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MULTIVALENT HARMONIC MAPPINGS CONVOLUTED WITH A MULTIVALENT ANALYTIC FUNCTION

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ABSTRACT. The object of this paper is to study certain geometric properties of a family of multivalent harmonic mappings in the plane convoluted with a multivalent analytic function in the open unit disc.

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1. INTRODUCTION

A continuous complex-valued mapping $f = u + iv$ is defined as *harmonic* in a simply connected complex domain $\mathbb{D}$ in the complex plane if it is satisfies $f_{z\bar{z}} \equiv 0$ on $\mathbb{D}$, i.e., u and v are real harmonic functions in $\mathbb{D}$. Such a harmonic function f can be expressed as the canonical representation $f = h + \bar{g}, \quad g(0) = 0,$ where h and g are analytic and g denotes the function $z \rightarrow \overline{g(z)}$. In [3], it was shown that the mapping $z \rightarrow \ f(z)$ is sense-preserving and locally univalent in $\mathbb{D}$ if and only if the Jacobian $J_f = |h'|^2 - |g'|^2 > 0$ in $\mathbb{D}$. We observe that if $f = h + \bar{g}$, then

$$h' = f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \overline{g'} = f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

are always (globally) analytic functions on $\mathbb{D}$. In general, we do not require f to be univalent in $\mathbb{D}$. The study of functions which are multivalent harmonic in the open unit disc $\Delta = \{z : |z| < 1\}$ was indicated in [1], [4]. For $p \geq 1$, denote by $H(p)$, the set of all multivalent harmonic functions $f = h + \bar{g}$ defined in Δ, where h and g are analytic functions defined in Δ and of the form

$$h\left(z\right) = \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1}, \quad g\left(z\right) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1}, \quad a_p = 1, \quad |b_p| < 1. \tag{1.1}$$

Various subclasses of $H(p)$ were studied in [1], [2] and others. Denote by $P_H(p)$ the class of all multivalent harmonic functions of the form

$$f\left(z\right) = p + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} \overline{b_{n+p-1} z^{n+p-1}}, \tag{1.2}$$

such that $Ref(z) > 0$, where $p \geq 1$ is an integer. For $p = 1$ various properties of the family $P_H(1)$ and the subclasses of $P_H(1)$ with real coefficients were studied in [5], [6], [7]. A well-known family

$$P\left(p\right) = \left\{ h(z) = p + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} : Reh(z) > 0, z \in \Delta \right\}$$

is a subclass of $P_H(p)$. We also define

$$P_H^0\left(p\right) = \{f = h + \bar{g} \in P_H(p) : g(0) = 0\}$$

and

$$R_H^0\left(p\right) = \left\{ f = h + \bar{g} \in H(p) : \frac{1}{p}\left(\frac{h'}{z^{p-1}} + \overline{\left(\frac{g'}{z^{p-1}}\right)}\right) \in P_H^0(p) \right\}.$$

If $f_j = h_j + \bar{g}_j, j = 1, 2$ are in the class $H(p)$, then we define convolution $f_1 * f_2$ of f_1 and f_2 in the natural way as $h_1 * h_2 + \overline{g_1 * g_2}$. If ϕ is a p-valent analytic function and $f = h + \bar{g}$ is in $H(p)$, we define

$$f \widetilde{*} \phi = f * (\phi + \bar{\phi}) = h * \phi + \overline{g * \phi}.$$

Clunie and Sheil-Small [3] considered $f \widetilde{*} \phi$ when ϕ is analytic and f is convex harmonic univalent in Δ. The object of this paper is to study certain geometric properties of family $R_H^0\left(p, \alpha\right)$ of mappings obtained as $f \widetilde{*} \phi_{p,\alpha}$ when $f \in R_H^0\left(p\right)$ and $\phi_{p,\alpha}$ is defined by

$$\phi_{p,\alpha}(z) = z^p + \sum_{n=2}^{\infty} \frac{p}{p + (n-1)\alpha} z^{n+p-1}, \tag{1.3}$$

where $p \geq 1$ is an integer and α is a complex number different from $-p, -\frac{p}{2}, -\frac{p}{3}, \cdots$. Note that the function $\phi_{p,\alpha}$ is analytic in Δ. Also, $R_H^0(p,0) = R_H^0(p)$.

