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# A METHOD FOR SOLVING SYSTEMS OF NONLINEAR EQUATIONS

JAVAD SHOKRI

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DEPARTMENT OF MATHEMATICS, URMIA UNIVERSITY, P.O.BOX 165, URMIA, IRAN
j.shokri@urmia.ac.ir

ABSTRACT. In this paper, we suggest and analyze a new two-step iterative method for solving nonlinear equation systems using the combination of midpoint quadrature rule and Trapezoidal quadrature rule. We prove that this method has quadratic convergence. Several examples are given to illustrate the efficiency of the proposed method.

*Key words and phrases:* Systems of nonlinear equations, Fixed point iteration, Newton's method.

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## 1. INTRODUCTION

In this paper we consider the problem of finding a real zero of a function $F : \Omega \subseteq \Re^n \to \Re^n$, that is, a real solution $\alpha \in \Re^n$, of the nonlinear equation system $F(x) = 0$, of n equations with $n$ unknown variables. This solution can be obtained as a fixed point of some function $G : \Re^n \to \Re^n$ by means of the fixed point iteration method

$$x_{k+1} = G(x_k), \quad k = 0, 1, \dots,$$

where $x_0$ is the initial estimation. The best known fixed point method is the classical Newton's method, given by

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \dots,$$

where $F'(x_k)$ is the Jaccobian Matrix of the function $F$ evaluated in $x_k$.

**Definition 1.1.** *Let $\{x_k\}_{k\geqslant 0}$ be a sequence in $\Re^n$ convergent to $\alpha$. If there exists $M$, $0 < M < 1$, $p = 1, 2, 3, \cdots$ and $k_0$ such that*

$$\|x_{k+1} - \alpha\| \leqslant \|x_k - \alpha\|^p, \quad k \geqslant k_0.$$

*Then, convergence is called (a) linear, if $p = 1$. (b) quadratic, if $p = 2$. (c) cubic, if $p = 3$, and so on.*

In practice, because of $\alpha$ is unknown, we analyze for each $p$ the behavior of the quotients

$$T_p = \frac{\|x_{k+1} - x_k\|}{\|x_k - x_{k-1}\|}, \quad k = 1, 2, 3, \cdots,$$

where $p = 1, 2, 3, \cdots$ , which are called convergence rate or convergence order.

If $p = 1$ and the convergence rate to $C_L$, $0 < C_L < 1$, is said that the sequence $\{x_k\}_{k\geqslant 0}$ has linear convergence to $\alpha$.

If $p = 2$ and there exists a $C_C$, $C_C > 0$, such that the convergence rate eventually tends to $C_C$, the sequence $\{x_k\}_{k\geqslant 0}$ is said to be quadratically convergence to $\alpha$. Similarly if $p = 3$, convergence order is said cubic($C_{Cu}$) and so on.

In suggested method, we have an adjustment on the classic Newton's method in order to accelerate the convergence or to reduce the number of operations and evaluations in each step of the iterative process. This method is based on the method which introduced by Noor [1] to solve nonlinear equation $f(x) = 0$. We suggest and analyze this iterative method which is obtained by using the combination of midpoint quadrature rule and Trapezoidal quadrature rule to solve systems of nonlinear equations. By using of numerical results, we show that the proposed method usually has high convergence order with respect to classical Newton's method.

This method is an implicit-type method. To implement this, we use Newton's method as predictor method and then use suggested method as corrector method. Several examples are given to illustrate the efficiency and advantage of this two-steep method. In Section 2, we describe the iterative method from [1] to solve system $F(x) = 0$. In Section 3, the quadratic convergence of this method has been proved. The proposed algorithm is illustrated by some examples in Section 4, and conclusion is in Section 5.

## 2. DESCRIPTION OF ITERATIVE METHOD

Consider the nonlinear equation $f(x) = 0$, we assume that $f(x)$ has a simple root $\alpha$ and is sufficiently differentiable function. Then

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{2.1}$$

By using the combination of midpoint quadrature rule and Trapezoidal rule, we have

$$\int_{x_n}^{x} f'(t)dt = \frac{x - x_n}{4}\Big[f'(x) + 2f'\Big(\frac{x_n + x}{2}\Big) + f'(x_n)\Big]. \tag{2.2}$$

From (2.1) and (2.2), one can write

$$f(x) = f(x_n) + \frac{x - x_n}{4}\Big[f'(x) + 2f'\Big(\frac{x_n + x}{2}\Big) + f'(x_n)\Big]. \tag{2.3}$$

Since $f(x) = 0$, so from (2.3), we have

$$0 = f(x_n) + \frac{x - x_n}{4}\Big[f'(x) + 2f'\Big(\frac{x_n + x}{2}\Big) + f'(x_n)\Big],$$

which produces the following iteration scheme

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f'(\frac{x_n+y_n}{2}) + f'(y_n)}, \tag{2.4}$$

where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In the next Section we extend this method to solve a system of nonlinear equations.

