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THE ε-SMALL BALL DROP PROPERTY

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ABSTRACT. We continue the investigation on classes of small sets in a Banach space that give alternative formulations of the Drop Property. The small sets here considered are the set having the small ball property, and we show that for sets having non-empty intrinsic core and whose affine hull contains a closed affine space of infinite dimension the Drop Property can be equivalently formulated in terms of the small ball property.

Key words and phrases: Drop Property, stream, small ball property, symmetrically compact set, intrinsic core.

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1. INTRODUCTION

Let $(X, \|\ \|)$ be a reflexive Banach space.
We remind a definition from [3]

Definition 1.1. *:* Let $\mathcal{K}$ be a non-empty class of subsets of X, and let C be a closed and convex subset of X. We shall say that $C \in (\mathcal{K}) - \mathbf{DP}(X)$ if, for every closed F with $F \cap C = \emptyset$, there exists $x_0 \in F$ such that $D(x_0, C) \cap F \in \mathcal{K}$.

When $\mathcal{K} = \widehat{X}$ is the class of singletons in 2^X, we shall simply use the symbol $\mathbf{DP}(X)$; $\mathbf{DP}(X)$ has been introduced by D. Kutzarova in [6].
In [8] and [3] it has been shown that some classes of "small" sets playing the role of $\mathcal{K}$ are such that $(\mathcal{K}) - \mathbf{DP}(X) = \mathbf{DP}(X)$.
More precisely, the equivalence has been shown when $\mathcal{K}$ is the class of compact subsets of X in [8], and for the class of microscopic and of scalarly microscopic subsets of X in [3], where they are also defined.
In this note we shall consider the class of sets in X having the *small ball property* as defined in [2], and we shall compare it with the class $\mathbf{DP}(X)$.

2. PRELIMINARIES

The Drop Property has been equivalently characterized in several ways in [5], [7], [8], [3]. Besides the characterization in term of different classes of small sets of [8] and [3], we shall need the following equivalence, that has been obtained in [5].

Definition 2.1. Given a closed bounded convex set C, a sequence $(x_n)_n$ in $X \setminus C$ such that $x_{n+1} \in D(x_n, C)$, for all $n \in I\!N$, is called a *stream*.
A stream $(y_n)_n$ is said a *dyadic stream* if it can be represented by the following inductive formula $y_1 = \dfrac{x + x_1}{2}$ and $y_n = \dfrac{y_{n-1} + x_n}{2}$, for $n \geq 2$, where $x \in X \setminus C$ and $(x_n)_n \subset C$.

In [5] the following result is given:

Theorem 2.1. *A bounded, closed and convex set C has the Drop Property iff every stream in $X \setminus C$ has a norm converging subsequence.*

In this paper we shall continue the investigation of classes of small sets and their role in defining alternative forms of the Drop Property, and we shall turn our attention to the class of sets with the small ball property. This class has been introduced and studied in [2].

Definition 2.2. A set C has the *small ball property* if for every $\varepsilon > 0$ there exists a sequence of positive scalars $r_n < \varepsilon$ with $\lim_n r_n = 0$ and a sequence of elements $(x_n)_n \subset X$ such that

$$C \subset \bigcup_{n=1}^{\infty} (x_n + r_n X_1)$$

(where X_1 denotes the closed unitary ball of the space).

Among the properties obtained in [2] we quote the following that we shall need in the sequel

Theorem 2.2. *A closed, convex and bounded subset of X has the small ball property iff it is compact.*

3. COMPARISON

Throughout this section C will denote a non empty, bounded, closed and convex set in X.
As mentioned in the Introduction, besides the already defined hyperspace $\mathbf{DP}(X)$ we shall consider also the hyperspace $(\mathcal{K}) - \mathbf{DP}(X)$ where $\mathcal{K}$ is the class of sets having the $\varepsilon-$ small ball property.
Since singletons clearly have the $\varepsilon-$small ball property, it is immediate to note the inclusion

$$\mathbf{DP}(X) \subset (\mathcal{K}) - \mathbf{DP}(X).$$

In this section we investigate the inverse inclusion. For the sake of simplicity in the sequel we shall also consider the following notation:

Definition 3.1. The set C is *symmetrically compact* if for every $y \in C$, every closed and symmetric subset V of $C - y$ is compact.

