hence

$$\|x_n - F(x)\|^p \le \|x_n - F(x_n)\|^p. \tag{2.10}$$

By (2.10),

$$\begin{aligned}\|x-F(x)\|^p - \|x-F(x_n)\|^p &\le \|x-F(x)\|^p + \|x-x_n\|^p - \|x_n - F(x_n)\|^p \\ &\le \|x-F(x)\|^p + \|x-x_n\|^p - \|x_n - F(x)\|^p \\ &\le \|x-x_n\|^p + \|x-F(x)-x_n+F(x)\|^p \\ &\le 2\|x-x_n\|^p.\end{aligned} \tag{2.11}$$

It follows from (2.11) and the convergence of $\{x_n\}$ to $x$ that

$$\lim_{n\to\infty} \|x-F(x_n)\|^p = \|x-F(x)\|^p.$$

Hence $\{F(x_n)\}$ is a maximizing sequence for $x$ and so, by the first part of the theorem

$$\lim_{n\to\infty} F(x_n) = F(x).$$

∎

**Theorem 2.4.** *Suppose $M$ is a closed bounded subset of a $p$-normed space $X$ and $x \in X\backslash M$ is a point of $p$-Frechet differentiability of $\Psi$ and if $y \in F(x)$ and $\Psi'(x)$ strongly exposes $B(X)$ at $\|x-y\|^{-1}(x-y)$, then every maximizing sequence for $x$ converges to $y$.*

*Proof.* Let $\{y_n\}$ be a maximizing sequence for $x$ and let $k = \inf_n \|x-y_n\|^{2p}$. Since $M$ is closed, $k > 0$. Hence there is $N$ such that for $n \ge N$ then $\Psi(x) - \|x-y_n\|^p < k \le \|x-y_n\|^{2p}$, and so we can choose a sequence $\{\alpha_n\}$ of positive numbers such that $\alpha_n \to 0$ and

$$\|x-y_n\|^{2p} > \alpha_n^2 > \Psi(x) - \|x-y_n\|^p \qquad (n \in \mathbb{N}).$$

Hence

$$(1+t)^p\|x-y_n\|^p > (1+t)^p(\Psi(x) - \alpha_n^2) \qquad (n \in \mathbb{N}, -1 < t < 1). \tag{2.12}$$

Let $0 < t < 1$. Since $y_n \in M$ we have

$$\Psi(x - t(y_n - x)) \ge \|x - t(y_n - x) - y_n\|^p = (1+t)^p\|x-y_n\|^p. \tag{2.13}$$

It follows from (2.12) and (2.13) that

$$\begin{aligned}\Psi(x - t(y_n - x) - \Psi(x) &\ge (1+t)^p\|x-y_n\|^p - \Psi(x) \\ &> \Psi(x)((1+t)^p - 1) - (1+t)^p\alpha_n^2.\end{aligned} \tag{2.14}$$

Fix $\varepsilon > 0$. By the definition of $\Psi'(x)$, there is $\delta > 0$ such that if $\|y\|^p < \delta$ then

$$|\Psi(x+y) - \Psi(x) - \langle \Psi'(x), y\rangle| \leq \varepsilon \|y\|^p. \tag{2.15}$$

Let $t_n^p = \alpha_n(\|x - y_n\|)^{-p} < 1$ and $\alpha_n < \delta$ for large $n$. Replacing $y$ by $t_n(x - y_n)$ in (2.15) and noting (2.14), we get

$$\begin{aligned}\varepsilon\|t_n(x-y_n)\|^p + \langle \Psi'(x), t_n(x-y_n)\rangle &\geq \Psi(x + t_n(x-y_n)) - \Psi(x)\\ &\geq \Psi(x)((1+t_n)^p - 1) - (1+t_n)^p\alpha_n^2,\end{aligned}$$

whence

$$\begin{aligned}&\frac{1}{\alpha_n}\langle \Psi'(x), t_n(x-y_n)\rangle\\ &\geq \frac{1}{\alpha_n}\big(\Psi(x)((1+t_n)^p - 1) - (1+t_n)^p\alpha_n^2 - \varepsilon\|t_n(y_n - x)\|^p\big)\\ &\geq \frac{1}{\alpha_n}\Psi(x)((1+t_n)^p - 1) - (1+t_n)^p\alpha_n - \varepsilon\\ &\geq \frac{1}{t_n^p\|(y_n - x)\|^p}\Psi(x)((1+t_n)^p - 1) - (1+t_n)^p\alpha_n - \varepsilon.\end{aligned} \tag{2.16}$$