2. Main Results

Theorem 2.1. *If $F = H + \bar{G} \in R_H^0(p,\alpha)$ and $Re\alpha > 0$, then there exists an $f \in R_H^0(p)$ so that*

$$F(z) = \frac{p}{\alpha}\int_0^1 \zeta^{\frac{p(1-\alpha)-\alpha}{\alpha}} f(z\zeta)d\zeta \quad , \quad z \in \Delta.$$

Proof. Note that

$$\phi_{p,\alpha}(z) = \frac{p}{\alpha}\int_0^1 \zeta^{\frac{p}{\alpha}-1}\frac{z^p}{1-z\zeta}d\zeta \quad , \quad |\zeta| \leq 1.$$

where $Re\alpha > 0$. Also for $f = h + \bar{g} \in R_H^0(p)$,

$$h(z) * \frac{z^p}{1-z\zeta} = z^p + \sum_{n=2}^{\infty}\zeta^{n-1}a_{n+p-1}z^{n+p-1} = \frac{h(z\zeta)}{\zeta^p}$$

$$g(z) * \frac{z^p}{1-z\zeta} = \sum_{n=1}^{\infty}\zeta^{n-1}b_{n+p-1}z^{n+p-1} = \frac{g(z\zeta)}{\zeta^p}.$$

Therefore,

$$H(z) = h(z) * \phi_{p,\alpha}(z) = \frac{p}{\alpha}\int_0^1 \zeta^{\frac{p(1-\alpha)-\alpha}{\alpha}} h(z\zeta)d\zeta$$

and

$$G(z) = g(z) * \phi_{p,\alpha}(z) = \frac{p}{\alpha}\int_0^1 \zeta^{\frac{p(1-\alpha)-\alpha}{\alpha}} g(z\zeta)d\zeta.$$

Hence

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= \frac{p}{\alpha}\int_0^1 \zeta^{\frac{p(1-\alpha)-\alpha}{\alpha}}\left(h(z\zeta) + \overline{g(z\zeta)}\right)d\zeta \\ &= \frac{p}{\alpha}\int_0^1 \zeta^{\frac{p(1-\alpha)-\alpha}{\alpha}} f(z\zeta)d\zeta. \end{aligned}$$

∎

We next give the interrelation between classes $R_H^0(p,\alpha)$ and $R_H^0(p)$.

Theorem 2.2. *If $F \in R_H^0(p,\alpha)$, then there exists a function f in $R_H^0(p)$ such that*

$$f(z) = \frac{\alpha}{p}\left[zF_z(z) + \bar{z}F_{\bar{z}}(z)\right] + (1-\alpha)F(z). \tag{2.1}$$

Conversely, if $f \in R_H^0(p)$, then there exists $F \in R_H^0(p,\alpha)$ such that F is a solution of (2.1).

Proof. Since $F \in R_H^0(p,\alpha)$, there exists $f \in R_H^0(p)$ so that $F(z) = \left(f(z)\tilde{*}\phi_{p,\alpha}(z)\right)$. Also, it is a routine manipulation to prove that

$$\frac{\alpha z}{p}\phi'_{p,\alpha}(z) + (1-\alpha)\phi_{p,\alpha}(z) = \phi_{p,0}(z).$$

Therefore we have

$$\begin{aligned} f(z) &= f(z)\tilde{*}\phi_{p,0}(z) = f(z)\tilde{*}\left(\frac{\alpha z}{p}\phi'_{p,\alpha}(z) + (1-\alpha)\phi_{p,\alpha}(z)\right) \\ &= \frac{\alpha}{p}\left(f(z)\tilde{*}z\phi'_{p,\alpha}(z)\right) + (1-\alpha)\left(f(z)\tilde{*}\phi_{p,\alpha}(z)\right) \\ &= \frac{\alpha}{p}\left[zF_z(z) + \bar{z}F_{\bar{z}}(z)\right] + (1-\alpha)F(z). \end{aligned}$$