The main result in this section is the following

Theorem 3.1. *If C is not symmetrically compact, and $C \in (\mathcal{K}) - \mathbf{DP}(X)$, then $C \in \mathbf{DP}(X)$.*

Proof. Since C is not symmetrically compact, there exists $\bar{x} \in C$ such that $C - \bar{x}$ contains a non-compact, closed, symmetric subset V. As $C - \bar{x}$ is convex, we can always assume that V is convex, and therefore circled.
Also C has the Drop Property iff $C - \bar{x}$ has the Drop Property, and $C \in (\mathcal{K}) - \mathbf{DP}(X)$ iff $(C - \bar{x}) \in (\mathcal{K}) - \mathbf{DP}(X)$.
Hence we can replace C with $C - \bar{x}$ namely we can assume without loss of generality that C contains a non compact, convex, circled set V.
Assume by contradiction that $C \notin \mathbf{DP}(X)$.

Claim 1. *There exists a dyadic stream in $X \setminus C$ with no converging subsequences.*
Proof. Observe first that, by Theorem 2.1 it is clear that a stream in $X \setminus C$ with no converging subsequences certainly exists; our statement is a little more restrictive, for we want the stream to be dyadic.
We have only two possible cases: if $C^o = \emptyset$, the existence of a dyadic stream with no converging subsequences is proven in [7], Theorem 3. Otherwise suppose $C^o \neq \emptyset$. Then, the equivalence between (i) and (ii) in Theorem 7 of [7] shows that C does not have property (α) (for C does not have the Drop Property). In this case, then, the existence of the required dyadic stream is proven in Proposition 1 of [7]. Hence in both cases Claim 1 holds.

Let then the dyadic stream of Claim 1 be defined as

$$x_o \in X \setminus C$$

$$x_{n+1} = \frac{x_n + a_{n+1}}{2},\ a_n \in C, n \in \mathbb{N}.$$

We shall now replace a_n with a suitable choice of $b_n \in C$.
To this aim, consider the following procedure: first pick some positive scalar $r_1 < \dfrac{1}{2}$ such that $(x_o + 2r_1 V) \cap C = \emptyset$; since $0 \in C$ and C is convex, $(1-r_1)a_1 \in C$ for every choice of $r_1 < \dfrac{1}{2}$; then choose r_1 such that $\dfrac{(1-r_1)a_1 + x_o}{2} \notin C$.
Define now $z_o = x_o$, $b_1 = (1 - r_1)a_1$ and $z_1 = \dfrac{z_o + b_1}{2}$.

Iterating this procedure, we can construct a decreasing sequence of positive scalars $(r_n)_n$ such that

$$r_n \leq \frac{r_{n-1}}{2} \left(\Longrightarrow r_n \leq \frac{1}{2^n} \right); \tag{3.1}$$

$$(z_n + 2r_{n+1}V) \cap C = \emptyset; \tag{3.2}$$

$$b_n = (1 - r_n)a_n; \tag{3.3}$$

$$z_n = \frac{b_n + z_{n-1}}{2} \notin C. \tag{3.4}$$

Claim 2. *The dyadic stream $(z_n)_n$ has no converging subsequences.*
Proof. In fact, setting $R = \mathrm{diam}(C)$, we have from (3.3) that

$$\|b_n - a_n\| = r_n \|a_n\| \leq \frac{1}{2^n} R. \tag{3.5}$$

Thus

$$\|x_1 - z_1\| = \frac{1}{2}\|a_1 - b_1\| = \frac{R}{2}; \tag{3.6}$$

then from (3.5) and (3.6)

$$\|x_2 - z_2\| \leq \frac{1}{2}\left(\|x_1 - z_1\| + \|a_2 - b_2\|\right) \leq \frac{R}{4} + \frac{R}{8} = \frac{1}{4}R\left(1 + \frac{1}{2}\right).$$

By induction then

$$\|x_n - z_n\| \leq \frac{1}{2^n} R \sum_{k=0}^{n} \frac{1}{2^k},$$

which implies that $(x_n)_n$ and $(z_n)_n$ would have the same converging subsequences.