It's known that Newton's iterative method for the nonlinear system $F(x) = 0$ where $F : \Omega \subseteq \Re^n \to \Re^n$ is considered as

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),$$

where $F'(x_n)$ is the Jacobian matrix in point $x_n$.

Then we can rewrite Eq. (2.4) to solve $F(x) = 0$, as following iteration scheme:

$$x_{n+1} = x_n - 4\Big[F'(x_n) + 2F'\Big(\frac{x_n + y_n}{2}\Big) + F'(y_n)\Big]^{-1}F(x_n), \tag{2.5}$$

where $y_n = x_n - F'(x_n)^{-1}F(x_n)$. This method is called Midpoint – Trapezoidal Newton's method (MTN).

Two following technical lemmas are needed to solve convergence theorem, whose proof can be found in [2] or [4].

**Lemma 2.1.** *Let $F : \Re^n \to \Re^n$ be a differentiable function such that*

$$\|F'(u) - F'(v)\| \leqslant \|u - v\|$$

*for any $u, v \in \Omega$ convex set. Then there exists $\gamma > 0$ such that for any $x, y \in \Omega$,*

$$\|F(y) - F(x) - F'(x)(y - x)\| \leqslant \frac{\gamma}{2}\|x - y\|^2.$$

**Lemma 2.2.** *(Banach) Let $A \in L(\Re^n)$ be nonsingular. If $E \in L(\Re^n)$ and $\|A^{-1}\|.\|E\| \leqslant 1$, then $A + E$ is nonsingular and*

$$\|(A + E)^{-1}\| \leqslant \frac{\|A^{-1}\|}{1 - \|A^{-1}\|.\|E\|}.$$

The Ostrowiski's Theorem in the following, is needed to convergence theorem.

**Theorem 2.1.** *Let $G : \Re^n \to \Re^n$ is differentiable function in $\alpha$, that is a solution of the system $x = G(x)$. Let $\{x_{k+1}\}_{k\geq 0}$ be the sequence of iterates obtained by means of fixed point iteration, $x_{k+1} = G(x_k), k = 0, 1, \ldots$. If the spectral radius of $G'(\alpha)$ is lower than $1$, then $\{x_k\}_{k\geqslant 0}$ converges to $\alpha$.*

*Proof.* See the proof in [3]. ∎

## 3. CONVERGENCE ANALYSIS

At the first we prove the following lemma needed to convergence theorem.

**Lemma 3.1.** *Let $F : \Re^n \to \Re^n$ be a differentiable function in $\alpha$, where $\alpha$ is a solution of the system of nonsingular equations $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and $F'(\alpha)$is nonsingular. Then the functions*

$$G(x) = x - C(x)^{-1}F(x),$$

*where $C(x) = \frac{1}{4}[F'(x) + 2F'(z) + F'(y)]$, $y = x - F'(x)^{-1}F(x)$ and $z = \frac{x+y}{2}$, is well-defined in a neighborhood of $\alpha$, is differentiable and*

$$G'(\alpha) = I - F'(\alpha)^{-1}F'(\alpha) = 0.$$

*Proof.* Firstly, let us prove that $C(x)$ is nonsingular for any $x$ in a neighborhood of $\alpha$. Let $\beta$ be $\beta = \|F'(\alpha)^{-1}\|$ and $\epsilon$ be such that $0 < \epsilon < (2\beta)^{-1}$ is satisfied. By continuity of $F'$ in $\alpha$ there exists a $\delta > 0$ such that $\|F'(x) - F'(\alpha)\| \leqslant \epsilon$ if $\|x - \alpha\| \leqslant \delta$.
Now by the convergence of classical Newton's method in [2] or [4], it can be assured that $\|y - \alpha\| \leqslant \delta$ and $\|z - \alpha\| \leqslant \delta$, then $\|F'(y) - F'(\alpha)\| \leqslant \epsilon$ and $\|F'(z) - F'(\alpha)\| \leqslant \epsilon$.
Then by using lemma (2.2), Banach's lemma, it is proved that $C(x)$ is nonsingular and