Conversely, suppose $f(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1}z^{n+p-1} + \sum_{n=2}^{\infty} \overline{b_{n+p-1}z^{n+p-1}}$ is in $R_H^0(p)$ and let

$$F(z) = z^p + \sum_{n=2}^{\infty} A_{n+p-1}z^{n+p-1} + \sum_{n=2}^{\infty} \overline{B_{n+p-1}z^{n+p-1}}, \tag{2.2}$$

be a solution of (2.1). On comparing both sides of (2.1), we have

$$A_{n+p-1} = \frac{p}{p+\alpha(n-1)}a_{n+p-1} \quad , \quad B_{n+p-1} = \frac{p}{p+\alpha(n-1)}b_{n+p-1}, \quad n \geq 2. \tag{2.3}$$

Then $F \in R_H^0(p,\alpha)$ because

$$\begin{aligned} F(z) &= \left(h(z) * \phi_{p,\alpha}(z)\right) + \overline{\left(g(z) * \phi_{p,\alpha}(z)\right)} \\ &= f(z)\tilde{*}\phi_{p,\alpha}(z). \end{aligned}$$

∎

Corollary 2.3. *A function* $F(z) = H(z) + \overline{G(z)}$ *where* H *and* G *are the form* (1.1), *is in the family* $R_H^0(p,\alpha)$ *if and only if* F *satisfies the condition*

$$Re\left\{\frac{(\alpha + p(1-\alpha))H'(z) + (\bar{\alpha} + p(1-\bar{\alpha}))\,G'(z)}{pz^{p-1}} + \frac{\alpha H''(z) + \bar{\alpha}G''(z)}{pz^{p-2}}\right\} > 0, \quad z \in \Delta. \tag{2.4}$$

Proof. If $F = H + \overline{G} \in R_H^0(p,\alpha)$, then by Theorem (2.2) there exists $f = h + \overline{g} \in R_H^0(p)$ such that (2.1) is satisfied and

$$h(z) = \frac{\alpha}{p}zH'(z) + (1-\alpha)H(z) \quad , \quad g(z) = \frac{\bar{\alpha}}{p}zG'(z) + (1-\bar{\alpha})G(z).$$

Since $f = h + \overline{g} \in R_H^0(p)$, it follows that

$$\begin{aligned} 0 &< Re\left(\frac{h'(z)}{z^{p-1}} + \left(\overline{\frac{g'(z)}{z^{p-1}}}\right)\right) < Re\left\{\frac{h'(z) + g'(z)}{z^{p-1}}\right\} \\ &= Re\left\{\frac{(\alpha + p(1-\alpha))H'(z) + (\bar{\alpha} + p(1-\bar{\alpha}))\,G'(z)}{pz^{p-1}} + \frac{\alpha H''(z) + \bar{\alpha}G''(z)}{pz^{p-2}}\right\}. \end{aligned}$$

Conversely, suppose that

$$\begin{aligned} F(z) &= H(z) + \overline{G(z)} \\ &= z^p + \sum_{n=2}^{\infty} \frac{p}{p+\alpha(n-1)}a_{n+p-1}z^{n+p-1} + \sum_{n=2}^{\infty} \overline{\frac{p}{p+\alpha(n-1)}b_{n+p-1}z^{n+p-1}} \end{aligned}$$

satisfies (2.4), where a_{n+p-1} and b_{n+p-1} are the coefficients of $f \in R_H^0(p)$. Then using the arguments in Theorem (2.2), it follows that the function

$$f(z) = h(z) + \overline{g(z)} = \frac{\alpha}{p}\left[zH'(z) + \bar{z}G'(z)\right] + (1-\alpha)\left[H(z) + \overline{G(z)}\right]$$