Let us set

$$F = \bigcup_{n=1}^{\infty} \left(z_n + \frac{r_n}{2} V \right).$$

Then clearly $F \cap C = \emptyset$.

Claim 3. *F is closed.*
Proof. The proof is rather similar to that of Lemma 1 in [3]: in fact given $(t_n)_n \subset F$ such that $t_n \to t$ define $\mathbb{N}_k = \left\{ p | t_p \in \left(z_k + \frac{r_k}{2} V \right) \right\}$. If $\mathbb{N}_k$ is infinite for some k_o then a subsequence of $(t_n)_n$ lies in $\left(z_{k_o} + \frac{r_{k_o}}{2} V \right)$ and therefore $t \in \left(z_{k_o} + \frac{r_{k_o}}{2} V \right) \subset F$.
We prove now that this is the only possible occurrence, namely that assuming that $\mathbb{N}_k$ is finite for every $k \in \mathbb{N}$ leads to a contradiction.
Indeed if $\mathbb{N}_k$ is finite or empty for every $k \in \mathbb{N}$ and we pick

$$y_k = \begin{cases} t_{p(k)} & \text{if } \mathbb{N}_k \neq \emptyset \text{ and } p(k) = \min \mathbb{N}_k \\ z_k & \text{if } \mathbb{N}_k = \emptyset, \end{cases}$$

then $(y_n)_n$ has a subsequence converging to t, and since $\|y_n - z_n\| \leq \frac{r_n}{2}$, from (3.1) it would follow that $(z_n)_n$ has a subsequence converging to t, which contradicts Claim 2.

Claim 4. *For every $t \in F$, say $t \in \left(z_n + \frac{r_n}{2}V\right)$ then*

$$\left(z_{n+1} + \frac{r_{n+1}}{2}V\right) \subset D(t, C).$$

Proof. Note that from Claim 4, there follows that $D(t, C) \cap F$ contains a non-compact closed and convex set; from Theorem 2.2 then C fails to fulfill the $(\mathcal{K})$-Drop Property, contradicting the assumption.

In order to prove the Claim, fix $t = z_n + \frac{r_n}{2}v$ for some $v \in V$. Put $W = \frac{V}{2}$, and write $t = z_n + r_n w$ with $w \in W$. To prove that $(z_{n+1} + r_{n+1}W) \subset D(t, C)$ means that for every $u \in W$ there exist $\alpha \in]0, 1[$ and $a \in C$ such that

$$z_{n+1} + r_{n+1}u = \alpha(z_n + r_n w) + (1 - \alpha)a.$$

We shall in fact prove that for every $u \in W$ there exists $a \in C$ such that

$$z_{n+1} + r_{n+1}u = \frac{(z_n + r_n w) + a}{2}$$

which, since $z_{n+1} = \frac{z_n + b_n}{2}$, is equivalent to

$$\frac{b_n}{2} + r_{n+1}u = \frac{r_n w + a}{2}.$$

Therefore we have to show that for every $u \in W$,

$$2r_{n+1}u - r_n w + b_n \in C. \tag{3.7}$$

Since

$$2r_{n+1}u = r_n\left(\frac{2r_{n+1}}{r_n}\right)u = \frac{2r_{n+1}}{r_n}r_n u + \left(1 - \frac{2r_{n+1}}{r_n}\right)0$$

we have that $2r_{n+1}u \in r_n W$. Hence we can write

$$2r_{n+1}u - r_n w = r_n(z - w)$$

for some $z \in W$. Therefore $(z - w) \in W - W = V$ since V is circled.
In conclusion

$$b_n + 2r_{n+1}u - r_n w = b_n + r_n(z - w) = (1 - r_n)a_n + r_n(z - w)$$

with $(z - w) \in V \subset C$, and $a_n \in C$; this proves (3.7) and therefore the total assertion. □

4. SYMMETRICALLY COMPACT SETS

In view of Theorem 3.1 it becomes clear that an answer to the natural question whether the inclusion $\mathbf{DP}(X) \subset (\mathcal{K}) - \mathbf{DP}(X)$ is strict or not can be given only in the framework of non compact symmetrically compact sets.
To prove that the answer to the question is not this easy we provide an example of a non compact symmetrically compact set.