$$\begin{aligned}
\|C(x)^{-1}\| &= \|[\frac{1}{4}(F'(x) + 2F'(z) + F'(y))]^{-1}\| \\
&= 4\|[(F'(x) - F'(\alpha)) + 2(F'(z) - F'(\alpha)) + (F'(y) - F'(\alpha)) + 4F'(\alpha)]^{-1}\| \\
&\leqslant \frac{4 \times \frac{1}{4}\|F'(\alpha)^{-1}\|}{1 - \|(4F'(\alpha))^{-1}\|.\|(F'(x) - F'(\alpha)) + 2(F'(z) - F'(\alpha)) + (F'(y) - F'(\alpha))\|} \\
&\leqslant \frac{\beta}{1 - \frac{1}{4}\beta 4\epsilon} = \frac{\beta}{1 - \epsilon\beta} \leqslant 2\beta,
\end{aligned}$$

for $\|x - \alpha\| \leqslant \delta$. So, the function $G(x)$ is well-defined in the neighborhood of $\alpha$, $S = \{x : \|x - \alpha\| \leqslant \delta\}$.
Now, by differentiability of $F$ in $\alpha$, it can be assumed that $\delta$ is small enough to

$$\|F(x) - F(\alpha) - F'(\alpha)(x - \alpha)\| \leqslant \epsilon\|x - \alpha\|, \quad \forall x \in S.$$

Then, for any $x \in S$,

$$\begin{aligned}
&\|G(x) - G(\alpha) - (I - C(\alpha)^{-1}F'(\alpha))(x - \alpha)\| \\
&\quad = \|C(\alpha)^{-1}F'(\alpha)(x - \alpha) - C(x)^{-1}F(x)\| \\
&\quad \leqslant \|C(x)^{-1}(F(x) - F(\alpha) - F'(\alpha)(x - \alpha))\| \\
&\qquad + \|(C(x)^{-1}(C(x) - C(\alpha))(C(\alpha)^{-1}F'(\alpha)(x - \alpha)\| \\
&\quad \leqslant \|C(x)^{-1}\|.\|F(x) - F(\alpha) - F'(\alpha)(x - \alpha)\| \\
&\qquad + \|C(x)^{-1}\|.\|C(x) - C(\alpha)\|.\|x - \alpha\| \\
&\quad \leqslant (2\beta\epsilon + 2\beta\epsilon)\|x - \alpha\|,
\end{aligned}$$

As $\epsilon$ is arbitrary and $\beta$ is constant, then it can be concluded from the previous inequalities that $G$ is differentiable in $\alpha$, and also

$$G'(\alpha) = I - C(\alpha)^{-1}F'(\alpha) = I - F'(\alpha)^{-1}F'(\alpha) = 0.$$

∎

To complete of discussion, in the following, we bring the proof of the quadratic convergence of (MTN) method.

**Theorem 3.1.** *Let $F : \Re^n \to \Re^n$ be differentiable at each point of an open neighborhood $\Omega$ of $\alpha \in \Re$, that is a solution of the system $F(x) = 0$. Let us suppose that $F'(x)$ is continuous and nonsingular in $\alpha$. Then the sequence $\{x_k\}_{k\geqslant 0}$ obtained using the iterative expression (2.5) converges to $\alpha$ and*

$$\lim_{k\to\infty} \frac{\|x_{k+1} - \alpha\|}{\|x_k - \alpha\|} = 0.$$

*Moreover, if there exists $\gamma > 0$ such that*

$$\|F'(x) - F'(\alpha)\| \leqslant \gamma\|x - \alpha\|,$$

*for any $x$ in $D$, then there exists a constant $M > 0$ such that*

$$\|x_{k+1} - \alpha\| \leqslant M\|x_k - \alpha\|^2, \quad \forall k \geqslant k_0,$$

*where $k_0$ depends on the initial estimation $x_0$.*

*Proof.* From Lemma 3.1 we can assure that

$$G(x) = x - C(x)^{-1}F(x),$$

where $C(x)^{-1} = 4[F'(x)+2F'(z)+F'(y)]^{-1}$, $y = x - F'(x)^{-1}F(x)$, $z = \frac{x+y}{2}$, is well-defined in a neighborhood of $\alpha$, is differentiable in $\alpha$ and $G'(\alpha) = I - F'(\alpha)^{-1}F'(\alpha) = 0$, and also that $\|C(x)^{-1}\| < 2\beta$, where $\beta = \|F'(\alpha)^{-1}\|$.

