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A DOUBLE INEQUALITY FOR DIVIDED DIFFERENCES AND SOME IDENTITIES OF THE PSI AND POLYGAMMA FUNCTIONS

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ABSTRACT. In this short note, from the logarithmically completely monotonic property of the function $(x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)}$, a double inequality for the divided differences and some identities of the psi and polygamma functions are presented.

Key words and phrases: Inequality, Divided difference, Identity, Psi function, Polygamma function, Logarithmically completely monotonic function.

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1. INTRODUCTION

Recall [2, 14] that a positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geq 0 \tag{1.1}$$

for all $k \in \mathbb{N}$ on I. For more detailed information, please refer to [2, 3, 4, 9, 10, 13, 17, 18] and the related references therein.

It is well-known that the classical Euler's gamma function $\Gamma(x)$ plays a central role in the theory of special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called the psi or digamma function, and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are known as the polygamma or multigamma functions.

For real numbers α and β with $\alpha \neq \beta$, $(\alpha, \beta) \neq (0,1)$ and $(\alpha, \beta) \neq (1,0)$ and for $t \in \mathbb{R}$, let

$$q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t} - e^{-\beta t}}{1 - e^{-t}}, & t \neq 0, \\ \beta - \alpha, & t = 0. \end{cases} \tag{1.2}$$

From necessary and sufficient conditions such that the function $q_{\alpha,\beta}(t)$ is monotonic, which were established in [5, 11, 12] and related references therein, the following logarithmically complete monotonicity was obtained.

Lemma 1 ([5, 7, 8, 15, 16])**.** *Let a, b and c be real numbers and $\rho = \min\{a,b,c\}$. Then the function*

$$H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{1.3}$$

is logarithmically completely monotonic in $(-\rho, \infty)$ if and only if

$$\begin{aligned} (a,b,c) \in D_1(a-c, b-c) &\triangleq \{(a,b,c) : (b-a)(1-a-b+2c) \geq 0\} \\ &\cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \geq 0\} \\ &\setminus \{(a,b,c) : a = c+1 = b+1\} \setminus \{(a,b,c) : b = c+1 = a+1\}, \end{aligned} \tag{1.4}$$

so is $H_{b,a,c}(x)$ in $(-\rho, \infty)$ if and only if

$$\begin{aligned} (a,b,c) \in D_2(a-c, b-c) &\triangleq \{(a,b,c) : (b-a)(1-a-b+2c) \leq 0\} \\ &\cap \{(a,b,c) : (b-a)(|a-b|-a-b+2c) \leq 0\} \\ &\setminus \{(a,b,c) : b = c+1 = a+1\} \setminus \{(a,b,c) : a = c+1 = b+1\}. \end{aligned} \tag{1.5}$$

Remark 1. The domains $D_1(\alpha, \beta)$ and $D_2(\alpha, \beta)$ defined by (1.4) and (1.5) can be described respectively by Figure 1 and Figure 2 below.

The first aim of this short note is to deduce a double inequality for the divided differences of the polygamma functions from Lemma 1 as follows.

Theorem 1. *Let $b > a \geq 0$ and $k \in \mathbb{N}$. Then the double inequality*

$$\frac{(k-1)!}{(x+\alpha)^k} \leq \frac{(-1)^{k-1}\big[\psi^{(k-1)}(x+b) - \psi^{(k-1)}(x+a)\big]}{b-a} \leq \frac{(k-1)!}{(x+\beta)^k} \tag{1.6}$$

for $x \in (-a, \infty)$ holds if $\alpha \geq \max\big\{a, \frac{a+b-1}{2}\big\}$ and $0 \leq \beta \leq \min\big\{a, \frac{a+b-1}{2}\big\}$.

The second aim of this short note is to show some identities of the psi and polygamma functions by using Theorem 1.

