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ON ε-SIMULTANEOUS APPROXIMATION IN QUOTIENT SPACES

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ABSTRACT. The purpose of this paper is to develop a theory of best simultaneous approximation to ε-simultaneous approximation. We shall introduce the concept of ε-simultaneous pseudo Chebyshev, ε-simultaneous quasi Chebyshev and ε-simultaneous weakly Chebyshev subspaces of a Banach space. Then, it will be determined under what conditions these subspaces are transmitted to and from quotient spaces.

Key words and phrases: ε-simultaneous approximation, ε-simultaneous pseudo Chebyshev subspace, ε-simultaneous weakly Chebyshev subspace.

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1. INTRODUCTION

The theory of best simultaneous approximation has been studied by many authors (for example, [1]-[8], [10]-[12]). The concept of ε-approximation has been studied by Singer [9]. In this paper, we introduce the concepts of ε-simultaneous pseudo Chebyshev, ε-simultaneous quasi Chebyshev and ε-simultaneous weakly Chebyshev subspaces of a Banach space. Then, it will be determined under what conditions these subspaces are transmitted to quotient spaces.

let X be a normed linear space, W a subset of X and S a bounded set in X. we define

$$d(S,W) = \inf_{w\in W}\sup_{s\in S}\|s-w\|.$$

An element $w_0 \in W$ is called a best simultaneous approximation to S from W whenever $d(S,W) = \sup_{s\in S}\|s-w_0\|$. The set of all best simultaneous approximation to S from W will be denoted by $S_W(S)$. In the case $S=\{x\}$ ($x\in X$), $S_W(S)$ is the set of all best approximations of x in W, $P_W(x)$. Thus, simultaneous approximation is generalization of best approximation in a sense.

Definition 1.1. *let X be a normed linear space, W a subset of X and S a bounded set in X. An element $w_0 \in W$ is called ε-simultaneous best approximation to S from W if*

$$\sup_{s\in S}\|s-w_0\| \le d(S,W)+\varepsilon.$$

The set of all ε-simultaneous best approximations to S from W will be denoted by $S_{W,\varepsilon}(S)$.

It is easy to see that $S_{W,\varepsilon}(S)$ is a non-empty, convex and bounded subset of W. In fact if $s\in S$, $0<\lambda<1$ and $w_1, w_2 \in S_{W,\varepsilon}(S)$, then

$$\|s-\lambda w_1-(1-\lambda)w_2\| \le \lambda\|s-w_1\| + (1-\lambda)\|s-w_2\| \le d(S,W)+\varepsilon$$

for all $s\in S$. Hence,

$$\sup_{s\in S}\|s-\lambda w_1-(1-\lambda)w_2\| \le d(S,W)+\varepsilon.$$

So, $\lambda w_1+(1-\lambda)w_2 \in S_{W,\varepsilon}(S)$. Also, $S_{W,\varepsilon}(S)$ is closed whenever W so is.

We recall that for an arbitrary non-empty convex set A in X the linear manifold spanned by A which is denoted by $\ell(A)$ is defined as follows

$$\ell(A) := \{\alpha x + (1-\alpha)y : x, y \in A; \alpha \text{ is a scalar}\}.$$

For every fixed $y\in A$ the set $\ell(A-y)$ is a linear subspace of X satisfying

$$\ell(A-y) = \ell(A) - y := \{x-y : x \in \ell(A)\}.$$

It is clear that for an arbitrary non-empty convex set A in X

$$\ell(\pi(A)) = \pi(\ell(A)),$$

where π is the canonical map in the correspondence quotient space. The dimension of A is defined by

$$dim A := dim \ell(A).$$

Then, for every $y\in A$ we have

$$dim A = dim\ell(A) = dim[\ell(A)-y] = dim\ell(A-y) = dim(A-y).$$

For more details see [9].

Definition 1.2. *let X be a normed linear space, W a subspace of X and S a bounded set in X. Then, W is called*
(i) ε-simultaneous pseudo Chebyshev subspace if $S_{W,\varepsilon}(S)$ is finite dimensional subset of W for all bounded subset S in X.
(ii) ε-simultaneous quasi Chebyshev subspace if $S_{W,\varepsilon}(S)$ is compact subset of W for all bounded subset S in X.
(iii) ε-simultaneous weakly Chebyshev subspace if $S_{W,\varepsilon}(S)$ is weakly compact subset of W for all bounded subset S in X.