Example 4.1. In $X = \ell^2$ let $K = X_1 \cap X^+$ where X^+ is the usual closed order cone in X. Then immediately K is not compact, for it contains the standard basis.
We shall show that K is symmetrically compact. Let $y \in K$ be fixed, $y = (y_i)_{i \in \mathbb{N}}$, and consider a symmetric closed and convex subset $V \subset K - y$. Then for every $v \in V, v = (v_i)_{i \in \mathbb{N}}$ we find for each $i \in \mathbb{N}$

$$|v_i| \leq y_i. \tag{4.1}$$

In fact, $v = k - y$ for some $k \in K$; As $-v \in V$ too, there exists $k' \in K$ such that $-v = k' - y$, whence $k' - y = y - k$, namely $k' = 2y - k$. Since $k' \in K \subset X^+$ every component is non negative, that is $2y_i - k_i \geq 0$ for every $i \in I\!N$, or else $k_i \leq 2y_i$, which proves (4.1).
From (4.1) the compactness of V follows; in fact it implies that V is contained in the order interval $[-y, y]$, and since X is discrete, this interval is compact ([1] Corollary 21.13, page 156).
Hence K is symmetrically compact.

Therefore it becomes interesting to investigate which sets are in fact symmetrically compact.
We point out again that a set C in X is **not** symmetrically compact, provided there exists $x_o \in C$ such that the translate $C - x_o$ contains a non compact symmetric set.
Then a first example of sets that are not symmetrically compact is the class of sets in X having non empty interior, provided the dimension of X is infinite.
A generalization of this fact can be given according to [4].
In fact, following Giles, we define, for a non empty $V \subset X$, the *affine hull* aff$(V) = v + \mathrm{span}(V - v)$ whatever is $v \in V$.
Then for a non empty V we define the *intrinsic core* icor(V) to be the set of elements $v \in V$ such that for every $x \in \mathrm{aff}(V)$, $x \neq v$ there exists some $r \in]0,1[$ for which the whole interval $[v, rv + (1-r)x]$ is contained in V (where the interval means the line segment joining the two elements).

Proposition 4.1. *If C is bounded, closed and convex, with* icor$(C) \neq \emptyset$*, and if aff(C) contains a closed affine space of infinite dimension, C is not symmetrically compact.*

Proof. Clearly, if icor$(C) \neq \emptyset$, then for each $x \in$ icor(C), $0 \in$ icor$(C - x)$; set $A = C - x$; obviously aff$(A) =$ span$A$, and A absorbs its own linear span.
For every $x \in A \setminus \{0\}$ consider

$$\rho(x) = \sup\{t \in]0,1[: tx \in A\}$$

e then define

$$r(x) = \rho(x) \wedge \rho(-x).$$

Since $0 \in$ icor(A), $\rho(x) > 0$ for every $x \in A \setminus \{0\}$, and hence $r(x)$ is positive (in fact, A absorbs span$A$, and therefore $\rho(-x)$ is strictly positive too). Consider now the set

$$K = \bigcup_{x \in \mathrm{span} A} [-r(x), r(x)].$$

Then $K \subset A$ and clearly K is symmetric. Since A is closed, bounded and convex, $V = \overline{\mathrm{co}} K$ is a closed, bounded, convex, circled subset of A, that absorbs span $A =$ span V. In other words V is a barrel in span A.
By our assumptions, span A contains a closed subspace of infinite dimension Y, and $V \cap Y$ is a barrel in the complete Banach space Y ; hence $V \cap Y$ is a neighbourhood of 0 in Y, and hence $V \cap Y$ is not compact. Thus V is not compact too. $\square$

As a consequence, we can derive the following

Corollary 4.1. *If C is a bounded, closed and convex set in $(\mathcal{K}) - \mathbf{DP}(X)$ with* icor$(C) \neq \emptyset$*, and if* aff(C) *contains a closed affine space, then $C \in \mathbf{DP}(X)$.*

Proof. Indeed, if aff(C) is of finite dimension, then C is compact, and therefore directly $C \in \mathbf{DP}(X)$. If aff(C) is of infinite dimension, then from Proposition 4.1 C is not symmetrically compact, so that Theorem 3.1 can be applied. $\square$

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