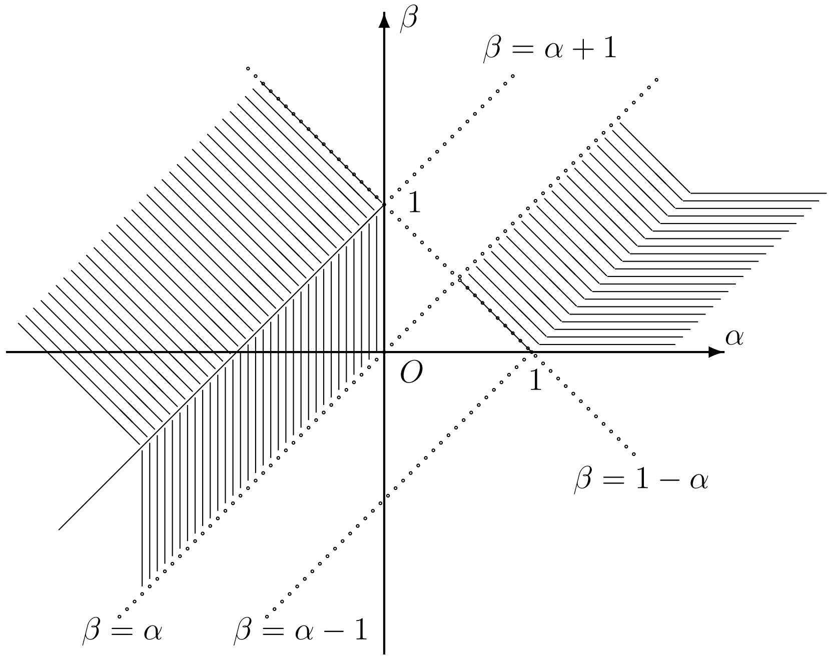

Figure 1: The (α, β)*-domain* $D_1(\alpha, \beta)$

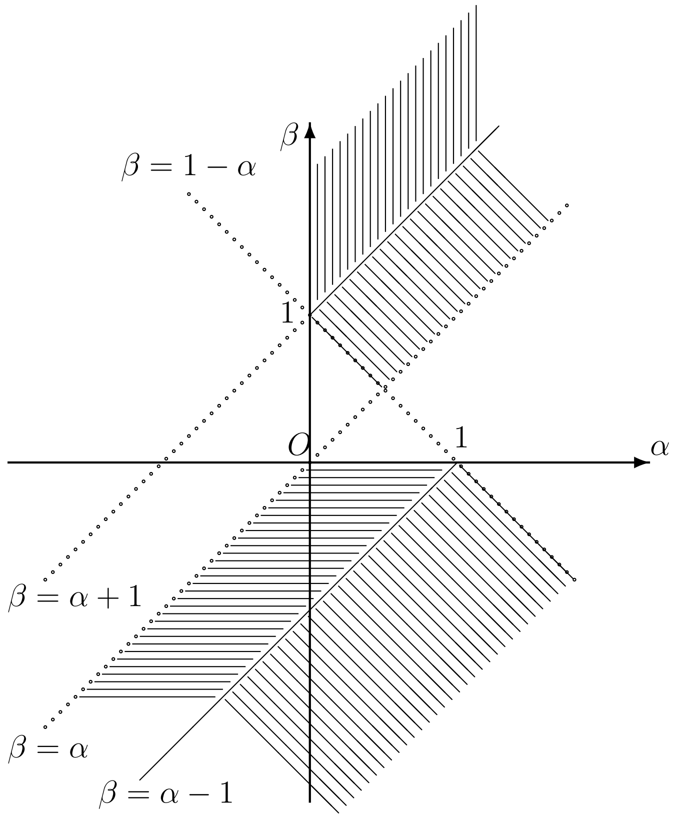

Figure 2: The (α, β)*-domain* $D_2(\alpha, \beta)$

Theorem 2. *For $v > 1$ and $\alpha > 1$, let $v_0 > 1$ denote the real root of equation $v^\alpha - v - 1 = 0$, then*

$$v_0^k\big[\psi^{(k-1)}(v_0^\alpha) - \psi^{(k-1)}(v_0)\big] = (-1)^{k-1}(k-1)!. \tag{1.7}$$

For $0 < v < 1$ and $\alpha < 0$, let $v_0 < 1$ be the real root of equation $v^\alpha - v - 1 = 0$, then identity (1.7) *is also valid.*

Corollary 1. *Let $k \in \mathbb{N}$. then the following identities of polygamma functions are valid:*

$$\psi^{(k-1)}\left(\left(\frac{\sqrt{5}+1}{2}\right)^2\right) - \psi^{(k-1)}\left(\frac{\sqrt{5}+1}{2}\right) = (-1)^{k-1}(k-1)!\left(\frac{\sqrt{5}-1}{2}\right)^k, \tag{1.8}$$

$$\begin{aligned}\psi^{(k-1)}&\left(\left(\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}}\right)^3\right) - \psi^{(k-1)}\left(\frac{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{18}}\right)\\&= (-1)^{k-1}(k-1)!\left(\frac{\sqrt[3]{18}}{\sqrt[3]{9-\sqrt{69}}+\sqrt[3]{9+\sqrt{69}}}\right)^k,\end{aligned} \tag{1.9}$$