Using (2.16), we get

$$\begin{aligned}\langle \Psi'(x), \|x-y_n\|^{-1}(x-y_n)\rangle &= \left\langle \Psi'(x), \frac{\|x-y_n\|^{-1}}{t_n} t_n(x-y_n)\right\rangle\\ &= \frac{\|x-y_n\|^{-p}}{t_n^p}\langle \Psi'(x), t_n(x-y_n)\rangle\\ &= \frac{1}{\alpha_n}\langle \Psi'(x), t_n(x-y_n)\rangle\\ &\geq \Psi(x)\frac{1}{t_n^p\|(y_n-x)\|^p}((1+t_n)^p - 1) - (1+t_n)^p\alpha_n - \varepsilon.\end{aligned} \tag{2.17}$$

Let $n \to \infty$ in (2.17). Then $\alpha_n \to 0$, $\|y_n - x\| \to \Psi(x)$ and $t_n \to 0^+$ such that

$$\lim_{n\to\infty}\langle \Psi'(x), \|x-y_n\|^{-1}(x-y_n)\rangle \geq 0 \tag{2.18}$$

Changing $t$ to $-t$ in (2.13) and (2.14) and utilizing a similar strategy as above we get

$$\lim_{n\to\infty}\langle \Psi'(x), \|y_n - x\|^{-1}(y_n - x)\rangle \geq 0,$$

whence

$$\lim_{n\to\infty}\langle \Psi'(x), \|x-y_n\|^{-1}(x-y_n)\rangle \leq 0. \tag{2.19}$$

By (2.18), (2.19) we have,

$$\lim_{n\to\infty}\langle \Psi'(x), \|x-y_n\|^{-1}(x-y_n)\rangle = 0.$$

By Lemma 2.2,

$$\langle d\Psi(x), \|x-y\|^{-1}(x-y)\rangle = 0$$

and then

$$\begin{aligned}
0 &= \lim_{\|t(x-y)\|\to 0} \|t(x-y)\|^{-p}\big(\Psi(x+t(x-y))) - \Psi(x) - \langle \Psi'(x), t(x-y)\rangle\big) \\
&= \lim_{t\to 0^+} \big(\|t(x-y)\|^{-p}(\Psi(x+(t(x-y))) - \Psi(x)) - \|t(x-y)\|^{-p}(\langle \Psi'(x), t(x-y)\rangle)\big) \\
&= \lim_{t\to 0^+} \left(\frac{t^{-p}(\Psi(x+(t(x-y)) - \Psi(x))}{\|(x-y)\|^p} - \langle \Psi'(x), \|x-y\|^{-1}(x-y)\rangle\right) \\
&= \frac{\lim_{t\to 0^+} t^{-p}(\Psi(x+(t(x-y)) - \Psi(x))}{\|(x-y)\|^p} - \langle \Psi'(x), \|x-y\|^{-1}(x-y)\rangle \\
&= \|x-y\|^{-p}\langle d\Psi(x), (x-y)\rangle - \langle \Psi'(x), \|x-y\|^{-1}(x-y)\rangle \\
&= \langle d\Psi(x), \|x-y\|^{-1}(x-y)\rangle - \langle \Psi'(x), \|x-y\|^{-1}(x-y)\rangle \\
&= \langle \Psi'(x), \|x-y\|^{-1}(x-y)\rangle,
\end{aligned}$$

therefore

$$\lim_{n\to\infty} \langle \Psi'(x), \|x-y_n\|^{-1}(x-y_n)\rangle = 0 = \langle \Psi'(x), \|x-y\|^{-1}(x-y)\rangle.$$

Since $\Psi'(x)$ strongly exposes $B(X)$ at $\|x-y\|^{-1}(x-y)$, we deduce that $\|x-y_n\|^{-1}(x-y_n) \to \|x-y\|^{-1}(x-y)$ which yields $y_n \to y$. ∎

## 3. HAHN–BANACH THEOREM AND ITS APPLICATION

This section is devoted to one version of the celebrated Hahn–Banach theorem. We follow the strategy and terminology of [9]. We begin with a lemma.

**Lemma 3.1.** *Let $0 < p < 1$, let $M$ be a linear subspace of a real vector space $X$ and let $\rho : X \to \mathbb{R}$ be a mapping such that*

(i) $\rho(x+y) \le \rho(x) + \rho(y) \quad (x, y \in X)$;
(ii) $\rho(tx) = t^p\rho(x) \quad (x \in X, 0 \le t \in \mathbb{R})$.