is in the class $R_H^0(p)$. Hence, by Theorem (2.2), $F \in R_H^0(p,\alpha)$. ∎

Theorem 2.4. *$R_H^0(p,\alpha)$ is convex.*

Proof. For $F_1 = H_1 + \overline{G_1}, F_2 = H_2 + \overline{G_2} \in R_H^0(p,\alpha)$,we have to show that

$$\lambda F_1 + (1-\lambda)F_2 = \lambda H_1 + (1-\lambda)H_2 + \overline{\lambda G_1 + (1-\lambda)G_2}$$

is in $R_H^0(p,\alpha)$ for any $\lambda \in [0,1]$. In view of Corollary (2.3), we only need to show that $\lambda F_1 + (1-\lambda)F_2$ satisfies (2.4). This follows because

$$\begin{aligned} & Re(\frac{(\alpha+p(1-\alpha))(\lambda H_1' + (1-\lambda)H_2') + (\bar\alpha + p(1-\bar\alpha))(\lambda G_1' + (1-\lambda)G_2')}{pz^{p-1}} \\ & \quad + \frac{\alpha(\lambda H_1'' + (1-\lambda)H_2'') + \bar\alpha(\lambda G_1'' + (1-\lambda)G_2'')}{pz^{p-2}}) \\ = \ & \lambda Re\left\{\frac{(\alpha+p(1-\alpha))H_1' + (\bar\alpha+p(1-\bar\alpha))G_1'}{pz^{p-1}} + \frac{\alpha H_1'' + \bar\alpha G_1''}{pz^{p-2}}\right\} \\ & +(1-\lambda)Re\left\{\frac{(\alpha+p(1-\alpha))H_2' + (\bar\alpha+p(1-\bar\alpha))G_2'}{pz^{p-1}} + \frac{\alpha H_2'' + \bar\alpha G_2''}{pz^{p-2}}\right\} > 0. \end{aligned}$$

Therefore, $R_H^0(p,\alpha)$ is convex. ∎

In order to show that $R_H^0(p,\alpha)$ is also compact, we need the following.

Lemma 2.5. *If $f = h + \bar g \in R_H^0(p)$, then $\frac{h'+g'}{pz^{p-1}} \in P(p)$. Conversely, if $\frac{h'+g'}{pz^{p-1}} \in P(p), h(0) = g(0) = \left(\frac{h'}{pz^{p-1}}\right)_{z=0} - 1 = \left(\frac{g'}{pz^{p-1}}\right)_{z=0} = 0$, then $f = h + \bar g \in R_H^0(p)$*

Proof. If $f = h + \bar g \in R_H^0(p)$, since

$$Re\left\{\frac{1}{p}\left[\frac{h'}{z^{p-1}} + \overline{\left(\frac{g'}{z^{p-1}}\right)}\right]\right\} > 0 \Rightarrow Re\left\{\frac{h'+g'}{pz^{p-1}}\right\} > 0$$

the required result is obtained.
Conversely, if $\frac{h'+g'}{pz^{p-1}} \in P(p)$ restricted by the given conditions on h and g, then $\frac{h'}{pz^{p-1}} + \overline{\left(\frac{g'}{pz^{p-1}}\right)} \in P_H(p)$. In view of the conditions of normalization, it follows that $f = h + \bar g \in R_H^0(p)$. ∎

Remark 2.1. *$P(p)$ is compact and so $R_H^0(p)$ is compact.*

Theorem 2.6. *$R_H^0(p,\alpha)$ is compact.*

Proof. If $\{F_n\}$ is a sequence of functions in $R_H^0(p,\alpha)$ where $F_n = H_n + \overline{G_n}$, then by Theorem (2.2),

$$\frac{\alpha}{p}\left(zH_n' + \overline{zG_n'}\right) + (1-\alpha)\left(H_n + \overline{G_n}\right) \in R_H^0(p).$$

Since $R_H^0(p)$ is compact and so if $F_n \to F = H + \bar G$, then

$$\frac{\alpha(zH' + \overline{zG'})}{pz^{p-1}} + (1-\alpha)\left(H + \overline{G}\right) \in R_H^0(p).$$