We shall use the following Lemmas throughout this paper.

Lemma 1.1. [2] *Let X be a normed linear space and M a proximinal subspace of X. Then, for each non-empty bounded set S in X we have*

$$d(S, M) = \sup_{s\in S} \inf_{m\in M} \|s - m\|.$$

Lemma 1.2. [2] *Let X be a normed linear space, M a proximinal subspace of X and S an arbitrary subset of X. Then, S is a bounded subset of X if and only if S/M is a bounded subset of X/M.*

Lemma 1.3. *Let W be a proximinal subspaces of normed space X, M a proximinal subspace of X and $M \subseteq W$. Then, for each non-empty bounded set S with $M \subseteq S \subseteq X$ we have*

$$d(S/M, W/M) = d(S, W).$$

Proof. It is easy to see that $d(S/M, W/M) \le d(S, W)$. Fix $w \in W$. Then, $\sup_{s\in S} \|s - w + M\| \ge \|s - w + M\|$ for all $s \in S$. Since M is proximinal, there exists $m_{s,w} \in M$ such that

$$\|s - w + M\| = \|s - w - m_{s,w}\| \ge \inf_{w'\in W} \|s - w'\|.$$

Thus, $\sup_{s\in S} \|s - w + M\| \ge \inf_{w'\in W} \|s - w'\|$ for all $s \in S$. Hence by Lemma 1.1,

$$\sup_{s\in S} \|s - w + M\| \ge \sup_{s\in S} \inf_{w'\in W} \|s - w'\| = \inf_{w'\in W} \sup_{s\in S} \|s - w'\| = d(S, W),$$

for all $w \in W$. Therefore,

$$d(S/M, W/M) = \inf_{w\in W} \sup_{s\in S} \|s - w + M\| \ge d(S, W).$$

∎

Lemma 1.4. *Let W be a proximinal subspaces of normed space X, M a proximinal subspace of X, S a bounded set in X, $M \subseteq W$ and $\varepsilon > 0$. Then,*

$$\pi(S_{W,\varepsilon}(S)) \subseteq S_{W/M,\varepsilon}(S/M),$$

where $\pi : X \to X/M$ is the canonical map.

Proof. If $w_0 \in S_{W,\varepsilon}(S)$, then by Lemma 1.3

$$\sup_{s\in S} \|s - w_0 + M\| \le \sup_{s\in S} \|s - w_0\| \le d(S, W) + \varepsilon = d(S/M, W/M) + \varepsilon.$$

So, $w_0 + M \in S_{S/M,\varepsilon}(S/M)$. ∎

Lemma 1.5. *Let W be a proximinal subspaces of normed space X, M a proximinal subspace of X, S a bounded set in X, $M \subseteq W$ and $\varepsilon > 0$. If $w_0 + M \in S_{W/M,\varepsilon}(S/M)$ and $m_0 \in S_M(S - w_0)$, then $w_0 + m_0 \in S_{W,\varepsilon}(S)$.*

Proof. By lemmas 1.1 and 1.3, we have

$$\sup_{s\in S}\|s-w_0-m_0\| = \inf_{m\in M}\sup_{s\in S}\|s-w_0-m\| = \sup_{s\in S}\inf_{m\in M}\|s-w_0-m\|$$
$$= \sup_{s\in S}\|s-w_0+M\| \le d(S/M, W/M)+\varepsilon = d(S,W)+\varepsilon.$$

So, $w_0+m_0 \in S_{W,\varepsilon}(S)$. ∎

Corollary 1.6. *Let W be a proximinal subspaces of normed space X, M a simultaneous proximinal subspace of X, S a bounded set in X, $M \subseteq W$ and $\varepsilon > 0$. Then,*

$$\pi(S_{W,\varepsilon}(S)) = S_{W/M,\varepsilon}(S/M),$$

where $\pi : X \to X/M$ is the canonical map.

Proof. By Lemma 1.4, we have

$$\pi(S_{W,\varepsilon}(S)) \subseteq S_{W/M,\varepsilon}(S/M).$$

Now, suppose that $w_0+M \in S_{W/M,\varepsilon}(S/M)$. Since M is simultaneous proximinal, there exists $m_0 \in M$ such that $m_0 \in S_M(S-w_0)$. Now by Lemma 1.5, $w_0+m_0 \in S_{W,\varepsilon}(S)$. So, $w_0+M \in \pi(S_{W,\varepsilon}(S))$. ∎

2. MAIN RESULTS

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let $\varepsilon > 0$ be given, M and W subspaces of normed space X such that M is finite dimensional and simultaneous proximinal and W is proximinal. Then the following are equivalent.*
(i) W/M is ε-simultaneous pseudo Chebyshev subspaces of X/M.
(ii) $W+M$ is ε-simultaneous pseudo Chebyshev subspaces of X.