*If $\varphi_0 : M \to \mathbb{R}$ is a linear mapping such that $\varphi_0(x) \le \rho^{\frac{1}{p}}(x)$ for all $x \in M$, then there is a linear mapping $\varphi : X \to \mathbb{R}$ such that $\varphi|_M = \phi_0$ and $\varphi(x) \le \rho^{\frac{1}{p}}(x) \qquad (x \in X)$.*

*Proof.* We use Zorn's lemma. Consider the partially ordered set $\mathcal{P}$, whose typical member is a pair $(Y, \psi)$, where (i) $Y$ is a linear subspace of $X$ which contains $X_0$; and (ii) $\psi : Y \to \mathbb{R}$ is a linear mapping which is an extension of $\varphi_0$ and satisfies $\psi(x) \le \rho^{\frac{1}{p}}(x) \quad \forall x \in Y$; the partial order on $\mathcal{P}$ is defined by setting $(Y_1, \psi_1) \le (Y_2, \psi_2)$ precisely when (a) $Y_1 \subset Y_2$, and (b) $\psi_2|_{Y_1} = \psi_1$.

Furthermore, if $\Gamma = \{(Y_i, \psi_i) : i \in I\}$ is any totally ordered set in $\mathcal{P}$, an easy verification shows that an upper bound for the family $\Gamma$ is given by $(Y, \psi)$, where $Y = \cup_{i\in I} Y_i$ and $\psi : Y \to \mathbb{R}$ is the unique necessarily linear mapping satisfying $\psi|_{Y_i} = \psi_i$ for all $i$.

Hence, by Zorn's lemma, the partially ordered set $\mathcal{P}$ has a maximal element, call it $(Y, \psi)$. The proof of the lemma will be completed once we have shown that $Y = X$.

Suppose that $Y \ne X$; fix $x_0 \in X - Y$, and let $Y_1 = Y + \mathbb{R}x_0 = \{y + tx_0 : y \in Y, t \in \mathbb{R}\}$. The definitions ensure that $Y_1$ is a subspace of $X$ which properly contains $Y$. Also, notice that any linear mapping $\psi_1 : Y_1 \to \mathbb{R}$ which extends $\psi$ is prescribed uniquely by the number $t_0 = \psi(x_0)$ (and the equation $\psi_1(y + tx_0) = \psi(x) + tt_0$).

We assert that it is possible to find a number $t_0 \in \mathbb{R}$ such that the associated mapping $\psi_1$ would - in addition to extending $\psi$ - also satisfy $\psi_1 \le \rho^{\frac{1}{p}}$. This would then establish the inequality $(Y, \psi) \le (Y_1, \psi_1)$, contradicting the maximality of $(Y, \psi)$; which would in turn imply that we must have had $Y = X$ in the first place, and the proof would be complete.

First, observe that if $y_1, y_2 \in Y$ are arbitrary, then

$$\begin{aligned}\psi(y_1) + \psi(y_2) &= \psi(y_1 + y_2)\\ &= \psi(y_1 - x_0 + y_2 + x_0)\\ &= \psi(y_1 - x_0) + \psi(y_2 + x_0)\\ &\leq \rho^{\frac{1}{p}}(y_1 - x_0) + \rho^{\frac{1}{p}}(y_2 + x_0)\end{aligned}$$

and consequently,

$$\sup_{y_1\in Y}\left[\psi(y_1) - \rho^{\frac{1}{p}}(y_1 - x_0)\right] \leq \inf_{y_2\in Y}\left[\rho^{\frac{1}{p}}(y_2 + x_0) - \psi(y_2)\right]. \tag{3.1}$$

Let $t_0$ be any real number which lies between the supremum and the infimum appearing in equation (3.1). We now verify that this $t_0$ does the job.

Indeed, if $t > 0$, and if $y \in Y$, then, since the definition of $t_0$ ensures that

$$\psi(y_2) + t_0 \leq \rho^{\frac{1}{p}}(y_2 + x_0) \quad \forall y_2 \in Y,$$

we find that

$$\begin{aligned}\psi_1(y + tx_0) &= \psi(y) + tt_0\\ &= t\left[\psi\left(\frac{y}{t}\right) + t_0\right]\\ &\leq t\rho^{\frac{1}{p}}\left(\frac{y}{t} + x_0\right)\\ &= \rho^{\frac{1}{p}}(y + tx_0).\end{aligned}$$

Similarly, if $t < 0$, then, since the definition of $t_0$ also ensures that

$$\psi(y_1) - t_0 \leq \rho^{\frac{1}{p}}(y_1 - x_0) \quad \forall y_1 \in Y,$$

we find that

$$\begin{aligned}\psi_1(y + tx_0) &= \psi(y) + tt_0\\ &= -t\left[\psi\left(\frac{y}{-t}\right) - t_0\right]\\ &\leq -t\rho^{\frac{1}{p}}\left(\frac{y}{-t} - x_0\right)\\ &= \rho^{\frac{1}{p}}(y + tx_0).\end{aligned}$$

Thus, $\psi(y + tx_0) \leq \rho^{\frac{1}{p}}(y + tx_0) \quad \forall y \in Y, t \in \mathbb{R}$, and the proof of the lemma is complete. ∎