Hence $R_H^0(p,\alpha)$ is compact because $F = H + \overline{G} \in R_H^0(p,\alpha)$ by Theorem (2.2). ∎

Theorem 2.7. *If $Re\alpha > 0$, then $R_H^0(p,\alpha) \subset R_H^0(p)$.*

Proof. Let $F \in R_H^0(p,\alpha)$ and $Re\alpha > 0$. Then there exists $f \in R_H^0(p)$ such that

$$F = H + \overline{G} = (h(z) * \phi_{p,\alpha}(z)) + \overline{(g(z) * \phi_{p,\alpha}(z))}.$$

Since $Re\alpha > 0$, it follows that

$$\begin{aligned} 0 &< Re\left\{\frac{h'}{z^{p-1}} + \overline{\left(\frac{g'}{z^{p-1}}\right)}\right\} \\ &= Re\left\{\frac{h'+g'}{z^{p-1}}\right\} \\ &= Re\left\{\frac{pH'+pG'}{pz^{p-1}} - \left(\frac{\alpha(p-1)H' + \overline{\alpha}(p-1)G'}{pz^{p-1}} - \frac{\alpha H'' + \overline{\alpha}G''}{pz^{p-2}}\right)\right\} \\ &< Re\left(\frac{H'+G'}{z^{p-1}}\right). \end{aligned}$$

However,

$$H(0) = G(0) = \left(\tfrac{H'}{pz^{p-1}}\right)_{z=0} - 1 = \left(\tfrac{G'}{pz^{p-1}}\right)_{z=0} = 0.$$

Hence by Lemma (2.5), it follows that $F = H + \overline{G} \in R_H^0(p)$. ∎

Lemma 2.8. [7] *If* $q(z) = z^p + \sum_{n=2}^{\infty} c_{n+p-1} z^{n+p-1}$ *and* $Re\left(\frac{q'(z)}{pz^{p-1}}\right) > 0$, *then* $|c_{n+p-1}| \leq \frac{2p}{n+p-1}$, $n \geq 1$. *The estimate is sharp.*

Theorem 2.9. *If* $F(z) = z^p + \sum_{n=2}^{\infty} A_{n+p-1} z^{n+p-1} + \sum_{n=2}^{\infty} \overline{B_{n+p-1} z^{n+p-1}}$ *is in* $R_H^0(p,\alpha)$, *then*

$$||A_{n+p-1}| - |B_{n+p-1}|| \leq \frac{2p^2}{(n+p-1)|p+(n-1)\alpha|}, \quad n \geq 1.$$

The estimate is sharp.

Proof. In view of Theorem (2.2), there exists

$$f(z) = h(z) + \overline{g(z)} = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1} + \sum_{n=2}^{\infty} \overline{b_{n+p-1} z^{n+p-1}} \in R_H^0(p)$$

so that $F = f \widetilde{*} \phi_{p,\alpha}$, where A_{n+p-1} and B_{n+p-1} are given by (2.3). From Lemma (2.5),

$$Re\left\{\frac{h'+g'}{pz^{p-1}}\right\} = Re\left\{1 + \sum_{n=2}^{\infty} \frac{n+p-1}{p}(a_{n+p-1} + b_{n+p-1}) z^{n-1}\right\} > 0,$$

and from Lemma (2.5), we have

$$|a_{n+p-1} + b_{n+p-1}| \leq \frac{2p}{n+p-1}. \tag{2.5}$$

From (2.3) and (2.5) we obtain

$$\begin{aligned} ||A_{n+p-1}| - |B_{n+p-1}|| &= \frac{p}{|p+(n-1)\alpha|} ||a_{n+p-1}| - |b_{n+p-1}|| \\ &\leq \frac{p}{|p+(n-1)\alpha|} |a_{n+p-1} + b_{n+p-1}| \\ &\leq \frac{2p^2}{(n+p-1)|p+(n-1)\alpha|}. \end{aligned}$$

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