$$\begin{aligned}\psi^{(k-1)}&\left(\frac{1}{8}\left(\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\sqrt{b-a}\right)^4\right)\\&- \psi^{(k-1)}\left(\frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\frac{\sqrt{b-a}}{2}\right)\\&= (-1)^{k-1}2^k(k-1)!\left(\sqrt{a-b+\frac{2}{\sqrt{b-a}}}+\sqrt{b-a}\right)^{-k},\end{aligned} \tag{1.10}$$

where $a = 4\sqrt[3]{\frac{2}{3(9+\sqrt{849})}}$ and $b = \sqrt[3]{\frac{9+\sqrt{849}}{18}}$.

Remark 2. In an e-mail to the second author on 24 November 2007, Dr. Abdolhossein Hoorfar at the University of Tehran pointed out that the identities in Theorem 2 and Corollary 1 are special cases of the following recurrence formula

$$\psi^{(n)}(z+1) - \psi^{(n)}(z) = (-1)^n n! z^{-n-1} \tag{1.11}$$

listed in [1, p. 260, 6.4.6]. This shows us that Lemma 1 and Theorem 1 above are generalizations of formula (1.11).

2. PROOFS OF THEOREMS

Proof of Theorem 1. From the logarithmically complete monotonicity of the function $H_{a,b,c}(x)$ in Lemma 1, it follows that

$$\begin{aligned}0 &\le (-1)^k[\ln H_{a,b,c}(x)]^{(k)}\\&= (-1)^k\left[\psi^{(k-1)}(x+a) - \psi^{(k-1)}(x+b) + \frac{(-1)^{k-1}(b-a)(k-1)!}{(x+c)^k}\right]\end{aligned} \tag{2.1}$$

for $(a,b,c) \in D_1(a,b,c)$, then the left-hand side inequality in (1.6) is deduced straightforwardly by standard arguments.

The right-hand side inequality in (1.6) can be deduced from $(-1)^k[\ln H_{b,a,c}(x)]^{(k)} \ge 0$ for $(a,b,c) \in D_2(a,b,c)$. ∎

Proof of Theorem 2. Inequality (1.6) in Theorem 1 can be rearranged as

$$\begin{aligned} \frac{(k-1)!}{[\max\{v,(u+v-1)/2\}]^k} &\le \frac{(-1)^{k-1}\big[\psi^{(k-1)}(u)-\psi^{(k-1)}(v)\big]}{u-v} \\ &= \frac{(-1)^{k-1}}{u-v}\int_v^u \psi^{(k)}(t)\,\mathrm{d}t \le \frac{(k-1)!}{[\min\{v,(u+v-1)/2\}]^k} \end{aligned} \tag{2.2}$$

for $u > v > 0$.

For $0 < v < 1$ and $\alpha < 0$, since the function $f_\alpha(v) = v^\alpha - v - 1$ satisfying

$$\lim_{v\to 1^+} f_\alpha(v) = -1 \quad \text{and} \quad \lim_{v\to 0^+} f_\alpha(v) = \infty, \tag{2.3}$$

the equation $v^\alpha - v - 1 = 0$ must have at least one root v_0 less than 1. Letting $u = v^\alpha > 1 > v$ and taking limit $v \to v_0$ in (2.2) leads to (2.13). Hence, identity (1.7) is proved for $0 < v < 1$ and $\alpha < 0$. ∎

Proof of Corollary 1. Substituting $u = v^2$ for $v > 1$ in (2.2) yields

$$\begin{aligned} \frac{(k-1)!(v^2-v)}{[\max\{v,(v^2+v-1)/2\}]^k} &\le (-1)^{k-1}\big[\psi^{(k-1)}(v^2)-\psi^{(k-1)}(v)\big] \\ &\le \frac{(k-1)!(v^2-v)}{[\min\{v,(v^2+v-1)/2\}]^k}. \end{aligned} \tag{2.4}$$

Since equation $v^2 - v - 1 = 0$ has a unique root $\frac{\sqrt{5}+1}{2}$ greater than 1, then, if $1 < v \le \frac{\sqrt{5}+1}{2}$,

$$(k-1)!\left(\frac{1}{v^{k-2}} - \frac{1}{v^{k-1}}\right) \le (-1)^{k-1}\big[\psi^{(k-1)}(v^2)-\psi^{(k-1)}(v)\big] \le \frac{(k-1)!2^k v(v-1)}{(v^2+v-1)^k}; \tag{2.5}$$

if $v \ge \frac{\sqrt{5}+1}{2}$, the above inequality reverses. Taking $v \to \frac{\sqrt{5}+1}{2}$ in (2.4) or (2.5) yields identity (1.8).