Proof. $(i) \Rightarrow (ii)$ Let S be an arbitrary bounded set in X and k_0 be an element of $S_{W+M,\varepsilon}(S)$. Then, by using Lemma 1.2 we have

$$\pi(\ell(S_{W+M,\varepsilon}(S)-k_0)) = \ell(\pi(S_{W+M,\varepsilon}(S)-k_0))$$
$$= \ell(S_{W/M,\varepsilon}(S/M)-(k_0+M)).$$

Since W/M is a ε-simultaneous pseudo Chebyshev subspaces of X/M, so

$$dim[\ell(S_{W/M,\varepsilon}(S/M)-(k_0+M))] < \infty.$$

Hence,

$$dim[\pi(\ell(S_{W+M,\varepsilon}(S)-k_0)] < \infty.$$

Since M is finite dimensional, so

$$dim[\ell(S_{W+M,\varepsilon}(S)-k_0)] < \infty.$$

Therefore, $W+M$ is ε-simultaneous pseudo Chebyshev subspace of X.
$(ii) \Rightarrow (i)$ Let S be an arbitrary bounded set in X. Since $W+M$ is ε-simultaneous pseudo Chebyshev subspaces of X, $S_{W+M,\varepsilon}(S)$ is finite dimensional. But $(W+M)/M = W/M$, so we have

$$dim[S_{W/M,\varepsilon}(S/M)] = dim[\ell(S_{W/M,\varepsilon}(S/M))] = dim[\ell(S_{(W+M)/M,\varepsilon}(S/M))]$$
$$= \dim[\ell(\pi(S_{W+M,\varepsilon}(S))] = \dim[\pi(\ell(S_{W+M,\varepsilon}(S))] < \infty.$$

Thus, W/M is ε-simultaneous pseudo Chebyshev subspace of X/M. ∎

Corollary 2.2. *Let $\varepsilon > 0$ be given, M and W subspaces of normed space X such that M is finite dimensional and simultaneous proximinal, W is proximinal and $M \subseteq W$. Then the following are equivalent.*
(i) W/M is ε-simultaneous pseudo Chebyshev subspaces of X/M.
(ii) W is ε-simultaneous pseudo Chebyshev subspaces of X.

Theorem 2.3. *Let $\varepsilon > 0$ be given, M and W subspaces of normed space X such that M is finite dimensional and simultaneous proximinal and W is proximinal. Then the following are equivalent.*
(i) W/M is ε-simultaneous quasi Chebyshev subspaces of X/M.
(ii) $W + M$ is ε-simultaneous quasi Chebyshev subspaces of X.

Proof. $(i) \Rightarrow (ii)$ Let S be a bounded set in X and $\{\ell_n\}$ a sequence in $S_{W+M,\varepsilon}(S)$. Then by Lemma 1.3, for each $n \geq 1$ we have

$$\sup_{s\in S} \|s - \ell_n + M\| \leq \sup_{s\in S} \|s - \ell_n\|$$
$$\leq d(S, W + M) + \varepsilon = d(S/M, (W + M)/M) + \varepsilon.$$

Hence $\ell_n + M \in S_{(W+M/M),\varepsilon}(S/M)$. Since $S_{(W+M)/M,\varepsilon}(S/M)$ is compact, there exist $\ell_0 \in W+M$ and a subsequence $\{\ell_{n_k}+M\}_{k\geq 1}$ of $\{\ell_n+M\}_{n\geq 1}$ such that $\ell_0+M \in S_{(W+M)/M,\varepsilon}(S/M)$ and $\{\ell_{n_k} + M\}_{k\geq 1}$ converges to $\ell_0 + M$. But, for all $k \geq 1$ we have

$$\|\ell_0 - \ell_{n_k} + M\| = \inf_{m\in M} \|\ell_0 - \ell_{n_k} - m\| = d(\ell_0 - \ell_{n_k}, M).$$