If the sequence $\{x_k\}_{k\geqslant 0}$ is obtained by means of fixed point iteration on $G$, using Theorem 2.1 it can be concluded that $\{x_k\}_{k\geqslant 0}$ converges to $\alpha$. Moreover, as $G$ is differentiable in $\alpha$,

$$\lim_{k\to\infty} \frac{\|G(x_k) - G(\alpha) - G'(\alpha)(x_k - \alpha)\|}{\|x_k - \alpha\|} = 0,$$

but $G'(\alpha) = 0$, so this limit is equivalent to:

$$\lim_{k\to\infty} \frac{\|G(x_k) - G(\alpha)\|}{\|x_k - \alpha\|} = \lim_{k\to\infty} \frac{\|x_{k+1} - \alpha\|}{\|x_k - \alpha\|} = 0.$$

Now, if $\|F'(x) - F'(\alpha)\| \leqslant \gamma\|x - \alpha\|$ for any $x$ in a neighborhood of $\alpha$, an analogous reasoning to the one made in the proof of Lemma 2.2 allows us to assure that, for any $x$ in the neighborhood of $\alpha$ and from $C(x) = \frac{1}{4}[F'(x) + 2F'(z) + F'(y)]$, $C(\alpha) = F'(\alpha)$ in Lemma 3.1,

$$\|F(x)-F(\alpha)-C(\alpha)(x-\alpha)\| = \|F(x)-F(\alpha)-F'(\alpha)(x-\alpha)\| \leqslant \frac{1}{2}\gamma\|x-\alpha\|^2.$$

So, by the convergence of classical Newton's method $\|F'(y) - F'(\alpha)\| \leqslant \gamma\|x - \alpha\|$, $\|F'(z) - F'(\alpha)\| \leqslant \gamma\|x - \alpha\|$ and $\|F'(x) - F'(\alpha)\| \leqslant \gamma\|x - \alpha\|$ for any $x$ in the neighborhood of $\alpha$ , so is obtained that,

$$\begin{aligned}
\|C(x) - C(\alpha)\| &= \|\tfrac{1}{4}(F'(x) + 2F'(z) + F'(y)) - F'(\alpha)\| \\
&= \tfrac{1}{4}\|F'(x) + 2F'(z) + F'(y) - 4F'(\alpha)\| \\
&= \tfrac{1}{4}\|(F'(x) - F'(\alpha)) + 2(F'(z) - F'(\alpha)) + (F'(y) - F'(\alpha))\| \\
&\leqslant \tfrac{1}{4}(\gamma + 2\gamma + \gamma)\|x - \alpha\| = \gamma\|x - \alpha\|
\end{aligned}$$

then is concluded that

$$\begin{aligned}
&\|G(x) - G(\alpha)\| = \|x - C(x)^{-1}F(x) - \alpha\| \\
&\quad = \|C(x)^{-1}[F(x) - F(\alpha) - C(\alpha)(x - \alpha)] - C(x)^{-1}[C(x) - C(\alpha)](x - \alpha)\| \\
&\quad \leqslant \|C(x)^{-1}[F(x) - F(\alpha) - C(\alpha)(x - \alpha)]\| + \|C(x)^{-1}[C(x) - C(\alpha)](x - \alpha)\| \\
&\quad \leqslant \|C(x)^{-1}\|.\|F(x) - F(\alpha) - C(\alpha)(x - \alpha)\| + \|C(x)^{-1}\|.\|C(x) - C(\alpha)\|.\|x - \alpha\| \\
&\quad \leqslant \gamma\beta\|x - \alpha\|^2 + 2\gamma\beta\|x - \alpha\|^2 = 3\gamma\beta\|x - \alpha\|^2,
\end{aligned}$$

in a neighborhood of $\alpha$. Thus,

$$\|x_{k+1}-\alpha\| \leqslant M\|x_k-\alpha\|^2,$$

is satisfied with $M = 3\gamma\beta$ only if, for any initial approximation $x_0$, a $k_0$ is chosen such that $x_k$ remains in the neighborhood of $\alpha$ for any $k$ from $k_0$ on. ∎

## 4. NUMERICAL EXAMPLES

In this section we will check the effectiveness of iterative method (2.5). These example show the high order convergence of proposed method(MTN) respect to classical Newton's method(CN), numerically. All computations were done using **mathematica**, stopping criteria $\|x_{n+1}-x_n\| + \|F(x_n)\| < \epsilon$ was used for computer programs. We use $\epsilon < 10^{-14}$.

In the following examples, we have iterates converge to a limit of a solution of the system of nonlinear equations. For MTN and CN, we analyze the number of iterations needed to converge to the solution (k), the error estimation in the last step ($\|x_{k+1}-x_k\|$) and the order of convergence is deduced from the convergence rate

$$T_p = \frac{\|x_{k+1}-x_k\|}{\|x_k-x_{k-1}\|^p} \quad p = 1, 2, 3.$$

**Example 4.1.** *Consider the following system of nonlinear equations:*

$$\begin{aligned} &e^{x_1}e^{x_2} + x_1\cos(x_2) = 0,\\ &x_1 + x_2 - 1 = 0. \end{aligned}$$

The initial approximation of the solution is $x_0 = (1, 3)^T$. Table 1 shows the values of the solution.