It is easy to see that equation $v^3 - v - 1 = 0$ has a unique real root

$$\sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} = \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}} = 1.324\cdots. \tag{2.6}$$

Substituting $u = v^3$ for $v > 1$ in (2.2) yields

$$\begin{aligned} \frac{(k-1)!(v^3-v)}{[\max\{v,(v^3+v-1)/2\}]^k} &\le (-1)^{k-1}\big[\psi^{(k-1)}(v^3)-\psi^{(k-1)}(v)\big] \\ &\le \frac{(k-1)!(v^3-v)}{[\min\{v,(v^3+v-1)/2\}]^k}. \end{aligned} \tag{2.7}$$

If $1 < v \le \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}$,

$$(k-1)!\left(\frac{1}{v^{k-3}} - \frac{1}{v^{k-1}}\right) \le (-1)^{k-1}\big[\psi^{(k-1)}(v^3)-\psi^{(k-1)}(v)\big] \le \frac{(k-1)!2^k v(v^2-1)}{(v^3+v-1)^k}; \tag{2.8}$$

if $v \ge \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}$, the above inequality reverses. Identity (1.9) follows from taking $v \to \frac{\sqrt[3]{9-\sqrt{69}} + \sqrt[3]{9+\sqrt{69}}}{\sqrt[3]{2}\,\sqrt[3]{9}}$ in (2.7) or (2.8).

It is not difficult to see that the quartic equation $v^4 - v - 1 = 0$ has a unique real root

$$\frac{1}{2}\sqrt{4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\,\right)}} - \sqrt[3]{\frac{9+\sqrt{849}}{18}} + \frac{2}{\sqrt{\sqrt[3]{\frac{9+\sqrt{849}}{18}} - 4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\,\right)}}}}} + \frac{1}{2}\sqrt{\sqrt[3]{\frac{9+\sqrt{849}}{18}} - 4\sqrt[3]{\frac{2}{3\left(9+\sqrt{849}\,\right)}}} = 1.220\cdots \tag{2.9}$$

Replacing u by v^4 for $v > 1$ in (2.2) gives

$$\begin{aligned}\frac{(k-1)!(v^4-v)}{[\max\{v,(v^4+v-1)/2\}]^k} &\le (-1)^{k-1}\big[\psi^{(k-1)}(v^4)-\psi^{(k-1)}(v)\big]\\ &\le \frac{(k-1)!(v^4-v)}{[\min\{v,(v^4+v-1)/2\}]^k}.\end{aligned} \tag{2.10}$$

If $1 < v \le \frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}}\, + \frac{1}{2}\sqrt{b-a}$, then

$$\begin{aligned}(k-1)!\left(\frac{1}{v^{k-4}}-\frac{1}{v^{k-1}}\right) &\le (-1)^{k-1}\big[\psi^{(k-1)}(v^4)-\psi^{(k-1)}(v)\big]\\ &\le \frac{(k-1)!2^k v(v^3-1)}{(v^4+v-1)^k};\end{aligned} \tag{2.11}$$

if $v \ge \frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}}\, + \frac{1}{2}\sqrt{b-a}$, the above inequality reverses. Identity (1.10) follows from taking $v \to \frac{1}{2}\sqrt{a-b+\frac{2}{\sqrt{b-a}}}\, + \frac{1}{2}\sqrt{b-a}$ in (2.10) or (2.11).

For $v > 1$ and $\alpha > 1$, since the function $f_\alpha(v) = v^\alpha - v - 1$ satisfying

$$\lim_{v\to 1^+} f_\alpha(v) = -1 \quad \text{and} \quad \lim_{v\to\infty} f_\alpha(v) = \infty, \tag{2.12}$$

the equation $v^\alpha - v - 1 = 0$ must have at least one root v_0 greater than 1. Letting $u = v^\alpha > v > 1$ and taking limit $v \to v_0$ in (2.2) leads to

$$\psi^{(k-1)}(v_0^\alpha) - \psi^{(k-1)}(v_0) = \frac{(-1)^{k-1}(k-1)!}{v_0^k}. \tag{2.13}$$

Identity (1.7) is proved for $v > 1$ and $\alpha > 1$. ∎

REFERENCES

[1] M. ABRAMOWITZ and I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, with corrections, Washington, 1965.