Since M is proximinal in X, there exists $m_{n_k} \in P_M(\ell_0 - \ell_{n_k})$. Hence,

$$\|\ell_0 - \ell_{n_k} - m_{n_k}\| = d(\ell_0 - \ell_{n_k}, M).$$

Therefore,

$$\lim_{k\to\infty} \|\ell_0 - \ell_{n_k} - m_{n_k}\| = 0. \tag{2.1}$$

Since $\ell_n \in S_{W+M,\varepsilon}(S)$ for all $n \geq 1$, $\{\ell_{n_k}\}_{k\geq 1}$ is a bounded sequence. Hence by (2.1), $\{m_{n_k}\}$ is a bounded sequence in M. Since M is a finite dimensional subspace of X, without loss of generality we can assume that $\{m_{n_k}\}_{k=1}^{\infty}$ converges to an element $m_0 \in M$. Let $\ell' = \ell_0 - m_0$. Thus, $\ell' \in W + M$ and we have

$$\|\ell' - \ell_{n_k}\| = \|\ell_0 - m_0 - \ell_{n_k}\| \leq \|\ell_0 - \ell_{n_k} - m_{n_k}\| + \|m_{n_k} - m_0\|, \forall k \geq 1.$$

Thus

$$\lim_{k\to\infty} \|\ell' - \ell_{n_k}\| = 0.$$

Since $\ell_{n_k} \in S_{W+M,\varepsilon}(S)$ for all $k \geq 1$ and $S_{W+M,\varepsilon}(S)$ is closed, ℓ' is an element of $S_{W+M,\varepsilon}(S)$. Therefore, $S_{W+M,\varepsilon}(S)$ is compact.
$(ii) \Rightarrow (i)$ Let S be an arbitrary bounded set in X. Then, $S_{W+M,\varepsilon}(S)$ is compact. But the canonic map is continuous, so $\pi(S_{W+M,\varepsilon}(S))$ is compact. Thus by Corollary 1.6,

$$\pi(S_{W+M,\varepsilon}(S)) = S_{(W+M)/M,\varepsilon}(S/M) = S_{W/M,\varepsilon}(S/M).$$

Therefore, W/M is ε-simultaneous quasi Chebyshev subspaces of X/M. ∎

Corollary 2.4. *Let $\varepsilon > 0$ be given, M and W subspaces of normed space X such that M is finite dimensional and simultaneous proximinal, W is proximinal and $M \subseteq W$. Then the following are equivalent.*
(i) W/M is ε-simultaneous quasi Chebyshev subspaces of X/M.
(ii) W is ε-simultaneous quasi Chebyshev subspaces of X.

Theorem 2.5. *Let $\varepsilon > 0$ be given, M and W subspaces of normed space X such that M is finite dimensional and simultaneous proximinal and W is proximinal. Then the following are equivalent.*
(i) W/M is ε-simultaneous weakly Chebyshev subspaces of X/M.
(ii) $W + M$ is ε-simultaneous weakly Chebyshev subspaces of X.

Proof. $(i) \Rightarrow (ii)$ Let S be a bounded set in X and $\{\ell_n\}$ a sequence in $S_{W+M,\varepsilon}(S)$. Then by Lemma 1.3, $\{\ell_n + M\}$ is a sequence in

$$S_{(W+M)/M,\varepsilon}(S/M) = S_{W/M,\varepsilon}(S/M).$$

Since $S_{W/M,\varepsilon}(S/M)$ is weakly compact, there exists a subsequence $\{\ell_{n_k} + M\}_{k=1}^{\infty}$ of $\{\ell_n + M\}_{n=1}^{\infty}$ such that $\{\ell_{n_k} + M\}_{k=1}^{\infty}$ converges weakly to an element $\ell_0 + M \in S_{W/M,\varepsilon}(S/M)$. Then, $\ell_0 + m_0 \in S_{W+M,\varepsilon}(S/M)$ for some $m_0 \in M$. But since M is proximinal, $T_f \in (X/M)^*$ for all $f \in X^*$. Therefore,

$$f(\ell_{n_k}) = T_f(\ell_{n_k} + M) \to T_f(\ell_0 + M) = T_f(\ell_0 + m_0 + M) = f(\ell_0 + m_0).$$

Hence,$\{\ell_{n_k}\}_{k\geq 1}$ converges weakly to $\ell_0 + m_0 \in S_{W+M,\varepsilon}(S)$. Thus, $S_{W+M,\varepsilon}(S)$ is weakly compact and hence $W + M$ is ε-simultaneous weakly Chebyshev subspaces of X.