**Example 4.2.** *Consider a second example as follows:*

$$\begin{aligned} &\ln(x_1^2) - 2\ln(\cos(x_2)) = 0,\\ &x_1\tan(\tfrac{x_1}{\sqrt{2}} + x_2) = \sqrt{2}. \end{aligned}$$

The initial approximation of the solution is $x_0 = (0.2, 0.2)^T$. Table 2 shows the iterative approximations of solutions.

**Example 4.3.** *A third example is as follows:*

$$\begin{aligned} &x_1^2 + x_2^2 = 1,\\ &x_1^2 - x_2^2 = \tfrac{1}{2}. \end{aligned}$$

The initial approximation of the solution is $x_0 = (0.2, 0.2)^T$. Table 3 shows the iterative approximations of solutions.

**Example 4.4.** *Consider the following system*

$$\begin{aligned} &x_1^2 + x_2^2 + x_3^2 = 9,\\ &x_1.x_2.x_3 - 1 = 0,\\ &x_1 + x_2 - x_3^2 = 0. \end{aligned}$$

Its initial approximation is $x_0 = (-2.5, 1, 1)^T$. Table 4 shows $x_i$s approximations.

**Example 4.5.** *The last example is taken as:*

$$\begin{aligned} x_2x_3 + x_4(x_2 + x_3) &= 0, \\ x_1x_3 + x_4(x_1 + x_3) &= 0, \\ x_1x_2 + x_4(x_1 + x_2) &= 0, \\ x_1x_2 + x_1x_3 + x_2x_3 &= 1. \end{aligned}$$

We solve this system by using initial approximation $x_0 = (0.5, 0.5, 0.5, 0.2)^T$. Table 5 shows the values of the solution.

**Table 1**
Approximations of $x_1$ and $x_2$ for example 4.1.

| $Method$ | $Approximated\ solution$ | $k$ | $Error$ | $O.C.$ |
|---|---|---|---|---|
| $MTN$ | $(-4.3816197548, 5.3816197548)$ | 6 | 0 | $C_{Cu}$ |
| $CN$ | $(-129.39710395, 130.3971039543)$ | 208 | $1.38 \times 10^{-16}$ | – |

**Table 2**
Approximations of $x_1$ and $x_2$ for example 4.2.

| $Method$ | $Approximated\ solution$ | $k$ | $Error$ | $O.C.$ |
|---|---|---|---|---|
| $MTN$ | $(0.9548041416, 0.3017961773)$ | 6 | 0 | $C_{Cu}$ |
| $CN$ | *no convergence* | – | – | – |

**Table 3**
Approximations of $x_1$ and $x_2$ for example 4.3.

| $Method$ | $Approximated\ solution$ | $k$ | $Error$ | $O.C.$ |
|---|---|---|---|---|
| $MTN$ | $(0.5, 0.8660254240)$ | 5 | $1.11 \times 10^{-16}$ | $C_{Cu}$ |
| $CN$ | $(0.5, 0.8660254240)$ | 8 | $1.11 \times 10^{-16}$ | $C_L$ |

**Table 4**
Approximations of $x_i$s for example 4.4.

| $Method$ | $Approximated\ solution$ | $k$ | $Error$ | $O.C.$ |
|---|---|---|---|---|
| $MTN$ | $(-2.09029464, 2.14025812, -0.22352512)$ | 5 | 0 | $C_{Cu}$ |
| $CN$ | $(-2.09029464, 2.14025812, -0.22352512)$ | 8 | $1.02 \times 10^{-16}$ | $C_L$ |

**Table 5**
Approximations of $x_i$s for example 4.5.

| $Method$ | $Approximated\ solution$ | $k$ | $Error$ | $O.C.$ |
|---|---|---|---|---|
| $MTN$ | $(0.57735020, 0.57735020, 0.57735020, -0.28867513)$ | 4 | 0 | $C_{Cu}$ |
| $CN$ | $(0.57735020, 0.57735020, 0.57735020, -0.28867513)$ | 5 | $1.66 \times 10^{-16}$ | $C_C$ |

## 5. CONCLUSION

In this paper, we suggested numerical solving method for nonlinear equation systems. This method is an implicit-type method. To implement this, we use Newton's method as predictor method and then use this method as corrector method. The method is discussed in detail. Several examples are given to illustrate the efficiency and advantage of this two-steep method.

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