[2] R. D. ATANASSOV and U. V. TSOUKROVSKI, Some properties of a class of logarithmically completely monotonic functions, *C. R. Acad. Bulgare Sci.*, **41** (1988), No. 2, pp. 21–23.

[3] C. BERG, Integral representation of some functions related to the gamma function, *Mediterr. J. Math.*, **1** (2004), No. 4, pp. 433–439.

[4] A. Z. GRINSHPAN and M. E. H. ISMAIL, Completely monotonic functions involving the gamma and q-gamma functions, *Proc. Amer. Math. Soc.*, **134** (2006), pp. 1153–1160.

[5] B.-N. GUO and F. QI, Properties and applications of a function involving exponential functions, *Commun. Pure Appl. Anal.*, **8** (2009), No. 4, pp. 1231–1249.

[6] F. QI, A double inequality for divided differences and some identities of psi and polygamma functions, *RGMIA Res. Rep. Coll.*, **10** (2007), No. 3, Art. 6. [Online: `http://www.staff.vu.edu.au/rgmia/v10n3.asp`]

[7] F. QI, A new lower bound in the second Kershaw's double inequality, *J. Comput. Appl. Math.*, **214** (2008), No. 2, pp. 610–616. [Online: `http://dx.doi.org/10.1016/j.cam.2007.03.016`]

[8] F. QI, A new lower bound in the second Kershaw's double inequality, *RGMIA Res. Rep. Coll.*, **10** (2007), No. 1, Art. 9. [Online: `http://www.staff.vu.edu.au/rgmia/v10n1.asp`]

[9] F. QI, Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions, *Nonlinear Funct. Anal. Appl.*, **12** (2007), No. 4, pp. 675–685.

[10] F. QI, Certain logarithmically N-alternating monotonic functions involving gamma and q-gamma functions, *RGMIA Res. Rep. Coll.*, **8** (2005), No. 3, Art. 5, pp. 413–422. [Online: `http://www.staff.vu.edu.au/rgmia/v8n3.asp`]

[11] F. QI, Monotonicity and logarithmic convexity for a class of elementary functions involving the exponential function, *RGMIA Res. Rep. Coll.*, **9** (2006), No. 3, Art. 3. [Online: `http://www.staff.vu.edu.au/rgmia/v9n3.asp`]

[12] F. QI, Three-log-convexity for a class of elementary functions involving exponential function, *J. Math. Anal. Approx. Theory*, **1** (2006), No. 2, pp. 100–103.

[13] F. QI and CH.-P. CHEN, A complete monotonicity property of the gamma function, *J. Math. Anal. Appl.*, **296** (2004), No. 2, pp. 603–607.

[14] F. QI and B.-N. GUO, Complete monotonicities of functions involving the gamma and digamma functions, *RGMIA Res. Rep. Coll.*, **7** (2004), No. 1, Art. 8, pp. 63–72. [Online: `http://www.staff.vu.edu.au/rgmia/v7n1.asp`]

[15] F. QI and B.-N. GUO, Wendel-Gautschi-Kershaw's inequalities and sufficient and necessary conditions that a class of functions involving ratio of gamma functions are logarithmically completely monotonic, *RGMIA Res. Rep. Coll.*, **10** (2007), No. 1, Art. 2. [Online: `http://www.staff.vu.edu.au/rgmia/v10n1.asp`]

[16] F. QI and B.-N. GUO, Wendel's and Gautschi's inequalities: Refinements, extensions, and a class of logarithmically completely monotonic functions, *Appl. Math. Comput.*, **205** (2008), No. 1, pp. 281–290. [Online: `http://dx.doi.org/10.1016/j.amc.2008.07.005`]

[17] F. QI, B.-N. GUO and CH.-P. CHEN, Some completely monotonic functions involving the gamma and polygamma functions, *J. Aust. Math. Soc.*, **80** (2006), pp. 81–88.

[18] F. QI, B.-N. GUO and CH.-P. CHEN, Some completely monotonic functions involving the gamma and polygamma functions, *RGMIA Res. Rep. Coll.*, **7** (2004), No. 1, Art. 5, pp. 31–36. [Online: `http://www.staff.vu.edu.au/rgmia/v7n1.asp`]