**Theorem 3.2.** *(Hahn–Banach theorem) Let $V$ be a $p$-normed space and let $V_0$ be a subspace of $V$. Suppose $\varphi_0 \in V_0^*$; then there exist a $\varphi \in V^*$ such that*
*(i)$\varphi|_{V_0} = \varphi_0$*
*(ii)$\|\varphi\| = \|\varphi_0\|$.*

*Proof.* Let $V_0$ be a (real) linear subspace of $V$. Now, apply the preceding lemma with $X = V, M = V_0$ and $\rho^{\frac{1}{p}}(x) = \|\varphi_0\|\|x\|$, to find that the desired conclusion follows immediately. ∎

**Remark 3.1.** One can follow the method of [9] to prove that the Hahn–Banach theorem holds for complex quasi-normed spaces.

**Corollary 3.3.** *Let $X$ be a $p$-normed space and $0 \neq x_0 \in X$. Then there exists $\varphi \in X^*$ such that $\|\varphi\| = 1$ and $\varphi(x_0) = \|x_0\|$.*

*Proof.* Set $X_0 = \mathbb{C}x_0 = \{\alpha x_0 : \alpha \in \mathbb{C}\}$, consider the linear functional $\varphi_0 \in X^*$ defined by $\varphi_0(\lambda x) = \lambda\|x_0\|$, and appeal to the Hahn–Banach theorem. ∎

**Theorem 3.4.** *Let $X$ be a $p$-normed space and let $M$ be a uniquely remotal subset of $X$ admitting a center. Suppose that for every $x$ in $M + r(M)^{\frac{1}{p}}B(X)$ the farthest point mapping $F : X \to M$ restricted to the line segment $[x, F(x)]$ is continued at $x$. Then $M$ is a singleton.*

*Proof.* We may assume that 0 is a center of $M$. Since $r(M) = \sup\{\|y - 0\|^p : y \in M\}$, we have $M \subseteq B\left(0, r(M)^{\frac{1}{p}}\right)$. For each $x \in M$, $0 \in B\left(x, r(M)^{\frac{1}{p}}\right) \subseteq M + r(M)^{\frac{1}{p}}B(X)$. Set $x_0 = F(0)$ and $g_n = F(\frac{x_0}{n})$. By the Hahn–Banach theorem for each $g_n - \frac{x_0}{n}$ there exists a functional $\Phi_n$ such that $\|\Phi_n\| = 1$, and $\left\langle \Phi_n, g_n - \frac{x_0}{n}\right\rangle = \left\|g_n - \frac{x_0}{n}\right\|$. Then

$$\begin{aligned}\left\langle \Phi_n, \frac{x_0}{n}\right\rangle &= \langle \Phi_n, g_n\rangle - \left\langle \Phi_n, g_n - \frac{x_0}{n}\right\rangle\\ &\le \|\Phi_n\|\|g_n\| - \left\|g_n - \frac{x_0}{n}\right\|\\ &= \|g_n\| - \left\|g_n - \frac{x_0}{n}\right\|,\end{aligned}$$

whence

$$\frac{1}{n}\Phi_n(x_0) \le \|g_n\| - \left\|g_n - \frac{x_0}{n}\right\|.$$

Since

$$\begin{aligned}\|g_n\|^p &\le \sup_n \|0 - g_n\|^p\\ &\le \sup_{y\in M} \|0 - y\|^p\\ &\le \sup_{z\in M} \left\|\frac{x_o}{n} - z\right\|^p\\ &= \left\|\frac{x_0}{n} - F\left(\frac{x_o}{n}\right)\right\|^p = \left\|\frac{x_0}{n} - g_n\right\|^p,\end{aligned}$$

we get $\Phi_n(x_0) \le 0$ for all $n$.

The function $F$ is continuous at $\frac{x_0}{n}$, so $F(\frac{x_0}{n}) \to F(0)$ hence $g_n - \frac{x_0}{n} \to x_0 - 0 = x_0$. Therefore $\Phi_n\left(g_n - \frac{x_0}{n}\right) \to \Phi_n(x_0)$,

$$\lim_{n\to\infty} \Phi_n(x_0) = \lim_{n\to\infty} \Phi_n\left(g_n - \frac{x_0}{n}\right) = \lim_{n\to\infty} \left\|g_n - \frac{x_0}{n}\right\| = \|x_0\|.$$

If $\|x_0\| > 0$, then $\Phi_n(x_0) > 0$ for some $n$ which is a contradiction. Hence $x_0 = 0$ and so for $y \in M$ we have

$$\|y - 0\|^p \le \|F(0) - 0\|^p = \|x_0\|^P = 0,$$

whence $M = \{0\}$. ∎

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