$(i) \Rightarrow (ii)$ Let S be an arbitrary bounded set in X and $\{w_n + M\}$ a sequence in $S_{W/M,\varepsilon}(S/M)$. Since M is proximinal and $S - w_n$ is a bounded set in X for all $n \geq 1$, there exists $m_n \in S_M(S - w_n)$ for all $n \geq 1$. But by Lemma 1.1, we have

$$\begin{aligned}\sup_{s\in S} \|s - w_n - m_n\| &= \inf_{m\in M} \sup_{s\in S} \|s - w_n - m\| = \sup_{s\in S} \inf_{m\in M} \|s - w_n - m\| \\ &= \sup_{s\in S} \|s - w_n + M\| \leq d(S/M, W/M) + \varepsilon \leq d(S, W + M) + \varepsilon.\end{aligned}$$

Therefore, $\{w_n + m_n\}$ is a sequence in $S_{W+M,\varepsilon}(S)$. Since $S_{W+M,\varepsilon}(S)$ is weakly compact, there exists a subsequence $\{w_{n_k} + m_{n_k}\}_{k=1}^{\infty}$ of $\{w_n + m_n\}_{n=1}^{\infty}$ such that $\{w_{n_k} + m_{n_k}\}_{k=1}^{\infty}$ converges weakly to an element $\ell_0 \in S_{W+M,\varepsilon}(S)$. By Corollary 1.6, $\ell_0 + M$ is an element of $S_{(W+M)/M,\varepsilon}(S/M) = S_{W/M,\varepsilon}(S/M)$. Note that for every $f \in (X/M)^*$, we have

$$f(w_{n_k} + M) = f(w_{n_k} + m_{n_k} + M) = f \circ \pi(w_{n_k} + m_{n_k}) \to f \circ \pi(\ell_0) = f(\ell_0 + M).$$

It follows that $\{w_{n_k} + M\}_{k=1}^{\infty}$ converges weakly to $w_0 + M$. Hence, $S_{W/M,\varepsilon}(S/M)$ is weakly compact subset of X/M. Therefore, W/M is ε-simultaneous weakly Chebyshev subspaces of X/M. ∎

Corollary 2.6. *Let $\varepsilon > 0$ be given, M and W subspaces of normed space X such that M is finite dimensional and simultaneous proximinal, W is proximinal and $M \subseteq W$. Then the following are equivalent.*
(i) W/M is ε-simultaneous weakly Chebyshev subspaces of X/M.
(ii) W is ε-simultaneous weakly Chebyshev subspaces of X.

REFERENCES

[1] A. S. HOLLAND, B. N. SHANEY, J. TZIMBALAEIO, On best simultaneous approximation, *J. Approx. Theory*, **17**(1976), pp. 187–188.

[2] M. IRANMANESH, H. MOHEBI, On best simultaneous approximation in quotient spaces, *Analysis in Theory and Applications*, **Vol. 23**, No. 1 (2007), pp. 35–49.

[3] C. LI, G. A. WATSON, On best simultaneous approximation, *J. Approx. Theory*, **91** (1997), pp. 332–348.

[4] K. LIM, Simultaneous approximation of compact sets by elements of convex sets in normed linear spaces, *J. Approx. Theory*, **12** (1974), pp. 332–351.

[5] H. MOHEBI, Downward sets and their best simultaneous approximation properties with applications, *Numerical Functional Analysis and Optimization*, **Vol. 25**, No. 7-8 (2004), pp. 685–705.

[6] T. D. NARANG, On best simultaneous approximation, *J. Approx. Theory*, **39** (1983), pp. 93–96.

[7] G. M. PHILLIPS, J. H. MC CABE, On best simultaneous Chebyshev approximation, *J. Approx. Theory*, **27** (1979), pp. 93–98.

[8] B. N. SAHNEY, S. P. SINGH, On best simultaneous approximation in Banach spaces, *J. Approx. Theory*, **35** (1982), pp. 222–224.

[9] I. SINGER, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, New York, 1970.

[10] S. TANIMOTO, A characterization of best simultaneous approximations, *J. Approx. Theory*, **59** (1989), pp. 359–361.

[11] S. TANIMOTO, On best simultaneous approximation, *Math. Japonica*, **48** (1998), pp. 275–279.

[12] G. A. WATSON, A characterization of best simultaneous approximations, *J. Approx. Theory*, **75** (1993), pp. 175–182.